Shape Optimization and the Pontryagin Principle

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Hamilton-Jacobi-Bellman and the Pontryagin Principle

$$\min_{\sigma\in A}\{g(X_T)+\int_0^T h(X_t,\sigma_t)\,dt\}\quad ,X_t'=f(X_t,\sigma_t)\quad ,X(0)=X_0$$

Value function

$$u(x,t)=\inf_{\sigma\in A,X(\tau)=x}\int_\tau^T h(X_t,\sigma_t)\,dt$$

solution to the HJB-equation (for smooth f,h,g)

$$u_t+\overbrace{\min_{\sigma\in A}\{u_x\cdot f(x,\sigma)+h(x,\sigma)\}}^{H(u_x,x)}=0\quad ,u(x,T)=g(x)$$

Differentiation along optimal paths σ_t^*,X_t^* gives Pontryagins Principle

$$\lambda_t'^*=-\lambda_t^* f_X(X_t^*,\sigma_t^*)+h_X(X_t^*,\sigma_t^*),\quad \lambda(T)=g'(X_T^*),\quad (\lambda_t^*\equiv u_x(X_t^*,t))$$

Also

$$\sigma^*=\mathsf{argmin}_{\sigma\in A}\{\lambda_t^*\cdot f(X_t^*,\sigma)+h(X_t^*,\sigma)\}$$

Electric Conduction

Minimize power-loss in conductive medium [Hoppe, Petrova, Schultz 2002]

$$\min_\sigma \left(\int_{\partial\Omega} j\varphi ds + \eta \int_\Omega \sigma dx \right)$$

$$\mathrm{div}(\sigma \nabla \varphi(x)) = 0 \quad x \in \Omega, \qquad \sigma \frac{\partial \varphi}{\partial n}|_{\partial\Omega} = j$$

$$\sigma : \Omega \to \{\sigma_-, \sigma_+\}$$

Lagrangian:

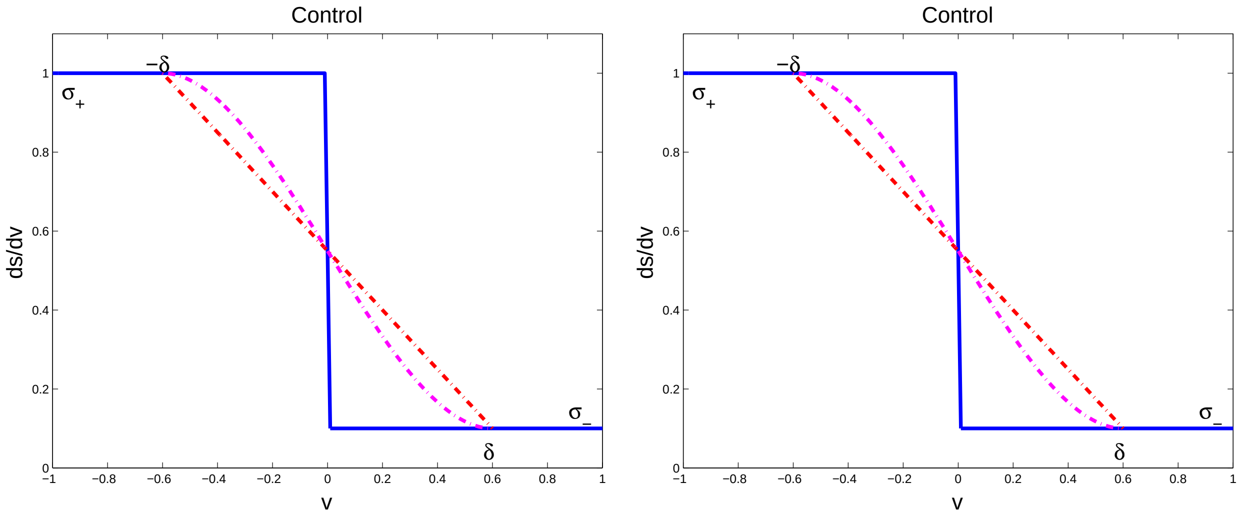
$$\mathcal{L} = \int_{\partial\Omega} j\varphi ds + \int_\Omega \eta\sigma + \mathrm{div}(\sigma\nabla\varphi(x))\lambda dx = \int_{\partial\Omega} j(\varphi + \lambda) ds + \int_\Omega \sigma \underbrace{(\eta - \nabla\varphi \cdot \nabla\lambda)}_{v} dx$$

Hamiltonian:

$$H(\varphi, \lambda) = \min_\sigma \mathcal{L}(\varphi, \lambda, \sigma) = \int_{\partial\Omega} j(\varphi + \lambda) ds + \int_\Omega \underbrace{\min_\sigma \{\sigma(\eta - \nabla\varphi \cdot \nabla\lambda)\}}_{s(v)} dx$$

Control:

$$s'(v) = \sigma^*(v) = \sigma_+ 1_{\{v<0\}} + \sigma_- 1_{\{v>0\}}$$



Regularized Hamiltonian

$$H^{\delta}(\varphi, \lambda) = \int_{\Omega} s_{\delta}(\eta - \nabla\varphi \cdot \nabla\lambda) dx + \int_{\partial\Omega} j(\varphi + \lambda) ds$$

The Pontryagin principle yields

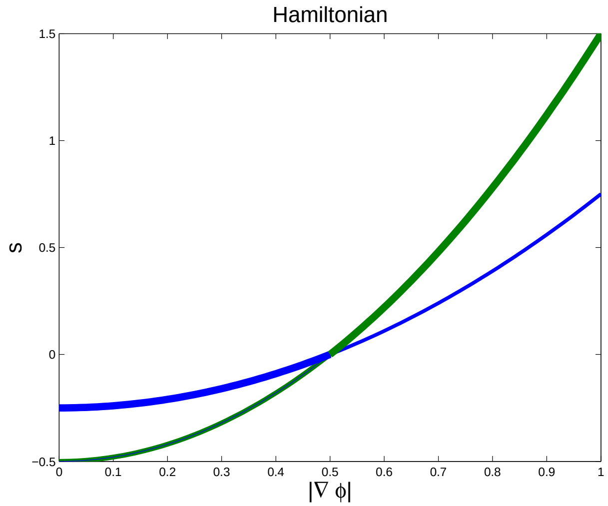
$$0 = \dot{\varphi} = H^{\delta}_{\lambda}$$

$$0 = \dot{\lambda} = -H^{\delta}_{\varphi}$$

Symmetry implies $\varphi = \lambda$ and

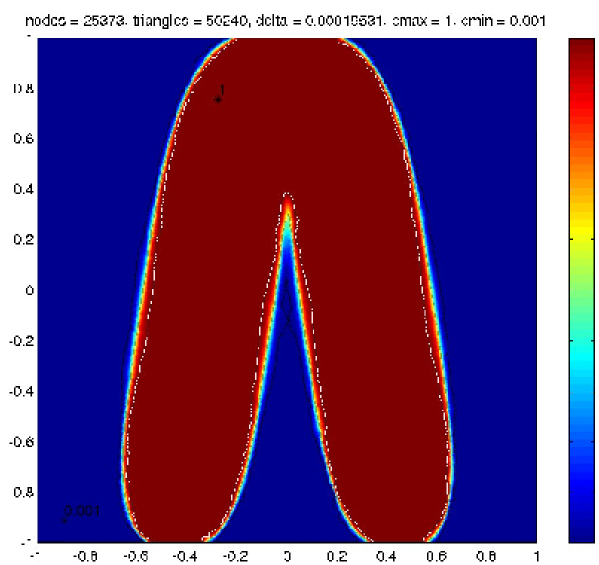
$$\text{div}(s'_{\delta}(\eta - |\nabla\varphi|^2)\nabla\varphi(x)) = 0 \quad x \in \Omega, \quad s'_{\delta}\frac{\partial\varphi}{\partial n}|_{\partial\Omega} = j$$

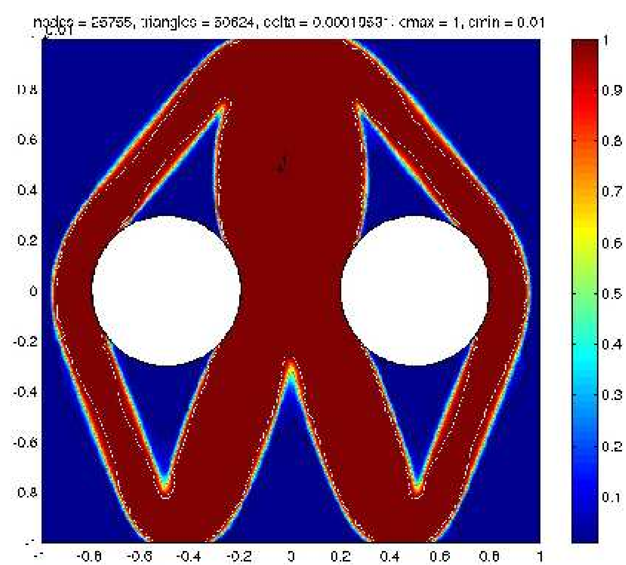
From symmetry we observe



The Hamiltonian is thus strictly convex and we attain the unique minimizer as $\delta \to 0$.

Examples:





Minimal Compliance

Minimal compliance of infinite bar [Kohn, Strang 1986, Allaire 2002]

$$\min_{\sigma}(-\int_{\Omega}\varphi\,dx+\eta\int_{\Omega}\sigma\,dx)$$

$$-\mathsf{div}(\sigma(x)\nabla\varphi(x))=1\quad\text{in }\Omega,\qquad\varphi=0\quad\text{on }\partial\Omega$$

$$\sigma:\Omega\to[\sigma_-,\sigma_+]$$

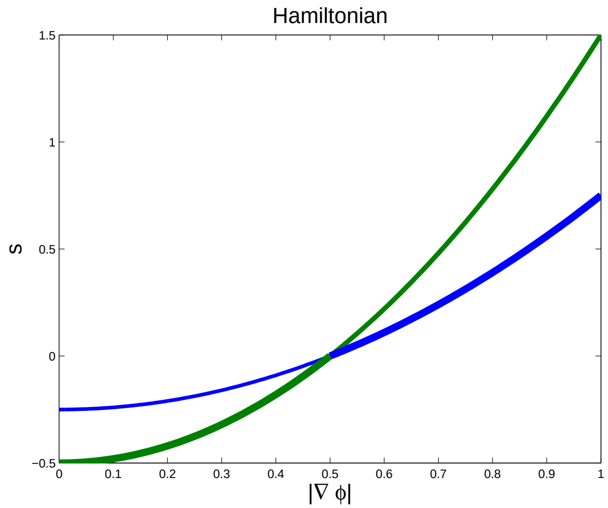
Hamiltonian:

$$H(\varphi,\lambda)=\int_{\Omega}(\lambda-\varphi)+\underbrace{\min_{\sigma}\{\sigma(\overbrace{\eta-\nabla\varphi\cdot\nabla\lambda}^{v})\}}_{s(v)}\,dx$$

Anti-symmetri $\lambda=-\varphi$ gives

$$H^{\delta}(\varphi)=\int_{\Omega}s_{\delta}(\eta+|\nabla\varphi|^2)-2\varphi\,dx$$

Observations from anti-symmetry



- Non-convex problem
- Lower bound on regularization
- Optimal to use a (quasi-) convexified functional since the value functions coincide.

The solution of the convexified problem is the local average (weak limit) of a minimizing sequence for the non-convex problem.

Examples:

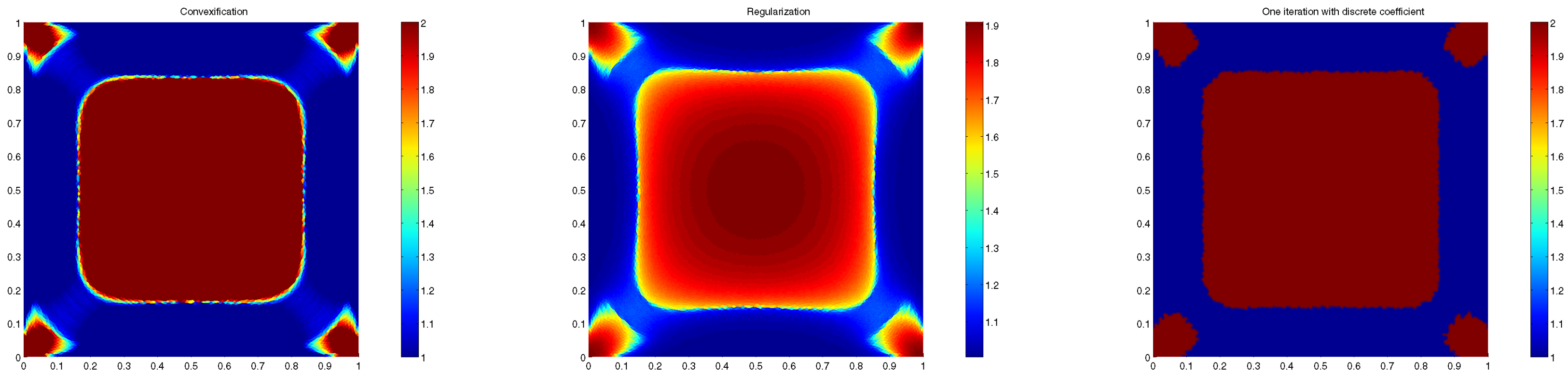


Figure 1: Inverse of the shear moduli on the cross section of an infinitely long bar. Left: Convexified reference solution. Middle: Solution from solving the regularized Hamiltonian system. The relative L2 error in the Hamiltonian is here less than 4%. Right: One additional iteration with σ restricted to $\{\sigma_-, \sigma_+\}$. The relative error is now less than 1%.

Impedance Tomography

Parameter estimation from measurements [Borcea 2002]

$$\min_{\sigma} \sum_{i=1}^{N} \int_{\Gamma_i} (\varphi_i - \bar{\varphi}_i)^2 \, ds$$

$$-\mathrm{div}(\sigma(x)\nabla\varphi_i(x)) = 0 \quad \text{in } \Omega, \quad \sigma\frac{\partial\varphi_i}{\partial n} = j_i \quad \text{on } \partial\Omega$$

$$\int_{\partial\Omega} j \, ds = 0, \quad \sigma : \Omega \to [\sigma_-, \sigma_+]$$

Hamiltonian

$$H(\varphi_1, \ldots, \varphi_N, \lambda_1, \ldots, \lambda_N) = \ldots + \int_{\Omega} \overbrace{\min_{\sigma}\{\sigma \underbrace{\sum_{i=1}^{N} -\nabla\varphi_i \cdot \nabla\lambda_i}_{v}\}}^{s(v)} \, dx$$

- Leads to coupled system of $2N$ equations
- Seems to behave as convex/non-convex problem depending on σ_+/σ_-
- Optimal choice of input currents?
- Averaging $s(v)$ may improve numerical stability

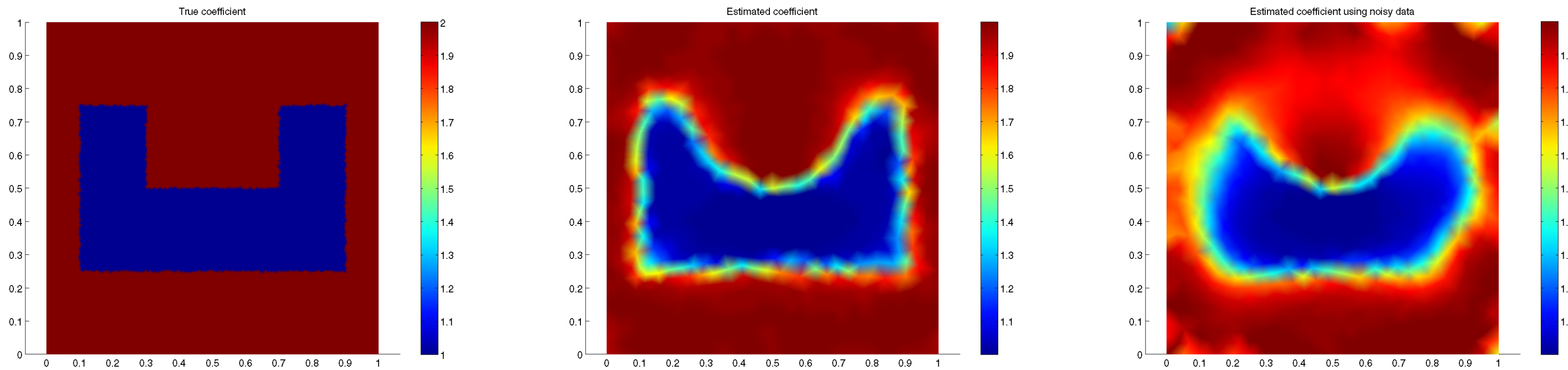


Figure 2: Left: True conductivity. Middle: Estimated conductivity from solving the regularized Hamiltonian system. Right: Estimated conductivity when 5% white noise is added to measurements. Data from four different experiments were used.

Optimal Currents

Optimal currents maximize

$$\frac{\|R(\sigma)j - R(\sigma^*)j\|}{\|j\|}$$

where $R(\sigma)$ is the Neumann-to-Dirichlet operator i.e. optimal currents are eigenfunctions corresponding to largest eigenvalues of R [Isaacson et al 1990].

Min-max formulation

$$\min_{\sigma} \max_{j} \int_{\partial\Omega} (\varphi - \bar{\varphi})^2 \, ds$$

$$-\text{div}(\sigma \nabla \varphi) = 0 \quad \text{in } \Omega, \quad \sigma \frac{\partial \varphi}{\partial n} = j \quad \text{on } \partial\Omega$$

$$-\text{div}(\sigma^* \nabla \bar{\varphi}) = 0 \quad \text{in } \Omega, \quad \sigma \frac{\partial \bar{\varphi}}{\partial n} = j \quad \text{on } \partial\Omega$$

$$\int_{\partial\Omega} j \, ds = 0, \quad \sigma : \Omega \to [\sigma_-, \sigma_+]$$

Hamiltonian

$$H(\varphi,\bar\varphi,\lambda,\bar\lambda) = \ldots + \int_{\partial\Omega} \overbrace{\max_{j\in J} j\,\underbrace{(\lambda+\bar\lambda)}_{\bar v}}^{\bar s(\bar v)}\,ds + \int_\Omega \overbrace{\min_\sigma\{\sigma(\underbrace{-\nabla\varphi\cdot\nabla\lambda}_{v})\}}^{s(v)}\,dx$$

Regularization gives Hamiltonian system

$$-\text{div}(s'_\delta(v)\nabla\varphi) = 0 \quad \text{in } \Omega, \quad s'_\delta(v)\frac{\partial\varphi}{\partial n} = \bar s'_\delta(\bar v) \quad \text{on } \partial\Omega$$

$$-\text{div}(s'_\delta(v)\nabla\lambda) = 0 \quad \text{in } \Omega, \quad s'_\delta(v)\frac{\partial\lambda}{\partial n} = -2(\varphi-\bar\varphi) \quad \text{on } \partial\Omega$$

$$-\text{div}(\sigma^*\nabla\bar\varphi) = 0 \quad \text{in } \Omega, \quad \sigma^*\frac{\partial\bar\varphi}{\partial n} = \bar s'_\delta(\bar v) \quad \text{on } \partial\Omega$$

$$-\text{div}(\sigma^*\nabla\bar\lambda) = 0 \quad \text{in } \Omega, \quad \sigma^*\frac{\partial\bar\lambda}{\partial n} = -2(\varphi-\bar\varphi) \quad \text{on } \partial\Omega$$

Algorithm:

1. Given iterates λ_n, φ_n and j_n we get $\bar{\varphi}_n$ and $\bar{\lambda}_n$ from measurements
2. Update $j_{n+1} = \bar{s}'_\delta(\lambda_n + \bar{\lambda}_n)$.
3. Get $\lambda_{n+1}, \varphi_{n+1}$ from one or more Newton steps on the non-linear system

$$
\begin{aligned}
-\text{div}(s'_\delta(v)\nabla\varphi) &= 0 && \text{in } \Omega \\
s'_\delta(v)\frac{\partial\varphi}{\partial n} &= j_{n+1} && \text{on } \partial\Omega \\
-\text{div}(s'_\delta(v)\nabla\lambda) &= 0 && \text{in } \Omega \\
s'_\delta(v)\frac{\partial\lambda}{\partial n} &= -2(\varphi - \bar{\varphi}_n) && \text{on } \partial\Omega
\end{aligned}
$$