On the Approximation Resistance of a Random Predicate

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Boolean CSPs

Constraint Satisfaction Problems where each input is a bit.

Same predicate appears P in all constraints.

A k-ary predicate P that accepts t of the 2^k inputs.

Examples of CSPs

k-Sat Disjunctions of k literals, $t = 2^k - 1$.

k-Lin Linear equations with k variables in each equation, $t = 2^{k-1}$.

Subspace ℓ dimensional subspace of k dimension, $t = 2^\ell$.

Equivalent predicates

Please note that negations are allowed for free, and so are permutations of the inputs.

We get families of equivalent predicates.

Max-CSPs

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Given a list of m k-tuples of literals find an assignment that makes as many as possible of the resulting k-tuples of bits satisfy P.

Approximation ratio

An algorithm has approximation ratio α if for any instance

$$\frac{\text{Value of found solution}}{\text{Value of optimal solution}} \geq \alpha$$

For randomized algorithms, expectation over internal coinflips, always worst case inputs.

Trivial result

It is easy to approximate Max-P within $t2^{-k}$.

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The trivial approximation ratio.

Approximation resistance

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A predicate P is approximation resistant on satisfiable instances if $\forall \epsilon > 0$ it is hard distinguish instances where we can satisfy all constraints from those where we can only satisfy a fraction $\epsilon + t2^{-k}$ of the constraints.

Hereditary properties

A predicates P is hereditary approximation resistant if whenever $P(x) \Rightarrow Q(x)$ then Q is also approximation resistant.

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Approximation resistance on satisfiable instances is possibly the ultimate hardness for a CSP.

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Are there such predicates?

Binary constraints

Constraints on two variables , $k = 2$.

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Extends to all domains sizes [H05] and binary constraints.

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Max-3-Sat is approximation resistant on satisfiable instances. What happens for the "not two ones predicate" on satisfiable instances?

The case $k = 4$

Partial classification by Hast [H05].

400 essentially different predicates.

- 79 approximation resistant.
- 275 not approximation resistant.
- 46 not classified.

# Acc	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Non-res	1	4	6	19	27	50	50	52	27	26	9	3	1	0	0
Res	0	0	0	0	0	0	0	16	6	22	11	15	4	4	1
Unkn	0	0	0	0	0	0	6	6	23	2	7	1	1	0	0

Satisfiability ignored.

Questions

How common is approximation resistance?

Can we find big classes of approximation resistant predicates?

What about a random predicate?

Random predicates

A random predicate from space $R_{p,k}$ accepts each input with probability p (and has $t \approx p2^k$).

Is a random predicate approximation resistant for $p = 1/2$?

Predicate P^1_{ST}

A predicate given by a subspace of dimension $l_1 + l_2$ with $k = l_1 + l_2 + l_1 l_2$.

Showed to be approximation resistant by Samorodnitsky and Trevisan [ST00] and hereditary so by Hast [H05].

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Gives many approximation resistant predicates but does not apply to a random predicate.

Predicate P_{ST}^2

A predicate given by a subspace of dimension d with $2^{d-1} < k \leq 2^d - 1$.

Assuming the Unique Games Conjecture (UGC) showed to be approximation resistant by Samorodnitsky and Trevisan [ST05].

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A very open conjecture.

Main result

Theorem: Assuming the unique games conjecture a random predicate from $R_{1/2,k}$ is with high probability, for sufficiently large k, approximation resistant.

Extends to $p = k^{-c}$ for $1/2 \leq c \leq 1$, $c \approx k2^{-d}$.

Proof 1

Assuming UGC P^2_{ST} is hereditary approximation resistant.

Extending the proof of Samorodnitsky and Trevisan.

Key Lemma

Lemma: For $S \subseteq [d]$ functions f_S such that

- One function (almost) unbiased, $|E[f_S(x)]| \leq \delta$.
- No two functions have high common influnce, $\max(inf_i(f_{S_1}), inf_i(f_{S_2})) \leq \epsilon$.

$$\left| E_{x_1 \ldots x_d} \left[\prod_{S \subseteq [d]} f_S(\prod_{i \in S} x_i) \right] \right| \leq \delta + (2^d - 2)\sqrt{\epsilon},$$

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New simpler more direct proof compared to [ST05].

Proof 2

Prove that if Q is random from $R_{1/2,k}$ then it is likely that there is a P^2_{ST}-equivalent predicate P' such that $P' \Rightarrow Q$.

Second moment method using only P^2_{ST}-equivalent predicates that are very different.

Summary

- Approximation resistance is a very strong notion of hardness.
- If the Unique Games Conjecture is true then a vast majority of predicates are approximation resistant.

Open problems

1. Prove result without the unique games conjecture.
2. Prove approximation resistance on satisfiable instances.
3. Classify more predicates with respect to approximation resistance.