Half course problem set

Lecturer: Massimo Lauria

http://www.csc.kth.se/~lauria/sos14/

Due: Wednesday, March 12th, 2014 at 23:59. Submit your solutions as a PDF file by e-mail to `lauria@csc.kth.se` with the subject line

`Half course problem set :` ⟨`your full name`⟩

Name the PDF file `PS1`⟨`YourFullName`⟩`.pdf` (with your name coded in ASCII without national characters and no spaces), and also state your name and e-mail address at the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system. Please try to be precise and to the point in your solutions and refrain from vague statements. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules stated on the course webpage always apply.

Collaboration: Discussions of ideas in groups of two people are allowed—and indeed, encouraged—but you should write down your own solution individually and understand all aspects of it fully. You should also acknowledge any collaboration. State at the beginning of the problem set if you have been collaborating with someone and if so with whom.

Reference material: Some of the problems are "classic" and hence it might be easy to find solutions on the Internet, in textbooks or in research papers. It is not allowed to use such material in any way unless explicitly stated otherwise. Anything said during the lectures or in the lecture notes should be fair game, though, unless you are specifically asked to show something that we claimed without proof in class. It is hard to pin down 100% formal rules on what all this means—when in doubt, ask the lecturer.

About the problems: Some of the problems are meant to be quite challenging and you are not necessarily expected to solve all of them. As a general guideline, **a total score of 100 should be sufficient to pass**. Any corrections or clarifications will be given at `sos-14@csc.kth.se` and any revised versions will be posted on the course webpage `http://www.csc.kth.se/~lauria/sos14/`.

Exercise 1 (30 points)**.** In Lecture 2[1] we have given two forms of semidefinite programs: we've used the first to describe primal programs,

$$\begin{aligned} \text{maximize} \quad & C \bullet X \\ \text{subject to} \quad & A_1 \bullet X = b_1 \\ & A_2 \bullet X = b_2 \\ & \vdots \\ & A_\ell \bullet X = b_\ell \\ & X \succeq 0 \end{aligned}$$

where A_i and C are symmetric matrices; we've used the second to express dual programs,

$$\begin{aligned} \text{minimize} \quad & b^T y \\ \text{subject to} \quad & y_1 A_1 + y_2 A_2 + \cdots + y_m A_m \succeq C \end{aligned}$$

where A_i and C are symmetric square matrices and y_is are scalar variables.

Are the two forms able to express the same problems? If they can please show how to translate each form into the other, otherwise please give an argument for the negative answer.

[1] `http://www.csc.kth.se/~lauria/sos14/lecturenotes/lecture2.pdf`

Exercise 2 (30 points). In the proof of Theorem 11 (see Lecture 2[2]) we have shown that if the dual has a strictly feasible solution then the primal optimum is attained. Prove the missing part, namely that when the primal program has strictly feasible solution, then the dual optimum is attained and $v_P = v_D$.

[2] http://www.csc.kth.se/~lauria/sos14/lecturenotes/lecture2.pdf

Exercise 3 (50 points). Recall Motzkin polynomial

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 \tag{1}$$

(15 points) use the inequality $\sqrt[3]{abc} \leq \frac{a+b+c}{3}$ to prove that $M(x, y, z) \geq 0$;

(15 points) show a sum of squares decomposition of

$$4x^2 + x^2y^2 + 2xyz + y^2z^2 + z^2$$

(20 points) show that $M(x, y, -1)$ has no sum of squares decomposition;

Exercise 4 (10 points). Consider the inference system that manipulates equations in Positivstellensatz Calculus, namely the system that has the inference rules

$$\frac{}{f_j} \qquad \frac{}{x_i^2 - x_i} \qquad \frac{p \quad q}{\alpha p + \beta q} \qquad \frac{p}{x_i p}$$

where x_i is a variable, f_j is the polynomial corresponding to an initial equation $f_j = 0$, and α and β are arbitrary real values. This inference system is called *polynomial calculus*.

A polynomial is called *multilinear* when none of its monomials has a variable raised to a power greater than one. Let's call *multilinear polynomial calculus* the inference system with the following rules

$$\frac{}{f_j} \qquad \frac{p \quad q}{\alpha p + \beta q} \qquad \frac{p_0 + x_i p_1}{x_i p_0 + x_i p_1}$$

where x_i is a variable, f_j is an initial multilinear polynomial, α and β are arbitrary real values, and p_0, p_1 are polynomials that do not contain the variable x_i.

Show that a multilinear polynomial calculus derivation that contains just polynomials of degree at most D can be translated into a polynomial calculus derivation of degree $D + 1$ of comparable size (i.e., at most a polynomial in the size of the original multilinear derivation).

Exercise 5 (30 points). Given a graph $G = (V, E)$, a k-clique is set of k vertices in V that are pairwise connected by an edge. We express the problem of finding a k-clique in a simple graph $G = (V, E)$ as the following integer program defined on variables x_v for $v \in V$,

$$\begin{aligned} \text{find} \quad & x \\ \text{subject to} \quad & x_v + x_w \leq 1 \qquad \text{for } \{v, w\} \notin E \\ & \sum_{v \in V} x_v \geq k \\ & x_v \in \{0, 1\}. \end{aligned} \tag{2}$$

Let Q_t be the Lasserre relaxation of rank t. Answer to the following questions, and provide a justification for each answer:

- (15 points) if G has no clique of size k, prove that Q_k is the empty polytope;
- (15 points) if G has cliques of size k but not of size $k+1$, describe the structure of Q_k;

Exercise 6 (40 points)**.** The first step in the degree lower bound for Positivstellensatz Calculus refutation of random 3-XOR is the translation of all variables x_i into variables y_i according to the substitution $x_i = \frac{1-y_i}{2}$.

A degree lower bound in this new representation implies, since the transformation is linear, that the degree lower bound must hold also for the Positivstellensatz Calculus refutation of the formulation on variables x_i.

What about the corresponding Lasserre hierarchies? The Lasserre hierarchies for the two formulations are expressed, respectively, as semidefinite programs over variables X_S and Y_S for $S \subseteq [n]$, with the intended meaning

$$\prod_{i \in S} x_i^{d_i} \longleftrightarrow X_S$$
$$\prod_{i \in S} y_i^{d_i} \longleftrightarrow Y_{\{i \,:\, d_i \text{ is odd}\}}.$$

Assume that $Y_S, S \subseteq [n]$ is a feasible solution for the rank t Lasserre relaxation of the second formulation of 3-XOR. Show that by setting

$$X_S := \frac{1}{2^{|S|}} \left[\sum_{T \subseteq S} (-1)^{|T|} Y_T \right] \tag{3}$$

for all $S \subseteq [n]$, we get a feasible solution for the rank t Lasserre relaxation of the original formulation over x variables.