Final problem set

Lecturer: Massimo Lauria

http://www.csc.kth.se/~lauria/sos14/

Due: Thursday, May 15th, 2014 at 23:59. Submit your solutions as a PDF file by e-mail to lauria@csc.kth.se with the subject line

`Final course problem set : ⟨your full name⟩`

Name the PDF file PS2⟨YourFullName⟩.pdf (with your name coded in ASCII without national characters and no spaces), and also state your name and e-mail address at the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system. Please try to be precise and to the point in your solutions and refrain from vague statements. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules stated on the course webpage always apply.

Collaboration: Discussions of ideas in groups of two people are allowed—and indeed, encouraged—but you should write down your own solution individually and understand all aspects of it fully. You should also acknowledge any collaboration. State at the beginning of the problem set if you have been collaborating with someone and if so with whom.

Reference material: Some of the problems are "classic" and hence it might be easy to find solutions on the Internet, in textbooks or in research papers. It is not allowed to use such material in any way unless explicitly stated otherwise. Anything said during the lectures or in the lecture notes should be fair game, though, unless you are specifically asked to show something that we claimed without proof in class. It is hard to pin down 100% formal rules on what all this means—when in doubt, ask the lecturer.

About the problems: Some of the problems are meant to be quite challenging and you are not necessarily expected to solve all of them. As a general guideline, **a total score of 120 should be sufficient to pass**. Any corrections or clarifications will be given at sos-14@csc.kth.se and any revised versions will be posted on the course webpage http://www.csc.kth.se/~lauria/sos14/.

Exercise 1 (10 points)**.** We consider a system of constraints

$$f_1 = 0, \dots, f_k = 0,\ h_1 \geq 0, \dots, h_m \geq 0, \tag{1}$$

where f_i and h_i are polynomials of $x_1, \dots, x_n$, where inequalities $0 \leq x_i \leq 1$ are among the h_js, and the equations $x_i^2 - x_i = 0$ are among the f_js.

In the fourth lecture[1] we defined a Positivstellensatz proof of a polynomial p as a an equation $p = f + h$ where

$$f = \sum_{s=1}^{k} f_s g_s, \quad h = \sum_{I \subseteq \{1,\dots,m\}} \Big(\prod_{i \in I} h_i\Big)\Big(\sum_j e_{I,j}^2\Big), \tag{2}$$

where $e_{I,j}$ and g_s are arbitrary polynomials.

Show that we can prove the same polynomials in the same degree if we substitute each initial equation $f_j = 0$ with two initial inequalities $f_j \geq 0$ and $-f_j \geq 0$.

[1] http://www.csc.kth.se/~lauria/sos14/lecturenotes/lecture4.pdf

Exercise 2 (20 points). In the eleventh lecture[2] we sketched the proof of Lemma 2. Please present a full proof of the lemma, using the result from (Gottlieb, 1966)[3] mentioned in the proof sketch.

[2] http://www.csc.kth.se/~lauria/sos14/lecturenotes/lecture11.pdf

[3] Daniel Henry Gottlieb. A certain class of incidence matrices. *Proceedings of the American Mathematical Society*, 17(6):1233–1237, Dec 1966

Positivity of the quadratic operator Q_ℓ

The next series of exercises integrates the main proof in the twelfth lecture[4] and the notation here follows the one of the lecture.

[4] http://www.csc.kth.se/~lauria/sos14/lecturenotes/lecture12.pdf

Consider a vector $u \in A_t$ (i.e., Ker D_{t-1}), and recall that u is an homogeneous multilinear polynomial of degree t, and that by definition for every $I \subseteq [n]$ with $|I| = t-1$ it holds that

$$\sum_{J \supset I} u_J = 0$$

where u_J is the coefficient of u corresponding to the monomial $\prod_{i \in J} x_i$, with J that ranges over all sets of size t.

Exercise 3 (15 points). Prove that for every $T \subseteq [n]$ with $|T| < t$ it holds that

$$\sum_{J \supset T} u_J = 0$$

where J ranges over all sets of size t.

Exercise 4 (35 points). Prove that for every $X \subseteq Y \subseteq [n]$ with $|X| \leq t$ it holds that

$$\sum_{J \cap Y = X} u_J = (-1)^{t-|X|} \sum_{X \subseteq J \subseteq Y} u_j. \tag{3}$$

where J ranges over all sets of size t.

Exercise 5 (20 points). Use Equation (3) to show Lemma 4 in Lecture 12, namely that for $m < \lfloor n/2 \rfloor$ and any $u \in \text{Ker}\, D_{t-1}$ it holds that

$$D_m C_m(u^{(m)}) = (n-m-t)(m-t+1)u^{(m)}.$$

Boolean functions and small set expansion

For the following exercises we follow the notation and the naming convention of the thirteenth lecture[5].

[5] http://www.csc.kth.se/~lauria/sos14/lecturenotes/lecture13.pdf

Exercise 6 (10 points). Compute the eigenvalues for the normalized adjacency matrix of the hypercube graph, i.e., the graph on vertex set $\{0,1\}^n$ such that $\{x,y\}$ is an edge when $|x \oplus y| = 1$.

Exercise 7 (20 points). Compute the small-set-expansion for the hypercube.

Exercise 8 (20 points). Consider the hypercube graph. Compute $\Phi(S)$ for $S \subseteq \{0,1\}^n$ in the following two cases:

(10 points) $S = \{0,1\}^m \times 1^{n-m}$ for $n > m > 0$;

(10 points) S is the set of all the binary sequences of length n at distance at most αn from the zero sequences. In particular compute the value for $\alpha = \frac{1}{2}$.

Exercise 9 (20 points)**.** Show that for any multilinear polynomial p of degree $d > 0$ over n variables it holds that

$$\left|\{x \in \{0,1\}^n | p(x) \neq 0\}\right| \geq \frac{2^n}{2^d}, \tag{4}$$

and give an example for every $d > 0$ for which the inequality is tight.

Noise operator

The next exercise deals with the **noise operator**. For any $\rho \in [-1,1]$ we define the operator T_ρ over real valued functions (i.e. from $\{-1,1\}^n$ to $\mathbb{R}$).

Given $x \in \{-1,1\}^n$ we say that the random y is ρ-correlated with x if y is distributed according to the following distribution.

$$y_i = \begin{cases} x_i & \text{with probability } \frac{1+\rho}{2} \\ -x_i & \text{with probability } \frac{1-\rho}{2}. \end{cases} \tag{5}$$

Then for any function $f : \{-1,1\}^n \to \mathbb{R}$ we define the linear operator

$$T_\rho f(x) = \mathbb{E}_y[f(y)] \tag{6}$$

where y is ρ-correlated to x. We observe that

$$T_1 f(x) = f(x) \qquad T_{-1} f(x) = f(-x) \qquad T_0 f(x) = \frac{\sum_{y \in \{-1,1\}^n} f(y)}{2^n}$$

We recall the notation of the p-norm operator

$$\|f\|_p = \left(\mathbb{E}_x |f(x)|^p\right)^{\frac{1}{p}}.$$

In the lecture we have seen the following version of the Hypercontractive inequality

$$\|T_{\frac{1}{\sqrt{3}}} f\|_4 \leq \|f\|_2.$$

This was done proceeding by induction on the number of variables and observing that the expected value of x^d for $x \in \{-1,1\}$ is either 0 or 1, according to the parity of d. That is just a particular version (for $q = 4$) of the more general inequality

$$\|T_{\frac{1}{\sqrt{q-1}}} f\|_q \leq \|f\|_2. \tag{7}$$

Exercise 10 (30 points)**.** Prove the special case

$$\|T_{\frac{1}{\sqrt{5}}} f\|_6 \leq \|f\|_2$$

which corresponds to $q = 6$.

Hölder inequality

We recall **Hölder inequality** which states

$$\mathbb{E}_x f(x) \cdot g(x) \le \|f\|_p \|g\|_q \tag{8}$$

for any pair $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. This inequality cannot be represented in Sum of squares proofs, such proofs rely on an alternative pointwise version of it, called **Young Inequality**,

$$XY \le \frac{1}{p} X^p + \frac{1}{q} Y^q \qquad \text{for all } X, Y \ge 0.$$

Non integer powers which are not representable in Sum of squares, thus we fix $p = \frac{a+b}{a}$ and $q = \frac{a+b}{b}$ with a, b positive integers and substitute $X = x^a$ and $Y = y^b$. Next exercise asks to prove the resulting inequality.

Exercise 11 (40 points)**.** Assume $a + b = 2^k$ for $k > 0$. Prove the inequality

$$x^a y^b \le \frac{a}{a+b} x^{a+b} + \frac{b}{a+b} y^{a+b}$$

with a sum of squares proof of degree $a + b$.

References

[Got66] Daniel Henry Gottlieb. A certain class of incidence matrices. *Proceedings of the American Mathematical Society*, 17(6):1233–1237, Dec 1966.