Affine space

Lecture notes by Alex Loiko

Problem 1

$$f_1 = x^5 + x^4 + x^3,\ (f_1)^* = x^5 + x^4y + x^3y^2$$
$$f_2 = 2x^3 + 4x^2 + 6,\ (f_2)^* = 2x^3 + 4x^2y + 6y^3$$
$$f_3 = x,\ (f_3)^* = x$$

Problem 2 Homogenize in $\mathbb{C}[x, y, z]$

$$\begin{aligned} f_1 &= x^5 + y^5x^4 + yx^3, & (f_1)^* &= x^5z^4 + y^5x^4 + x^3yz^5 \\ f_2 &= 2x^3 + 4yx^2 + 6y^4, & (f_2)^* &= 2x^3z + 4yx^2z + 6y^4 \\ f_3 &= xy^7, & (f_3)^* &= xy^7 \end{aligned}$$

Problem 3 Dehomogenize wtr to y

$$F_1 = x^5 + yx^4 + 7y^2x^3,\ (F_1)_* = x^5 + x^4 + 7x^3$$
$$F_2 = x^3 + 4yx^2 + 60y^3,\ (F_2)_* = x^3 + 4x^2 + 60$$
$$F_3 = xy^7,\ (F_3)_* = x$$

Problem 4 If $F_3 = xy^7$, then $(F_3)_* = x$ and $((F_3)_*)^* = x \neq F_3$

Problem 5 We have $F = y^t(F_*)^*$ for some t. Write

$$F = \sum_i^n a_i x^i y^{d-i}, d = \deg(F)$$

Then $F_*(x) = F(x, 1)$ and that is

$$\sum_i^n a_i x^i 1^{d-i} = \sum_i^n a_i x^i = \sum_i^n \frac{a_i x^i y^{d-i}}{y^{d-i}}$$

When we homogenizie it back we have

$$(F_*)^* = \sum_i^n a_i x^i y^{d'-i}$$

for some d' not necessary equal to d and we can choose $t = y^{d-d'}$.

Definition 1. *Let $f \in \mathbb{C}[x,y]$, f is not a constant polynomial. Then f is* ***reducible*** *if there exist $h, g \in \mathbb{C}[x,y]$, both non-constant such that*

$$f = h \cdot g$$

We say that f is irreducible if there does not exist such polynomials h, g. Thus a polynomial can be constant, reducible and irreducible.

For example $x+1$ in $\mathbb{C}[x,y]$ is irreducible, $x^2 - y^2$ in $\mathbb{C}[x,y]$ is reducible since $x^2 - y^2 = (x-y)(x+y)$. $x^2 + y^2$ is reducible in $\mathbb{C}[x,y]$ but not in $\mathbb{R}[x,y]$.

Definition 2. *$\mathbb{A}^1(\mathbb{R})$ is called* ***real affine line****. It is a set. We can think of it as a line that contains all the real points. We can have $\mathbb{A}^1(0)$ or $\mathbb{A}^1(i)$ in $\mathbb{A}^1(\mathbb{C})$. Here comes the formal definition:*

$$A^1(\mathbb{R}) = \{r | r \in \mathbb{R}\}$$

$$A^1(\mathbb{C}) = \{w | w \in \mathbb{C}\}$$

$\mathbb{A}^2(\mathbb{R})$ is called the ***affine real plane****. Similarly we have $\mathbb{A}^2(\mathbb{C})$ is called the* ***affine*** COMPLEX ***plane*** *Formally,*

$$\mathbb{A}^2(\mathbb{R}) = \{(r_1, r_2) | r_1, r_2 \in \mathbb{R}\}$$

$$\mathbb{A}^2(\mathbb{C}) = \{(w_1, w_2) | w_1, w_2 \in \mathbb{C}\}$$

An affine space is not the same as a vector space. Here comes some examples. We study $\mathbb{A}^1(\mathbb{R})$. To do it we consider polynomials in $\mathbb{R}[x]$. We pick $p(x) = x+1$. To that polynomial we associate its root $x_0 = -1$. Now pick $h(x) = x^2 - 1$. It has two roots $\{x_1, x_2\} = \{-1, 1\}$. We associate them to that polynomial. The assosiated set of a polynomial may be empty if the polinomial doesn't have real roots.

Now consider $\mathbb{A}^2(\mathbb{R})$ and the polynomial $p(x,y) = x+1$. It has roots $(-1, y_0)$ for any y_0. $h(x,y) = x^2 - 1$. It has roots $(-1, y)$ and $(1, y)$ for any y.

Definition 3. *Let $f \in \mathbb{R}[x]$; we call $V(f)$ the* ***affine real variety*** *associated to f where*

$$V(f) \subset \mathbb{A}^1(\mathbb{R}), V(f) = \{r \in \mathbb{A}^1(\mathbb{R}) : f(r) = 0\}$$

Similarly, we define $V(f)$ for two-variable polynomials.

We always have

$$V(1) = \emptyset$$

In $\mathbb{A}^1(\mathbb{R})$

$$V(x+1) = \{-1\}$$

In $\mathbb{A}^2(\mathbb{R})$

$$V(x^2 + y^2 - 1) = \text{ unit circle}$$

In $\mathbb{A}^2(\mathbb{R})$,

$$V(x^2 + y^2 + 1) = \emptyset$$

The reason for introducing $\mathbb{C}$ is to make sure any polynomial of $\deg n$ always has n roots in $\mathbb{A}^1(\mathbb{C})$.

Theorem 1. ***FTA***
If

$$f(x) \in \mathbb{C}[x]$$

then there always exist a number

$$w \in \mathbb{C} f(w) = 0$$

Corollary 1. *If* $\deg(f) = d$*,* $f \in \mathbb{C}[x]$*, there exist* $\beta_1 \dots \beta_n \in \mathbb{C}$

$$f(x) = \alpha(x - \beta_1) \dots (x - \beta_d)$$

Consider a homogenoous polynomial $f(x, y)$ in $\mathbb{C}[x, y]$ with degree d. Write it as

$$f(x, y) = \sum_{i=0}^{d} \alpha_i x^i y^{d-i}$$

Then

$$f_* = f(x, 1) = \sum_{i=0}^{d} \alpha_i x^i \in \mathbb{C}[x]$$

and

$$f_* = \alpha(x - \beta_1)(x - \beta_2) \dots (x - \beta_d)$$

We can prove $(fg)^* = f^* g^*$ which immideately proves that $(f_*)^* = y^t f$ by **Problem 5**

Going back to varietys and affine spaces, we have in $\mathbb{A}^2(\mathbb{R})$ that

$$V(f(x, y)) = \text{ lines passing through the origin}$$