Decimal game

Solutions by Alex Loiko

Problem 1. *Given a countable set S, players A and B play a game. Starting with the interval $I = [0, 1]$, each player in turn shrinks the interval to a new interval inclosed in the previous one. If the sequence of these intervals collapse to a single number in S, A wins, if it collapses to a number not in S, B wins and if it does not converge it is a draw.*

I will use a variant of Cantor's diagonal process to find a winning strategy for player B. At each stage in the game, we have an interval given by its upper and lower bounds u and l. Write u, l in decimal notation and consider the decimal representation of an arbitrary number x with $l \leq x \leq u$. Some of the decimals of x will be fixed for all such x, some will only assume certain values and some may assume any values.

Example 1. *Let $u = 0,0001, l = 0,000099$. Then for any x with $l \leq x \leq u$ we have*

$$x = 0,000x_4x_5x_6\dots$$

The first 3 decimals are fixed for all x, x_4, x_5 may only assume values in $\{0, 9\}$ but starting from x_6 there is no bound on what values x_i may take. Also note that $u - l = 10^{-6}$. A player may want to guarantee the occurrence of the sequence 12345 in the decimal expansion of the resulting number. The player can accomplish this by choosing $(l', u') = (0.0000991234556, 0.0000991234558)$.

It is not difficult to see that any decimal after k will assume any values for k such that $u - l > 10^{-k}$. We are thus free at any point in the game to shrink the interval to $[l', u']$ in a way so that all numbers $x \in [l', u']$ will contain an arbitrary sequence of arbitrary length in their decimal expansion.

Now, let's diagonalize!

S is countable and can be enumerated by

$$\begin{aligned} S_1 &= s_1.s_{11}s_{12}s_{13}\dots \\ S_2 &= s_2.s_{21}s_{22}s_{23}\dots \\ S_3 &= s_3.s_{31}s_{32}s_{33}\dots \\ &\vdots \end{aligned}$$

If player B's strategy on the n :th turn is to fix the first free (not bounded) single decimal on the resulting number in a way it does not coincide with the corresponding (at the same position) number in S_n the resulting number will not be part of the set S because it will differ from any $S_i, i \geq 1$ at at least one point. If we want, we can specify exactly which number B chooses by setting x_k where k is the smallest decimal B is allowed to change freely to

$$x_k = \begin{cases} 0 \text{ if } s_{n,k} \neq 0 \\ 1 \text{ otherwise} \end{cases}$$

It is clear that the sequence will collapse since we fix at least one new decimal every second move.