Investigations on the properties of the arithmetic derivative

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Abstract

We investigate the properties of arithmetic differentiation, an attempt to adapt the notion of differentiation to the integers by preserving the Leibniz rule, (ab)' = a'b+ab'. This has proved to be a very rich topic with many different aspects and implications to other fields of mathematics and specifically to various unproven conjectures in additive prime number theory. Our paper consists of a self-contained introduction to the topic, along with a couple of new theorems, several of them related to arithmetic differentiation of rational numbers, a topic almost unexplored until now.

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1 An arithmetic derivative

The arithmetic derivative function, from here and throughout the entire text denoted by n', is a function $n' : \mathbb{N} \to \mathbb{N}$ defined recursively by

Definition 1.0.1. • p' = 1 for all prime numbers

• (ab)' = a'b + ab' for all natural numbers a, b

We will begin by computing the arithmetic derivative (henceforth sometimes referred to as AD) for two interesting special cases.

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Theorem 1.0.1.

1' = 0

Proof. Using the Leibniz rule it is possible to prove that

$$1 = 1^2 \Rightarrow 1' = (1^2)' \Leftrightarrow 1' = 1 \cdot 1' + 1' \cdot 1 \Leftrightarrow 1' = 2 \cdot 1' \Rightarrow 1' = 0$$

Theorem 1.0.2.

0' = 0

Proof. This proof is similar to the previous one.

$$0 = 2 \cdot 0 \Rightarrow 0' = 2' \cdot 0 + 2 \cdot 0' \Leftrightarrow 0' = 2 \cdot 0' \Rightarrow 0' = 0$$

We will shortly prove that n^\prime is well-defined. The proof depends on the following theorem.

Theorem 1.0.3. The solutions of the functional equation

$$L: \mathbb{N} \to \mathcal{S}$$
$$L(a) + L(b) = L(ab) \tag{1}$$

in which S is an arbitrary ring under the usual operations + and \cdot are given by

$$L(n) = \sum_{i=1}^{k} \alpha_i f(p_i) \tag{2}$$

where $\prod_{i=1}^{k} p_i^{\alpha_i} = n$ is the canonical prime factorization of n and $f : \mathbb{P} \to S$ is any function from the set \mathbb{P} of all primes to S.

Proof. First we show that all solutions to (1) are of the form (2). It follows by induction that

$$L\left(\prod_{i=1}^{k} a_i\right) = \sum_{i=1}^{k} L(a_i)$$

and

$$L\left(a^{b}\right) = bL(a)$$

Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ be an arbitrary integer. We find that

$$L(n) = \sum_{i=1}^{k} \alpha_i L(p_i)$$

We define $f : \mathbb{P} \to \mathcal{S}$ as f(p) = L(p) for every prime p. Then

$$L(n) = \sum_{i=1}^{k} \alpha_i f(p_i)$$

Next we prove that for every function $f : \mathbb{P} \to \mathcal{S}$ is $L(n) = \sum_{i=1}^{k} \alpha_i f(p_i)$ a solution to (1)

We let $a = \prod_{i=1}^{k} p_i^{\alpha_i}, \ b = \prod_{i=1}^{k} p_i^{\beta_i}$ in (1)

LHS =
$$L(a) + L(b) = L\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) + L\left(\prod_{i=1}^{k} p_i^{\beta_i}\right) =$$

$$\sum_{i=1}^{k} (\alpha_i + \beta_i) f(p_i) = L\left(\prod_{i=1}^{k} p_i^{\alpha_i + \beta_i}\right) = \text{RHS}$$

Definition 1.0.2. We call f the prime function of L. A solution L to (1) in 1.0.3 we call an arithmetically logarithmic function.

Theorem 1.0.4. The derivative n' defined in (1.0.1) is well-defined.

Proof. Let $ld(n) = \frac{n'}{n}$ then the conditions on ld are:

- $ld(p) = \frac{1}{p}$ for all prime numbers p
- $ab \cdot ld(ab) = ab \cdot ld(a) + ab \cdot ld(b) \Leftrightarrow ld(a) + ld(b) = ld(ab)$

According to 1.0.2 *ld* is an arithmetically logarithmic function with the prime function $f(p) = \frac{1}{p}$. Then, by 1.0.3, it is well defined and can be written as

$$ld\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) = \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$$
 or $n' = n \sum_{i=1}^{k} \frac{\alpha_i}{p_i}$

2 General properties of the derivative

2.1 Inequalities

Here we will present some general properties of the arithmetic derivative. All of these theorems were originally proved in [1] and are presented here for two reasons: we will use several of the theorems and definitions later and the proofs provide interesting examples of previous work in the field. **Theorem 2.1.1.** Let n be a natural number and k be the smallest prime factor in n. Then or every natural number n,

$$\frac{n \cdot \log_k n}{k} \geq n'$$

with equality iff n is a power of k.

Proof. If

$$n = \prod_{i=1}^{m} p_i^{a_i}$$

is the unique prime factorization of n, then, according to (1.0.4)

$$n' = n \cdot \sum_{i=1}^{m} \frac{\alpha_i}{p_i} \le n \sum_{i=1}^{m} \frac{\alpha_i}{k}$$
$$= n \sum_{i=1}^{m} \frac{1}{k}$$

where the last sum iterates from one to the sum of all α_i . The last expression is not greater than

$$n \cdot \frac{1}{k} \log_k n$$

because $\sum_{i=1}^{m} \alpha_i \leq \log_k n$ with equality iff n is a perfect power of k.

Theorem 2.1.2. For every natural non-prime n with k prime factors,

$$n' \geq kn^{\frac{k-1}{k}}$$

Proof. If

$$n = \prod_{i=1}^{k} p_i$$

is the unique prime factorization of n, where a prime factor may appear several times, then

$$n' = n \sum_{i=1}^{k} \frac{p'_i}{p_i} = n \sum_{i=1}^{k} \frac{1}{p_i} \ge nk \left(\prod_{i=1}^{k} \frac{1}{p_i}\right)^{1/k} = n' \ge kn^{\frac{k-1}{k}}$$

according to the AG inequality.

Theorem 2.1.3. The arithmetic derivative is uniquely defined over the integers by the rule (-x)' = -(x')

Proof. First, we attempt to find the derivative of -1. After observing that $(-1)^2 = 1$, this is easy. $(-1)^2 = 1 \rightarrow ((-1)^2)' = 1' \leftrightarrow 2 \cdot (-1) \cdot (-1)' = 0$ (according to the Leibniz rule) $\leftrightarrow (-1)' = 0$.

Now we can use this new knowledge to derive any negative integer. For every positive $k, (-k)' = ((-1)k)' = (-1)'k + (-1)k' = 0 \cdot k - (k') = -(k')$

(-k)' = -(k') for every integer k, or in other words, the arithmetic derivative is an odd function.

Theorem 2.1.4. If we wish to preserve the Leibniz rule, then the arithmetic derivative is uniquely defined over the rational numbers by the rule (a/b)' = $(a'b - b'a)/b^2$.

Proof. If we wish to preserve the Leibniz rule, then 1' must be equal to 0. From this we get the following equality for every non-zero integer n.

 $(n/n)' = 0 \Leftrightarrow n' \cdot (1/n) + n \cdot (1/n)' = 0 \Leftrightarrow (1/n)' = -n'/n^2$. Now we can show that $(a/b)' = a' \cdot (1/b) + a \cdot (1/b)' = \frac{a'/b - ab'}{b^2 p} = \frac{a'b - ab'}{b^2}$. Now we will prove that this formula is well-defined. It is sufficient to show

that $\frac{ac}{bc}' = \frac{a}{b}'$.

$$\left(\frac{ac}{bc}\right)' = \frac{(ac)'bc - ac(bc)'}{(bc)^2} = \frac{(a'c + ac')bc - ac(b'c + bc')}{(bc)^2} = \frac{c^2(a'b - ab')}{(bc)^2} = \frac{a'b - b'a}{b^2} = \left(\frac{a}{b}\right)'$$

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3 Further properties of the derivative

3.1 The rational derivative is unbounded

It would be interesting to find a upper and lower bound for n' like the ones described in (2.1.1) and (2.1.2) when n is an arbitrary rational number.

Definition 3.1.1.

$$P(a,b) = \begin{cases} True & if \quad \forall \ L \in \mathbb{Q} \ \exists \ x \in (a,b) : \ x' \ge L \\ False & else \end{cases}$$

Or more simply that the function is true when for arbitrarily large L the rational interval (a, b) contains another rational number which when differentiated is not smaller than L. With this definition made we will address the following theorem.

Theorem 3.1.1. In any rational interval there exists a rational number with arbitrary large or small derivative.

Proof. This proof is rather long and depends on several lemmas.

Lemma 3.1.1.

$$P\left(\frac{1}{2},1\right)$$
 is True

Proof. We construct a sequence $\left\{a_i = \frac{2^i}{p_i}\right\}_{i=2}^{\infty}$ where p_i is the smallest prime between 2^{i-1} and 2^i . Such a p_i always exists according to Bertrand's postulate. Observe the sequence of all numbers a'_i . By the rules of arithmetical differentiation (2.1.4),

$$a_i' = \left(\frac{2^i}{p_i}\right)' = \left(\frac{2^{i-1} \cdot i}{p_i} - \frac{2^i}{p_i^2}\right)$$

since $2^{i-1} < p_i < 2^i$, we easily find that $a'_i > (\frac{i}{2} - \frac{1}{2^{i-2}})$ which obviously becomes arbitrary large as *i* increases. All a_i 's lies between 1/2 and 1, so our proof is complete.

Lemma 3.1.2. $P(a,b) \Rightarrow P(ka,kb)$ for positive rationals a, b and k.

Proof. We need to prove that for all N, there are numbers in (ka, kb) with derivative $\geq N$. We choose a rational $c \in (a, b)$ with $c' \geq \frac{N-k'a}{k}$ (such a c always exists according to the definition of P). It is evident that ka < kc < kb. By the rules of differentiation we have that $(kc)' = k'c+c'k \geq k'c+N-k'a \geq N$ from the inequality on c' and because c > a.

Lemma 3.1.3. P(a, 2a) holds.

Proof. This follows directly from (3.1.1) and (3.1.2).

Lemma 3.1.4. P(a, a + 1) is true for all positive rationals a.

Proof. We prove this by contradiction. Assume that P(a, a+1) is false for some a. Then it follows from (3.1.3) that P(a + 1, 2a) is true ((3.1.3) basically says that between a and 2a there are numbers with large derivatives. The assumption says that these numbers are not in (a, a+1)). By using (3.1.2) with $k = \frac{a}{a+1}$ we know that $P\left(a, 2a \cdot \frac{a}{a+1}\right)^n$ is true. Inductively repeating this procedure shows that $P\left(a, 2a \cdot \left(\frac{a}{a+1}\right)^n\right)$ is also true. We did earlier assume that P(a, a+1) was not. That now leads to contradiction since $2a \cdot \left(\frac{a}{a+1}\right)^n < a+1$ for sufficiently large values of n (remember that a is positive so $0 < \frac{a}{a+1} < 1$ and $r^n \to 0$ as n goes to infinity for all 0 < r < 1). But wait! It's not! Because of the fact that $2a \cdot \left(\frac{a}{a+1}\right)^n \to 0$ as n grows large, it will eventually become less than a and we can no longer use the

$$[\mathbf{not} \ P(a, a+1) \land P(a, B)] \Rightarrow P(a+1, B)$$

argument. But if we prove that there exists a value of n such that $a < 2a \cdot \left(\frac{a}{a+1}\right)^n < a+1$ everything would be all right again. In fact it does. Let $n \in \mathbb{N}$ be the greatest number such that $a < 2a \cdot \left(\frac{a}{a+1}\right)^n$. This means that

$$a \ge 2a \cdot \left(\frac{a}{a+1}\right)^{n+1}$$

This is equivalent to

$$\left(\frac{a+1}{a}\right) \cdot a \geq 2a \cdot \left(\frac{a}{a+1}\right)^n \Leftrightarrow a+1 \geq 2a \cdot \left(\frac{a}{a+1}\right)^n$$

which is exactly what we wanted to prove. We have a contradiction and P(a, a+1) is true for every positive rational a.

Lemma 3.1.5. P(a, a + c) is true for all positive rational a, c.

Proof. By lemma (3.1.4), $U\left(\frac{a}{c}, \frac{a}{c} + 1\right)$ holds. By lemma (3.1.2) with k = c, this gives us that P(a, a + c) is true.

If we define Q(a, b) to denote the boolean function "there exists numbers in (a, b) with arbitrary *small* derivatives", it can similarly be shown that corresponding versions of lemma (3.1.1), (3.1.2), (3.1.3), (3.1.4) and (3.1.5) are also true for Q. We encourage our readers to do this exercise.

Lemma 3.1.6.

$$P(a,b) \Leftrightarrow Q(-b,-a)$$

Proof. By P(a, b) we know that for each N there is a number x in (a, b) with derivative larger than N. Then $(-x)' \leq -N$ which leads to Q(-b, -a) since $-x \in (-b, -a)$. The reverse is proven similarly.

Using (3.1.5) and (3.1.6) it is possible to deduce P and Q is true for all a, b such that a < b.

That is the end of the proof.

3.2 Some properties of the Λ function

Definition 3.2.1. For all natural numbers n, we define $\Lambda(n)$ as the smallest natural number m less than or equal to n such that $m' = \max(0', 1', 2' \dots n')$.

Theorem 3.2.1. $\Lambda(2^a) = 2^a$ for every positive natural a.

Proof. According to theorem (2.1.1), $n' \leq \frac{n \log_2 n}{2}$ with equality iff n is a perfect power of 2. This means that all smaller natural numbers will have a smaller derivative, thereby proving this theorem.

Theorem 3.2.2. For every natural number m there exists a natural number N such that for every $n \ge N$, $2^m | \Lambda(n)$.

Proof. We prove this by contradiction. We assume that there exists an m such that for every N there exists an n > N such that $2^m \nmid \Lambda(n)$ and $\Lambda(n) = n$.

We write $n = 2^a \cdot B$ where B is odd and, by assumption, a < m. According to the rules of arithmetic differentiation,

$$n' = a2^{a-1}B + 2^{a}B' \\ \leq a2^{a-1}B + 2^{a}\frac{B\log_{3}B}{2}$$

The inequality is valid because of theorem (2.1.1) and the fact that the smallest prime factor in B is at least 3 (since B is odd).

$$= a2^{a-1}\frac{n}{2^{a}} + 2^{a}\frac{\frac{n}{2^{a}}\log_{3}\frac{n}{2^{a}}}{3}$$

$$= \frac{an}{2} + \frac{n\log_{3}(n/2^{a})}{3}$$

$$= n\left(\frac{a}{2} + \frac{\log_{3}(n) - a \cdot \log_{3} 2}{3}\right)$$

$$= n\left(a\left(\frac{1}{2} - \frac{\log_{3} 2}{3}\right) + \frac{\log_{3} n}{3}\right)$$

$$< n\left(m\left(\frac{1}{2} - \frac{\log_{3} 2}{3}\right) + \frac{\log_{3} n}{3}\right)$$

The last inequality is true since a < m. Now let f(n) be the last expression minus $(2^{\lfloor \log_2 n \rfloor})' = \lfloor \log_2 n \rfloor 2^{\lfloor \log_2 n \rfloor - 1}$ or

$$f(n) = n\left(m\left(\frac{1}{2} - \frac{\log_3 2}{3}\right) + \frac{\log_3 n}{3}\right) - \lfloor \log_2 n \rfloor 2^{\lfloor \log_2 n \rfloor - 1}$$

If we can prove that f(n) will always assume negative values for sufficiently large n, we are done. We will prove the stronger

$$\lim_{n \to +\infty} f(n) = -\infty$$

Now we repeatedly apply floor inequalities and logarithm rules: $\lfloor x \rfloor > x - 1$.

$$\begin{split} \lim_{n \to +\infty} f(n) &= \lim_{n \to +\infty} n \left(m \left(\frac{1}{2} - \frac{\log_3 2}{3} \right) + \frac{\log_3 n}{3} \right) - \frac{\lfloor \log_2 n \rfloor 2^{\lfloor \log_2 n \rfloor}}{2} \\ &< \lim_{n \to +\infty} n \left(m \left(\frac{1}{2} - \frac{\log_3 2}{3} \right) + \frac{\log_3 n}{3} \right) - \frac{\lfloor \log_2 n \rfloor 2^{\log_2 n}}{4} \\ &= \lim_{n \to +\infty} n \left(m \left(\frac{1}{2} - \frac{\log_3 2}{3} \right) + \frac{\log_3 n}{3} \right) - \frac{n \lfloor \log_2 n \rfloor}{4} \\ &< \lim_{n \to +\infty} n \left(m \left(\frac{1}{2} - \frac{\log_3 2}{3} \right) + \frac{\log_3 n}{3} \right) - \frac{n \lfloor \log_2 n \rfloor}{4} \\ &= \lim_{n \to +\infty} n \left(m \left(\frac{1}{2} - \frac{\log_3 2}{3} \right) + \frac{\log_3 n}{3} \right) - \frac{n \lfloor \log_2 n \rfloor}{4} \end{split}$$

If we can prove that the expression inside the parenthesis becomes negative as $n \to \infty$ we are done. If

$$\lim_{n \to \infty} \left(\frac{\log_3 n}{3} - \frac{\log_2 n}{4} \right) = -\infty$$

this is obviously true. Note that

$$\frac{\log_3 n}{3} - \frac{\log_2 n}{4} = \log_2 n \left(\frac{1}{3\log_2 3} - \frac{1}{4}\right) \approx \log_2 n \cdot (-0.0396901)$$

according to the logarithm laws. This means that the entire expression becomes negative as $n \to \infty$. But this means that

$$\lim_{n \to \infty} f(n) = -\infty$$

and gives us that $(2^{\lfloor \log_2 n \rfloor})' > n'$ for sufficiently large n satisfying the assumptions, which contradicts the assumption that $\Lambda(n) = n$. This ends the proof. \Box

References

- [1] Remarks on the arithmetic derivative *E. J. Barbeau* Canadian Mathematical Bulletin 4, 1961
- [2] How to differentiate a number V. Ufnarovski, B. Åhlander Journal of Integer Sequences, Vol. 6, 2003.
- [3] An Introduction to the Theory of Numbers G. H. Hardy, E. M. Wright Oxford, 1960