Everywhere below V is a vector space over a field F.

Problem 1 Let $P = \{p(x) \in F[x] | \deg p < n\}$. Assume $f, g \in P$. Prove that if there exists n distinct scalars $a_1, a_2, \dots a_n \in F$ with $f(a_i) = g(a_i)$ for $1 \le i \le n$, then

$$f \equiv g$$

Hint:
Write the n equalities as matrix multiplication. (Or not. There are other ways to prove this)

Problem 2 Let P be as in the last problem and let $a_1, a_2 \dots a_n$ be n distinct numbers in F. Then let

$$g_i(x) = \prod_{k \ne i} \frac{x - a_k}{a_i - a_k}$$

for $1 \le i \le n$. Prove that $g_1(x), g_2(x), \dots g_n(x)$ is a basis in P and that the coefficients of $f(x) \in P$ in this basis are $(f(a_1), f(a_2), \dots f(a_n))$.
Hint:
Apply the first problem.

Problem 3 Let $f : V \longrightarrow F$ be a nonzero linear function, see (1) and assume $\dim V = n$. Prove that

$$\dim \ker f = \dim\{v | f(v) = 0\} = n - 1$$

without using any fancy linear transformation or matrix theorems[1].
Hint:
Find a basis in $\ker f$ and extend it to V.

Problem 4 (Related to the previous problem) Assume $\dim V = n$ and let W be a subspace of V with $\dim W = n - 1$. Prove that there is a linear function $f : V \to F$ with $\ker f = W$.
Hint:
Same as **Problem (3)**.

Remark 1. *Note that "have equal values at all points" or "equal at $n-1$" points are not enough in **Problem (1)**. Take $n = 6$, $f(x) = x^5$ and $g(x) = x$ and $F = \mathbb{Z}_5$. f and g are equal at all points in $\mathbb{Z}_5$ but are not the same polynomial.*

1 Linear functions

Given a vector space V over a field F, we call a function $f : V \longrightarrow F$ **linear** if it satisfies

$$f(\alpha v + \beta w) = \alpha f(v) + \beta f(w) \qquad \textit{(lin)}$$

for all $\alpha, \beta \in F$ and all $v, w \in V$. Given a basis $B = b_1, b_2 \dots b_n$ of V, we write $v \in V$ as a linear combination $v = \lambda_1 b_1 + \dots \lambda_n b_n$ for $\lambda_i \in F$. By *(lin)*, the value of f at v is

$$f(v) = f(\lambda_1 b_1 + \dots + \lambda_n b_n) = \lambda_1 f(b_1) + \dots + \lambda_n f(b_n)$$

and the value of f on V only depends on the value of f on B.

[1]It's of course OK if you prove them

Example 1. *Let $V = \mathbb{R}^2$ and $F = \mathbb{R}$. Then any linear function $V \longrightarrow F$ must be of the form*

$$f(v) = f((a, b)) = f(a \cdot (1, 0) + b \cdot (0, 1)) = af((1, 0)) + bf((0, 1))$$

Let $\alpha = f((1, 0)), \beta = f((0, 1))$,

$$af((1, 0)) + bf((0, 1)) = a\alpha + b\beta$$

Example 2. *Let V be an n-dimensional vector space over an arbitrary field F. By **Theorem 1** of [1], there is a basis $B = (b_1 \dots b_n)$ of B. Then $f(v) = f(\sum_i \lambda_i b_i) = \sum_i \lambda_i f(b_i)$ If f is nonzero (that is, there is a $v \in V$ such that $f(v) \neq 0$), then all $f(b_i)$ cannot be 0. There has to be at least **some** b_i with $f(b_i) \neq 0$.*

Definition 1. *Given a linear function $f : V \longrightarrow F$, we define $\ker f = \{v \in V | f(v) = 0\}$. $\ker f$ is a subset of V. It is also a vector space because*

$$\begin{aligned} a, b \in \ker f &\Longrightarrow f(a + b) = f(a) + f(b) \\ &= 0 + 0 = 0 \Longrightarrow a + b \in \ker(f) \\ \lambda \in F, a \in \ker f &\Longrightarrow f(\lambda a) = \lambda f(a) = 0 \\ &\Longrightarrow \lambda a \in \ker f \end{aligned}$$

which proves (cl).

Remark and more hints on problems 3 and 4
In 3, you have to prove that

$$\dim \ker f = n - 1$$

given that $\dim V = n$ and that f is nonzero (that is, not zero on all points) By **Theorem 1** of [1], V has a basis $B = b_1 \dots b_n$ and by **Example (2)** $f(v)$ can be represented as

$$f(v) = f(\lambda_1 b_1 + \ldots + \lambda_n b_n) = \lambda_1 f(b_1) + \ldots + \lambda_n f(b_n)$$

so what we want to prove is that given the solution set in $(\lambda_1, \ldots, \lambda_n)$ of

$$\lambda_1 f(b_1) + \ldots + \lambda_n f(b_n) = 0$$

the set of all $(\lambda_1 b_1 + \ldots + \lambda_n b_n)$ is an $(n - 1)$-dimensional vector space. At the moment, the only way we can prove this is by **explicitly constructing a basis** for this vector space, proving that it is indeed a basis (it should be linearly independent) and counting the number of basis elements to $n - 1$. Without loss of generality, $f(b_n) \neq 0$. Now finish the proof!

In 4, you are given a subspace W of V with dimension $n-1$ and are told to find a linear function. By definition (and **Theorem 1** of [1]) there is a basis B' of size $n - 1$ for W. By **Theorem 3** of [1] that basis can be extended to a basis for V. Now find the linear function.

References

[1] http://www.csc.kth.se/~loiko/rep02.pdf