Repetition of 1:st lecture

We defined **field, vector spaces**, proved some elementary implications from the field and vector space axioms, gave examples to vector spaces. Here are the definitions for reference.

Definition 1. *A set F with binary operations $+$ and $\cdot$ is a **field** if*

$$\begin{array}{rcl} \forall a,b \in F: & a+b \in F, \quad a \cdot b \in F & \textit{(cl)} \\ \forall a,b,c \in F: & (a+b)+c = a+(b+c), \quad (a\cdot b)\cdot c = a \cdot (b \cdot c) & \textit{(assoc)} \\ \forall a,b \in F: & a+b = b+a, \quad a \cdot b = b \cdot a & \textit{(comm)} \\ \exists 0 \in F : \forall a \in F: & a+0 = a & \textit{(add. id)} \\ \exists 1 \neq 0 \in F : \forall a \in F \setminus \{0\}: & 1 \cdot a = a & \textit{(mul. id)} \\ \forall a \in F : \exists -a \in F: & a + (-a) = 0 & \textit{(add. inv)} \\ \forall a \in F \setminus \{0\} : \exists a^{-1} \in F: & a \cdot a^{-1} = 1 & \textit{(mul. inv)} \\ \forall a,b,c \in F: & a\cdot(b+c) = a\cdot b + a \cdot c & \textit{(distr)} \end{array}$$

Note that it follows from (add. id) and (mul. id) that F has at least two elements.

We also briefly talked about field characteristic. For reference, here is the full definition.

Definition 2. *In a field, we can define **multiplication with positive integer** as repeated addition,*

$$n \cdot a = \begin{cases} a \text{ when } n = 1 \\ (n-1)\cdot a + a \text{ otherwise} \end{cases}$$

*Having done this, we define the **characteristic** of the field F as the minimal $n \in \mathbb{Z}_+$ for which*

$$n \cdot 1 = 0$$

or, if $n \cdot 1 \neq 0$ for any n, we say that the characteristic of F is zero. Notation: $\mathrm{char}(F)$.

Definition 3. *A vector space is a set V of elements called **vectors**, a field F of elements called **scalars** and two operations*

$$\begin{array}{rl} +: & V \times V \longrightarrow V \\ \cdot: & F \times V \longrightarrow V \end{array}$$

satisfying

$$\begin{array}{rcl} & L \text{ is an abelian groups under } + & \textit{(abel)} \\ \forall a,b \in F, v \in V: & a\cdot(b\cdot v) = (a\cdot b)\cdot v & \textit{(assoc)} \\ \forall a,b \in F, v,w \in V: & a\cdot(v+w) = a\cdot v + a\cdot w, \quad (a+b)\cdot v = a\cdot v + b \cdot v & \textit{(distr)} \\ \forall v \in V: & 1 \cdot v = v & \textit{(unit)} \end{array}$$

We proved various elementary facts in the spirit of

Proposition 1. *V is a vector space over a field F. Then*

$$\forall v \in V : 0_F \cdot v = 0_V$$

because

$$\begin{array}{rcll} 0_V & = & (0_F v + (-0_F v)) & \textit{(inv)} \\ & = & (0_F + 0_F)v + (-0_F v) & \textit{(add. id)} \\ & = & (0_F v + 0_F v) + (-0_F v) & \textit{(distr)} \\ & = & 0_F v + (0_F v + (-0_F v)) & \textit{(assoc)} \\ & = & 0_F v + 0 & \textit{(inv)} \\ & = & 0_F v & \textit{(add. id)} \end{array}$$

We discovered the importance of *(unit)*, gave examples of vector spaces and tried to solve the following problem:

Problem 1. *Does there exist a (non-zero) finite vector space V over an infinite field F? If yes, what values of* $\mathrm{char}(F)$ *are allowed?*