3:rd lecture

We discussed the home problems:

Old problem 1 *If two $n-1$-degree polynomials coincide at n points, their coefficients are equal.* Let $f = \sum_i f_i x^i, g = \sum_i g_i x^i$ be the polynomials. If they are equal at the points $a_1, \ldots a_n$, we have $\sum_i (f_i - g - i) a_j^i = 0$ for all a_j. Write this equations as matrix multiplication,

$$\begin{pmatrix} 1 & a_1 & \ldots & a_1^{n-1} \\ \vdots & & & \vdots \\ 1 & a_n & \ldots & a_n^{n-1} \end{pmatrix} \begin{pmatrix} f_0 - g_0 \\ \vdots \\ f_{n-1} - g_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

short for $VX = 0$. Then the matrix V, see below, is invertible and the only solution in X is $X = 0$ so all coefficients are equal.

Old problem 2 *Prove that a certain set of n polynomials g_i of degree $n-1$ is a basis for the set P_n of all polynomials of degree $\leq n-1$.* The polynomials g_i were defined as

$$g_i(x) = \prod_{j \neq i, 1 \leq j \leq n} \frac{x - a_j}{a_i - a_j}$$

and a_i - n different scalars. Pick a polynomial $p \in P_n$. Then it can be seen that $A(p)(x) = \sum_i p(a_i) g_i(x)$ coincides with p on the points $a_1, \ldots a_n$ and by problem 1, this shows that $A(p) = p$. Thus any polynomial in P_n can be written as a linear combination of the g_i and since there are n of them they form a basis for the (n-dimensional) vector space P_n.

Old problem 3 *Prove that the kernel of a nonzero linear functional from an n-dimensional vector space has dimension $n-1$.* Pick any basis $b_1 \ldots b_n$. Then $f\left(\sum \lambda_i b_i\right) = \sum \lambda_i f(b_i)$. WLOG $f(b_n) \neq 0$. Then $v_k = 1 \cdot b_k - f(b_k) f(b_n)^{-1} b_n$, $1 \leq k \leq n-1$ form a basis for $\ker f$.

Old problem 4 *Prove that any $n-1$ dimensional subspace W of an n-dimensional vector space V is the kernel of a linear functional.* Easy: let $b_1, \ldots b_n$ be a basis of W, extend it to a basis $b_1 \ldots b_n$ of V and let $f : W \longrightarrow V : f(v) = f\left(\sum \lambda_i b_i\right) = \lambda_n$. Then $\ker f = W$.

We proved that that the Vandermonde matrix,

$$V(a_1, \ldots, a_n) = \begin{pmatrix} 1 & a_1 & \ldots & a_1^{n-1} \\ \vdots & & & \vdots \\ 1 & a_n & \ldots & a_n^{n-1} \end{pmatrix}$$

is invertible iff the a_i are all distinct. We used this to solve problems 1 and 2 and to prove that for any scalars b_i, $1 \leq i \leq n$ there is exactly one polynomial $p \in P_n$ with $p(a_i) = b_i$ in any field F.