Square dissection

Solutions by Alex Loiko

Problem 1. *Prove that n squares can always be cut by straight lines to finitely many pieces in a way so that it is possible to form a new square of the resulting pieces without overlaps and gaps.*

Lemma 1. *Any rectangle with sides a, b,*

$$\frac{1}{2} \leq a/b \leq 2$$

can be cut to pieces that can be assembled without overlaps and gaps to a square with side $\sqrt{ab}$.

Proof. Try the interactive applet at
http://www.csc.kth.se/~loiko/the_square_ruler_and_compass.html
As seen in the figure, let's label the rectangle $ABCD$ and the square $PQRS$. Set $a = |BC|, b = |AB|$ and assume WLOG that $a \leq b$ (otherwise we can just relabel the sides). Find a point E in the interior of CD with $|BE| = \sqrt{ab}$. (It is clear that E exists and is unique since we can find the length $|CE| = \sqrt{b^2 - ab} > 0$). Now draw the perpendicular onto BE from A. I claim that it will intersect BE at point F inside of $ABCD$. To prove it it is enough to prove $|BF| > |BE|$ which is true since $|BF| = \sqrt{|AB|^2 - |AF|^2} = \sqrt{b^2 - ab} < \sqrt{ab} \Leftrightarrow b < 2a$ from similar triangles $\triangle AFB \sim \triangle BCE$.

Now we will prove that it is possible to construct a square out of ABF, $AFED$ and BCE by calculating lengths and angles. We have:

$$\begin{aligned}
|AF| &= |AB|\frac{|BE|}{|BC|} = \sqrt{ab} \text{ from } \triangle AFB \sim \triangle BCE \\
\angle FAB &= \frac{\pi}{2} - \angle ABF = \angle EBC \\
\angle ABF &= \angle CEB = \angle DAF
\end{aligned}$$

Arrange the pieces as shown in the figure to form $PQRS$. There will be no gaps because $\angle SPU + \angle UPQ = \angle EBC + \angle DAF = \angle EBC + \pi/2 - \angle EBC = \pi/2$,

$$\begin{aligned}
\angle RST + \angle USP &= \angle FAB + \angle CEB = \pi/2 \\
\angle STR + \angle UTQ &= \angle ABF + \angle DEF = \pi
\end{aligned}$$

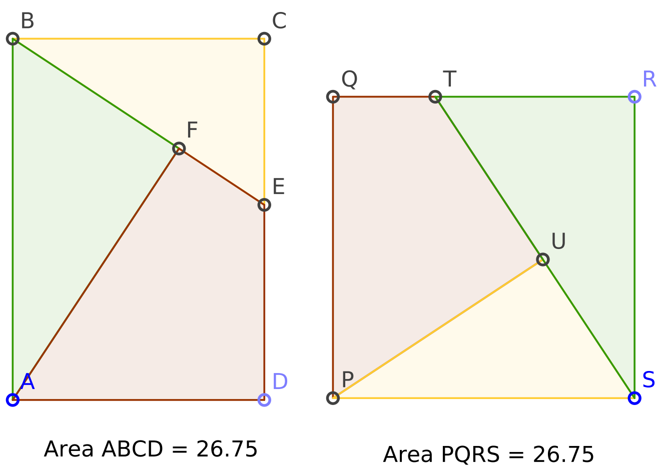

Figure 1: An example dissection

No overlaps because

$$|PU| = |BC| = |AD| = a \text{ (the yellow and red fit together)}$$
$$|SU| + |UT| = |CE| + |ED| = b = |AB| = |ST|$$
$$|QT| + |TR| = |FE| + |BF| = |BE| = \sqrt{ab}$$

And we have a square! $\square$

Now another lemma. This one is fairly trivial but I'll state it anyway to avoid confusing the reader.

Lemma 2. *Define $A \sim B$ to mean that* ***rectangle*** *A* ***can be transformed to rectangle*** *B for any rectangles A, B. Then $\sim$ is an equivalence relation.*

Proof. If $A = B$ we don't have to cut anything, if there is a dissection from A to B it can be done backwards and the composition of dissections is again a dissection. $\square$

Almost done!

Lemma 3. *Any rectangle can be transformed to a square of equal area where* ***transformed*** *means* ***dissected into finitely many pieces and the pieces rearranged...***

Proof. Define

$$P(r) := \text{ the statement is true for a rectangle with side-ratio } r$$

We have alredy proven $P(r)$ for $1/2 \leq r \leq 2$ in **Lemma 1**. We have also motivated

$$P(r) \Leftrightarrow P\left(\frac{1}{r}\right)$$

(just relabel the sides). Now we will prove

$$P(r) \Leftrightarrow P(4r)$$

Take a rectangle with side ratio r, cut it in half, put the halves on top of each other, apply **Lemma 2** and you have a rectangle with side ratio $4r$. That proves the equivalence.

By induction we have

$$P(r) \Leftrightarrow P(4^n r)$$

for any $n \in \mathbb{Z}$.

Now for any $s \in \mathbb{R}^+$, there always exist an $m \in \mathbb{Z}$ with $-1/2 \leq \log_4 s + m \leq 1/2$ (just take $m = \lfloor 1/2 - \log_4 s \rfloor$) Then $4^m s \in [-1/2, 1/2]$ and $P(s)$ is true by $P(4^m s) \Leftrightarrow P(s)$

$\square$

Instead of going over to squares we will focus on rectangles for another lemma.

Lemma 4. *Any rectangle can be transformed to any other rectangle of equal area.*

Proof. By **Lemma 4** both rectangles can be transformed to the same square. Take one, dissect to square, take the square and dissect backwards to the other.

$\square$

And now we are ready to prove the main result. It will be done by induction. Take two squares A and B. Paint a square with area equal to the sum $Area(A)+Area(B)$. Put A in one corner of the resulting square. There there will be an L-shaped space not filled by square and having area equal to $Area(B)$. That shape can be divided into two rectangles with area P, Q. We now split B into two rectangles having area P and Q, apply **Lemma 4** and get a transformation of two arbitrary squares to a new larges square. That was the base case.

The induction step is simple. Assume n squares can be assembled to a large one. Add one square. Use the assumption to reduce n of the squares to one. Use the base case to transform the two remaining squares to a large one. Done!