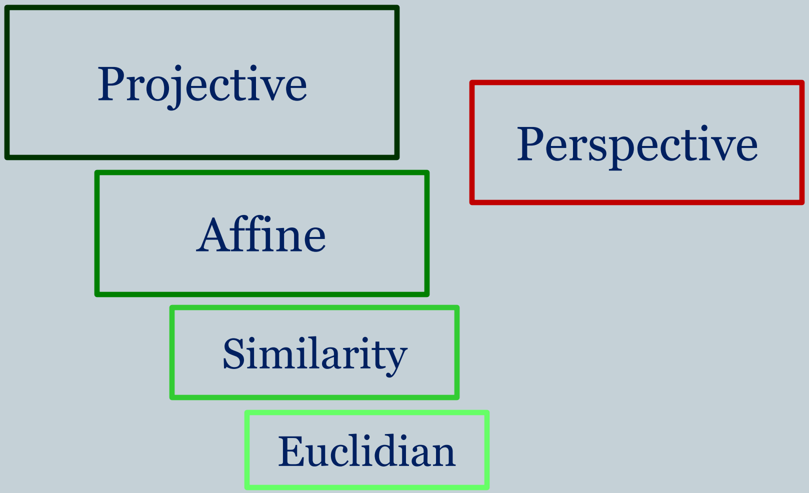
Hierarchy of Transformations



- Group: Projective, Affine, Similarity, Isometrics
- Perspective not closed.
- Invariant: an alternative way to describe transformations in terms of elements or quantities (of a geometric configuration) that are preserved.
  - E.g. rotation + translation



Class I: Isometries

- Transformations which preserves Euclidian distance (iso = same, metric = measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

  - Orientation Preserving (Euclidian Transformation or displacement) $\varepsilon = 1$
  - Orientation Reversing $\varepsilon = -1$

- Rotation(R) + Translation (t): 3 DOF
- Special cases when R = I or t = 0
- Invariants: Distance, Length, Area
- Orientation Preserving: Group, Reversing is not Closed

$$\mathrm{x}' = \mathbf{H}_E\,\mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

Class II: Similarities

- Similarity: An isometry composed with an isotropic (only position) scaling.
- Euclidian with isotropic scaling:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\,\mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

- Equi-form
- Euclidian transform + scale factor = 4 DOF
- Invariants: Angle(Parallel lines), ratio of lengths, ration of areas
- Metric Structure: Structure is defined up to a similarity

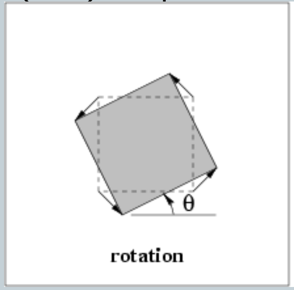
Class III: Affine transformations

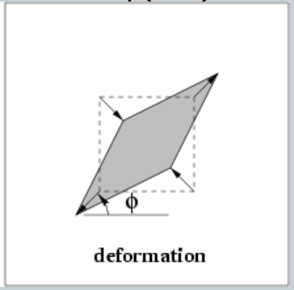
- Affinities: non-singular linear transformation followed by a translation:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- Block form:

$$\mathrm{x}' = \mathbf{H}_A \,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x}$$





- SVD:

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

- 2 Rotations + 2 scale factors + translation = 6 DOF
- Non-isotropic scaling (adds scale ratio and orientation on top of similiarity : 2 DOF)
- Scaling in oriented orthogonal directions.

Class III: Affine transformations

- Invariants
  - Parallel lines: Intersection of parallel lines remains at infinity.
  - Ratio of lengths of parallel line segments
  - Ratio of area

Class VI: Projective transformations

- General non-singular linear transformation of homogeneous coordinates.

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Block form:

$$\mathrm{x}' = \mathbf{H}_P\, \mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \mathrm{x}$$

- 9 elements, disregarding scale factor (not v) : 8 DOF
- Invariant to ratio of ratios or cross ratio of lengths

Class VI: Projective transformations

- Comparison to Affinities
- Shape deformation (Scaling)
  - Affinity: homogeneous over the plane, depends only on orientation
  - Projectivity: Depends also on position (Perspective)
- v not null vector -> non-linear behavior in inhomogeneous system -> e.g. points at infinity can become finite, vanishing points, horizon
  - Affinity:

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

  - Projectivity:

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Class VI: Projective transformations

- Decomposition

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$$

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

- Decomposition unique (if chosen s>0)
- Example

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Class VI: Projective transformations

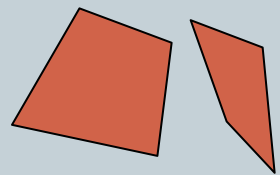
- Objective: Partially determine the transformation (e.g. determine the length ratios or shape up to similarity)
- Inverse decomposition $H^{-1} = H_P^{-1} H_A^{-1} H_S^{-1}$
- Projectivity is invertible thus

$$\mathbf{H} = \mathbf{H}_P \mathbf{H}_A \mathbf{H}_S = \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

Summary of Transformations

Projective
8dof

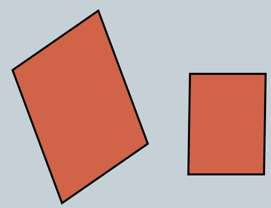
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

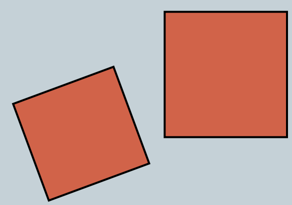
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity $\mathbf{l}_\infty$

Similarity
4dof

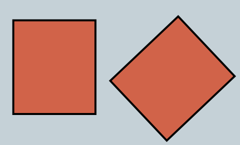
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



lengths, areas.

The number of Invariants

- Number: Functinally independent invariant quantities.

- The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation

- e.g. configuration of 4 points in general position has 8 dof (2/pt) and so 4 similarity, 2 affinity and zero projective invariants

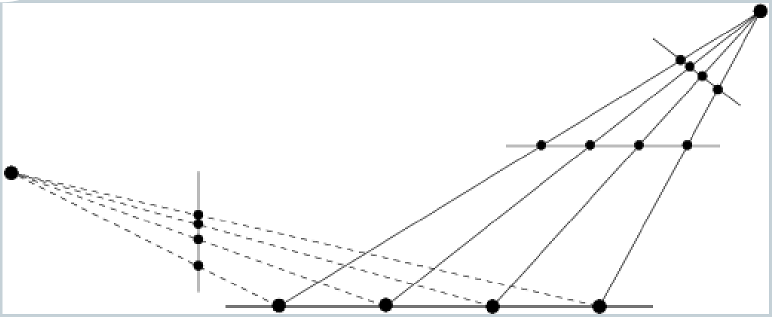
The projective geometry of 1D

- Projective geometry of a line $(x_1, x_2)^{\mathsf{T}}$
- Ideal point: $x_2 = 0$
- Projective transformation $\bar{\mathrm{x}}' = \mathbf{H}_{2\times 2}\,\bar{\mathrm{x}}$ 3DOF (2x2-1)
- The cross ratio $Cross(\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2, \bar{\mathrm{x}}_3, \bar{\mathrm{x}}_4) = \dfrac{|\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2||\bar{\mathrm{x}}_3, \bar{\mathrm{x}}_4|}{|\bar{\mathrm{x}}_1, \bar{\mathrm{x}}_3||\bar{\mathrm{x}}_2, \bar{\mathrm{x}}_4|}$ $|\bar{\mathrm{x}}_i, \bar{\mathrm{x}}_j| = \det\begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$

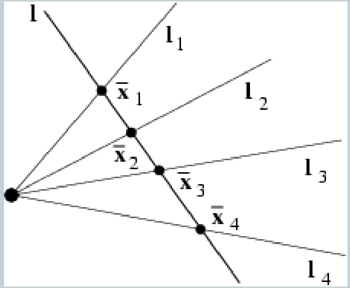
  - Not dependant on homogeneous representative of x
  - Particular representative ($x_2 = 0$) => $|\bar{\mathrm{x}}_i, \bar{\mathrm{x}}_j|$ is signed distance.
  - Valid if one of the points is ideal
  - Cross ratio is invariant to the projective coordinate

The projective geometry of 1D

- Illustration:



- Dual: Concurrent lines



- Representing points in p2

On a 1-dimensional image