Projective Geometry and Transformations of 3D

- Chapter 3 -

Meeting 2 - 11.02.2010

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based on M. Pollefeys' presentation:

www.cs.unc.edu/~marc/mvg/course04.ppt

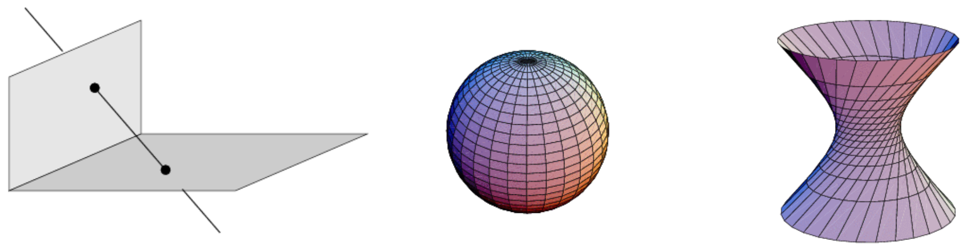
Content

Chapter 3: properties and entities of projective 3D space

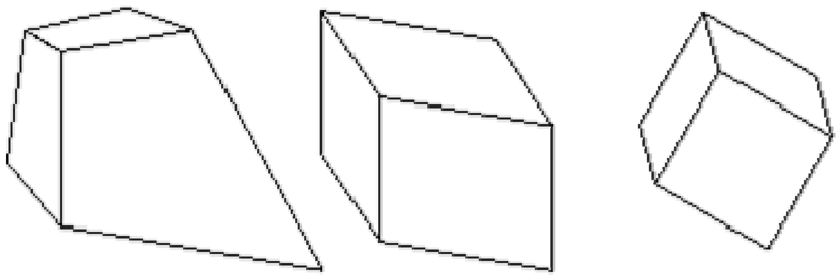
- many of these are generalizations of those of projective plane in 2D, but …

Content:

- Points, lines, planes and quadrics



- Transformations



- Π_∞ -maybe later

Notation: 3D points & Projective Transformation

3D point

Point in 3D ($\mathbf{R}^3$): $(X, Y, Z)^\mathsf{T} \in \mathbf{R}^3$

Homogeneus vector: $\mathrm{X} = (X_1, X_2, X_3, X_4)^\mathsf{T} \in \mathbf{P}^3$

Point in $\mathbf{R}^3$ with inhomogeneous coordinates: $\mathrm{X} = \left(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4} \right)^\mathsf{T} = (X, Y, Z)^\mathsf{T} \quad (X_4 \neq 0)$

Projective Transformation

$\mathrm{X}' = \mathbf{H}\,\mathrm{X}$ (15dof= 4x4-1)

Planes

3D Plane

Plane in 3D ($\mathbf{R}^3$): $\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$

Homogeneus representation: $\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$

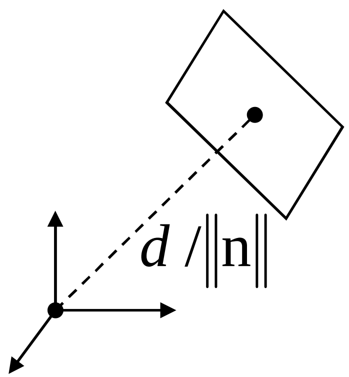
Point on plane: $\boldsymbol{\pi}^\mathsf{T} \mathrm{X} = 0$

Euclidian Representation

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0 \Leftrightarrow \mathrm{n}.\tilde{\mathrm{X}} + d = 0$$

$$\mathrm{n} = (\pi_1, \pi_2, \pi_3)^\mathsf{T} \qquad \pi_4 = d$$

$$\tilde{\mathrm{X}} = (X, Y, Z)^\mathsf{T} \qquad X_4 = 1$$



Dual: points ↔ planes, lines ↔ lines

Relations between points, lines and planes

- In $\mathbf{P}^3$ geometric relations between planes, points and lines:

1. A plane is defined uniquely by:
   A. the join of 3 points (in general position);
   B. the join of a line and point (in general position).
2. Three distinct planes intersect in a unique point.
3. Two distinct plane intersect in a unique line.

   - points & planes: 1A – 2
   - lines & points & planes : 1B – 3

Three points define a plane

Find π, when $X_1, X_2, X_3 \in \pi \Leftrightarrow X_1^T\pi = 0, X_2^T\pi = 0, X_3^T\pi = 0$ $\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix} \pi = 0$

- Solutions:

1. Solve π as right null-space of $\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix}$

2. Implicitly from coplanarity condition:

if $X \in \pi$ then

$\det[X\ X_1 X_2 X_3] = 0$

$$\det \begin{bmatrix} X_1 & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = (D_{234}, -D_{134}, D_{124}, -D_{123})^T$$

Three planes define a point

- Find X when $X \in \pi_1, \pi_2, \pi_3 \Leftrightarrow \pi_1^T X = 0, \pi_2^T X = 0, \pi_3^T X = 0$

$$\begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix} X = 0$$

  - Solutions:

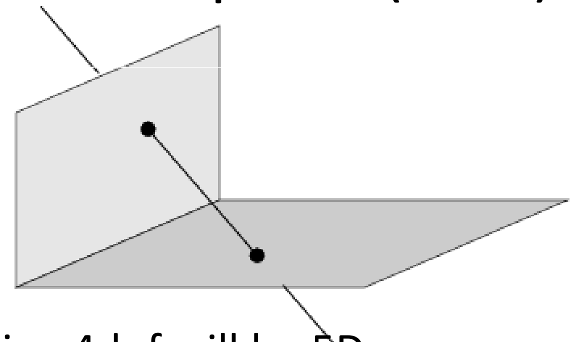
  1. Solve X as right nullspace of $\begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix}$

Dual: points ↔ planes

- Under projective transformation $X' = \mathbf{H} X$:
  - Plane: $\pi' = \mathbf{H}^{-T} \pi$

Lines

- Line (definition):
  - Join of two points
  - Intersection of two planes
- Line (4 dof) = 2 x Point on a plane (2dof)



  - Natural way of representing 4dof will be 5D space
  - Goal: line representation in 3D
  - Possible representations:
    - Null-space and span representation
    - Plücker matrices
    - Plücker line coordinates

Lines - Span Representation (1)

- Line is a *pencil* (one-parameter family) of collinear points, and is defined by any two of these points
- Line is a *span* of two vectors
  - ***A, B*** – two non-coincident space points

$$W = \begin{bmatrix} A^T \\ B^T \end{bmatrix}$$

span(S) = collection of all (finite) linear combinations of the elements of a set *S*

=> span($\mathbf{W}^T$) is the pencil of points $\lambda A + \mu B$ on the line

- Dual representation of line: the intersection of two planes

$$W^* = \begin{bmatrix} P^T \\ Q^T \end{bmatrix}$$

P,Q- are a basis for null-space

=> span($\mathbf{W}^{*T}$) is the pencil of planes $\lambda' P + \mu' Q$ with the line as axis

- $W^* W^T = W W^{*T} = 0_{2\times 2}$, 0_{2x2} is a 2x2 null matrix

Lines - Span Representation (2)

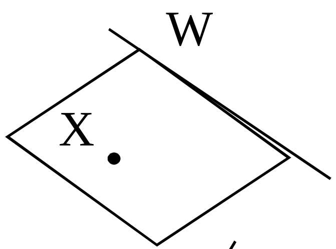
- Example: X-axis can be represented as:

$$W = \begin{bmatrix} A^T \\ B^T \end{bmatrix} \qquad \lambda A + \mu B \qquad W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{origin} \\ \text{ideal point} \end{matrix}$$

$$W^* = \begin{bmatrix} P^T \\ Q^T \end{bmatrix} \qquad \lambda P + \mu Q \qquad W^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{matrix} \text{XY -plane} \\ \text{XZ - plane} \end{matrix}$$

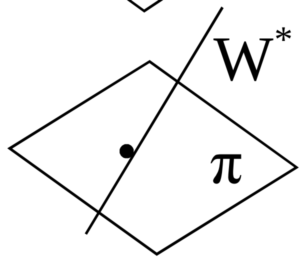
- Join and incidence properties:
  - Plane Π=?

$$\mathbf{M} = \begin{bmatrix} W \\ X^T \end{bmatrix} \qquad \mathbf{M}\pi = 0$$



  - Point **X**=?

$$\mathbf{M} = \begin{bmatrix} W^* \\ \pi^T \end{bmatrix} \qquad \mathbf{M}X = 0$$



Line - Plücker matrices (1)

- Line is represented by Plücker matrix = 4x4 skew-symmetric homogeneous matrix
- Line joining points **A** and **B**, is represented by matrix **L**:

$$l_{ij} = A_i B_j - B_i A_j$$

$$\mathrm{L} = \mathrm{A}\mathrm{B}^{\mathsf{T}} - \mathrm{B}\mathrm{A}^{\mathsf{T}}$$

- Properties of L:
  1. L has rank 2, in fact $\mathrm{L}\mathrm{W}^{*\mathsf{T}} = 0_{4\times 2}$
  2. L has 4 dof
  3. $\mathrm{L} = \mathrm{A}\mathrm{B}^{\mathsf{T}} - \mathrm{B}\mathrm{A}^{\mathsf{T}}$ is generalization of $\mathrm{l} = \mathrm{x} \times \mathrm{y}$ in $\mathbf{P}^2$
  4. L independent of choice A and B
  5. Transformation $\mathrm{X}' = \mathrm{H}\mathrm{X} \rightarrow \mathrm{L}' = \mathrm{H}\mathrm{L}\mathrm{H}^{\mathsf{T}}$
- Example: X-axis is represented as

$$\mathrm{L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Line - Plücker matrices (2)

- Dual Plücker representation:
  - Line is formed by the intersection of two planes **P** and **Q**

$$L^* = PQ^T - QP^T$$

  - Transformation: $X' = HX \rightarrow L^{*'} = H^{-T} L H^{-1}$
  - Relation between **L** and **L***: $l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l^*_{34} : l^*_{42} : l^*_{23} : l^*_{14} : l^*_{13} : l^*_{12}$
- Join and incidence properties:

$\pi = L^* X$ (plane through line and point)

$L^* X = 0$ (point on line)

$X = L\pi$ (intersection point of plane and line)

$L\pi = 0$ (line in plane)

$[L_1, L_2, \ldots]\pi = 0$ (coplanar lines)