Quantum Computation - Lecture 12 - Nonlocal Games

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TCS-KTH

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- Classical Value:

$$\omega_c(G(V, \pi)) = max_{a,b} \sum_{s,t} \pi(s, t) V(s, t, a(s), b(t))$$

Where the maximum is taken over all functions $a : S \to A$ and $b : T \to B$.

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- The results a and b are sent back to the referee.
- The referee accepts if $V(s, t, a, b) = 1$

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- $\mathcal{A}$ represents Alice's part of $|\psi\rangle$ and $\mathcal{B}$ represents Bob's part.
- Two collections of positive semidefinite $n \times n$ matrices.

$$\{X_s^a | s \in S, a \in A\} \text{ and } \{Y_t^b | t \in T, b \in B\}$$

satisfying

$$\sum_{a \in A} X_s^a = I \text{ and } \sum_{b \in B} Y_t^b = I$$

for every choice of $s \in S$ and $t \in T$ where I denotes the $n \times n$ identity matrix.

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- Given a question $s \in S$ for Alice and a question $t \in T$ for Bob, such a strategy causes Alice to answer with $a \in A$ and Bob to answer with $b \in B$ with probability $\langle\psi|X_s^a \otimes Y_t^b|\psi\rangle$

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- The quantum value of G, denoted $\omega_q(G)$ is the supremum of the winning probabilities over all quantum strategies of Alice and Bob.
- $$\omega_q = \sum_{s,t,a,b} \pi(s,t)V(s,t,a,b)\langle\psi|X_s^a \otimes Y_t^b|\psi\rangle$$

Observables

- Let $\Pi_1, ..., \Pi_k$ be a collection of projection matrices for which $\sum_i \Pi_i = I$, and suppose we associate the outcomes of the measurements with collection of real numbers $\{\lambda_1, ..., \lambda_k\}$. Then the observable corresponding to this measurement is given by

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- In the case of binary answers we will associate the real numbers $\{+1, -1\}$ with the values $\{0, 1\}$. Thus the observable corresponding to a measurement $\{\Pi_0, \Pi_1\}$ will be $A = \Pi_0 - \Pi_1$.

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- Classical value of $G(V, \pi)$: $\omega_c(G) = 3/4$

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- The fact that the strategy is optimal follows from Tsirelson's inequality.

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- Thus we write $V(s, t, a \oplus b)$

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- There exist collections $\{|u_s\rangle | s \in S\}$ and $\{|v_t\rangle | t \in T\}$ of unit vectors such that $\langle u_s | v_t \rangle = c_{s,t}$ for all $(s,t) \in S \times T$.

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$$\omega_q(G) - \tau(G) = \frac{1}{2}\max_{|u_s\rangle,|v_t\rangle\in\mathbb{R}^m}\sum_{s,t}\pi(s,t)(V(s,t|0) - V(s,t|1))\langle u_s|v_t\rangle$$

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  - On Input (s,t) the probability that Alice and Bob's answers are equal is

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- By Tsirelson's theorem one can find observables A_s and B_t such that $\langle\psi|A_s\otimes B_t|\psi\rangle = a_s b_t$

- Grothendieck's constant: K_G is the smallest number such that for all integers $N \geq 2$ and all $N \times N$ real matrices M if $\| \sum_{s,t} M(s,t) a_s b_t \| \leq 1$ for all numbers $a_1, ..., a_N$ and $b_1, .., b_N$ in $[-1, 1]$ then

$$\| \sum_{s,t} M(s,t) \langle u_s | v_t \rangle \| \leq K_G$$

for all unit vectors $|u_1\rangle, ...|u_N\rangle$ and $|v_1\rangle ... |v_N\rangle$ in $\mathbb{R}^n$.

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- $$1.679 \leq K_G \leq \frac{\pi}{2 \log(1 + \sqrt{2})} \simeq 1.7822$$

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  - Suppose $|S| = |T| = N$. Define an $N \times N$ matrix

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  - then $\| \sum_{s,t} M(s,t) a_s b_t \| \leq 1$
  - Then $\omega_q - \tau(G) = [\omega_c(G) - \tau(G)] \max_{|u_s\rangle,|v_t\rangle} M(s,t)\langle u_s | v_t \rangle \leq K_G[\omega_c(G) - \tau(G)]$