A derivation for $p \Rightarrow p$

1	$p \Rightarrow (\underbrace{(p \Rightarrow p)}_{Q} \Rightarrow p)$	axiom
2	$(p \Rightarrow (\underbrace{(p \Rightarrow p)}_{Q} \Rightarrow \underbrace{p}_{R})) \Rightarrow ((p \Rightarrow \underbrace{(p \Rightarrow p)}_{Q}) \Rightarrow (p \Rightarrow \underbrace{p}_{R}))$	axiom
3	$(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow p)$	1, 2, MP
4	$p \Rightarrow (\underbrace{p}_{Q} \Rightarrow p)$	axiom
5	$p \Rightarrow p$	3, 4, MP

Past Linear Temporal Logic $PLTL$
Expressive completeness of LTL

The definition of $PLTL$: (.U.) and (.S.) under the strict interpretation

$$\varphi \quad ::= \quad \bot \mid p \mid \varphi \Rightarrow \varphi \mid (\varphi \mathsf{U} \varphi) \mid (\varphi \mathsf{S} \varphi)$$

$s, n \models (\varphi \mathsf{U} \psi)$ if there exists a $k > 0$ s.t. $s, n+i \models \varphi$ for $i \in \{1, \ldots, k-1\}$ and $s, n+k \models \psi$

$s, n \models (\varphi \mathsf{S} \psi)$ if there exists a $k \in \{1, \ldots, n\}$ s.t. $s, n-i \models \varphi$ for $i \in \{1, \ldots, k-1\}$, and $s, n-k \models \psi$

$\{1, \ldots, 0\}$ stands for $\emptyset$.

$$\circ\varphi \rightleftharpoons (\bot \mathsf{U} \varphi)$$

$(.\mathsf{U}_{LTL}.)$ - $(.\mathsf{U}.)$ under the non-strict interpretation.

$$(\varphi \mathsf{U}_{LTL} \psi) \rightleftharpoons \psi \vee (\varphi \wedge (\varphi \mathsf{U} \psi)), \qquad (\varphi \mathsf{U} \psi) \rightleftharpoons \circ(\varphi \mathsf{U}_{LTL} \psi).$$

Abbreviations and variants of the notation

$\Box$ and $\Diamond$ have strict variants too:

$$\Diamond\varphi \rightleftharpoons (\top \mathsf{U} \varphi), \qquad \Box\varphi \rightleftharpoons \neg\Diamond\neg\varphi$$

The beginning of time:

$$\mathsf{I} \rightleftharpoons \neg(\top \mathsf{S} \top)$$

Abbreviations about the past:

$$\ominus A \rightleftharpoons (\bot \mathsf{S} A), \qquad \diamondminus A \rightleftharpoons (\top \mathsf{S} A), \qquad \boxminus A \rightleftharpoons \neg\diamondminus\neg A$$

Alternative "computer" notation for the temporal operators:

$\circ$	$\Diamond$	$\Box$	$\ominus$	$\diamondminus$	$\boxminus$
X	F	G	Y	P	H

Exercise 1 Describe the extensions to be made to the model-checking algorithm in order to enable model-checking past LTL formulas.

Gabbay's separation theorem

A formula is

past (future) if it is (.U.)-free ((.S.)-free);

strictly future (past) if it has the form $\circ\varphi$ ($\ominus\varphi$) with a future (past) φ;

boolean combination of $\varphi_1, \ldots, \varphi_n$ if it has the form

$$\varphi ::= \bot \mid \varphi_1 \mid \ldots \mid \varphi_n \mid \varphi \Rightarrow \varphi;$$

separated if it is a boolean combination of past and future formulas.

Exercise 2 Every future (past) formula is equivalent to a boolean combination of propositional variables and strictly future (past) formulas.

Theorem 1 (Gabbay's separation theorem) Every $PLTL$ formula is equivalent to a separated formula.

Remark 1 The theorem applies to models with unbounded past as well.

Proof of Gabbay's separation theorem: lemmata

Lemma 1 The following equivalences are valid in $PLTL$:

$$(\alpha \wedge \beta \mathsf{U} \gamma) \Leftrightarrow (\alpha \mathsf{U} \gamma) \wedge (\beta \mathsf{U} \gamma)$$

$$(\alpha \wedge \beta \mathsf{S} \gamma) \Leftrightarrow (\alpha \mathsf{S} \gamma) \wedge (\beta \mathsf{S} \gamma)$$

$$(\gamma \mathsf{U} \alpha \vee \beta) \Leftrightarrow (\gamma \mathsf{U} \alpha) \vee (\gamma \mathsf{U} \beta)$$

$$(\gamma \mathsf{S} \alpha \vee \beta) \Leftrightarrow (\gamma \mathsf{S} \alpha) \vee (\gamma \mathsf{S} \beta)$$

Proof: Direct check. $\dashv$

Proof of Gabbay's separation theorem: Lemmata

Lemma 2 [key lemma] a, q, α and β be propositional variables. Then each of the formulas

$$(q\mathsf{S}a \wedge (\alpha\mathsf{U}\beta)) \qquad (q \vee (\alpha\mathsf{U}\beta)\mathsf{S}a \wedge (\alpha\mathsf{U}\beta))$$

$$(q\mathsf{S}a \wedge \neg(\alpha\mathsf{U}\beta)) \qquad (q \vee (\alpha\mathsf{U}\beta)\mathsf{S}a \wedge \neg(\alpha\mathsf{U}\beta))$$

$$(q \vee (\alpha\mathsf{U}\beta)\mathsf{S}a) \qquad (q \vee \neg(\alpha\mathsf{U}\beta)\mathsf{S}a \wedge (\alpha\mathsf{U}\beta))$$

$$(q \vee \neg(\alpha\mathsf{U}\beta)\mathsf{S}a) \qquad (q \vee \neg(\alpha\mathsf{U}\beta)\mathsf{S}a \wedge \neg(\alpha\mathsf{U}\beta))$$

has an equivalent one in which (.U.) occurs only in the subformula $(\alpha\mathsf{U}\beta)$ which itself is not in the scope of a (.S.).

Corollary 1 Let α and β be purely propositional, and χ and θ be boolean combinations of $(\alpha\mathsf{U}\beta)$ and past formulas. Then $(\theta\mathsf{S}\chi)$ is equivalent to a boolean combination of $(\alpha\mathsf{U}\beta)$ and past formulas.

Proof: Convert θ, χ to CNF, DNF, resp., apply Lemma 1, then Lemma 2. $\dashv$

Proof the key lemma, (1): $(q\mathsf{S}a \wedge (\alpha\mathsf{U}\beta))$

$(q\mathsf{S}a \wedge (\alpha\mathsf{U}\beta))$ is equivalent to

$$[(q\mathsf{S}a) \wedge (\alpha\mathsf{S}a) \wedge \alpha \wedge (\alpha\mathsf{U}\beta)] \vee$$

$$[\beta \wedge (\alpha\mathsf{S}a) \wedge (q\mathsf{S}a)] \vee$$

$$(q\mathsf{S}\beta \wedge q \wedge (\alpha\mathsf{S}a) \wedge (q\mathsf{S}a))$$

Proof: $t_0 \models (q\mathsf{S}a \wedge (\alpha\mathsf{U}\beta))$ iff there exist $t_1 < t_0$ and $t_2 > t_1$ such that $t_1 \models a$, $t_2 \models \beta$, $t \models q$ for $x \in (t_1, t_0)$, and $t \models \alpha$ for $t \in (t_1, t_2)$. 3 possibilities:

$$t_2 > t_0: \quad \underbrace{a}_{t_1}, \alpha \wedge q, \ldots, \alpha \wedge q, \underbrace{\alpha}_{t_0}, \alpha, \ldots, \alpha, \underbrace{\beta}_{t_2} \qquad (\alpha \wedge q\mathsf{S}a) \wedge \alpha \wedge (\alpha\mathsf{U}\beta)$$

$$t_2 = t_0: \quad \underbrace{a}_{t_1}, \alpha \wedge q, \ldots, \alpha \wedge q, \underbrace{\beta}_{t_2=t_0} \qquad \beta \wedge (\alpha \wedge q\mathsf{S}a)$$

$$t_2 < t_0: \quad \underbrace{a}_{t_1}, \alpha \wedge q, \ldots, \alpha \wedge q, \underbrace{\beta \wedge q}_{t_2}, q \ldots, q, \underbrace{.}_{t_0} \qquad (q\mathsf{S}\beta \wedge q \wedge (\alpha \wedge q\mathsf{S}a))$$

$\dashv$

Some more equivalences for the proof of the key lemma

$$\models_{PLTL} \neg(\alpha \mathsf{U} \beta) \quad \Leftrightarrow \Box\neg\beta \vee (\neg\beta\mathsf{U}\neg\beta \wedge \neg\alpha)$$

$$\Leftrightarrow \Box\neg\beta \vee (\alpha \wedge \neg\beta\mathsf{U}\neg\beta \wedge \neg\alpha)$$

$$\models_{PLTL} \neg(\alpha \mathsf{S} \beta) \quad \Leftrightarrow \boxminus\neg\beta \vee (\neg\beta\mathsf{S}\neg\beta \wedge \neg\alpha)$$

$$\Leftrightarrow \boxminus\neg\beta \vee (\alpha \wedge \neg\beta\mathsf{S}\neg\beta \wedge \neg\alpha)$$

Proof of the key lemma (2): $(q\mathsf{S}a \wedge \neg(\alpha\mathsf{U}\beta))$

$$\models_{PLTL} (q\mathsf{S}a \wedge \neg(\alpha\mathsf{U}\beta)) \Leftrightarrow \underbrace{(q\mathsf{S}a \wedge \Box\neg\beta)}_{A} \vee \underbrace{(q\mathsf{S}a \wedge (\neg\beta\mathsf{U}\neg\beta \wedge \neg\alpha))}_{B}$$

A direct check shows that $\models_{PLTL} A \Leftrightarrow (\neg\beta \wedge q\mathsf{S}a) \wedge \neg\beta \wedge \Box\neg\beta$.

By the equivalence (1) about $(q\mathsf{S}a \wedge (X\mathsf{U}Y))$,

$$\begin{aligned} \models_{PLTL} B \Leftrightarrow \quad & \underbrace{[(q\mathsf{S}a) \wedge (\neg\beta\mathsf{S}a) \wedge \neg\beta \wedge (\neg\beta\mathsf{U}\neg\beta \wedge \neg\alpha)]}_{C} \vee \\ & [\neg\beta \wedge \neg\alpha \wedge (\neg\beta\mathsf{S}a) \wedge (q\mathsf{S}a)] \vee \\ & (q\mathsf{S}\neg\beta \wedge \neg\alpha \wedge q \wedge (\neg\beta\mathsf{S}a) \wedge (q\mathsf{S}a)) \end{aligned}$$

By distributivity, $\models A \vee C \Leftrightarrow (\neg\beta \wedge q\mathsf{S}a) \wedge \neg\beta \wedge (\Box\neg\beta \vee (\neg\beta\mathsf{U}\neg\beta \wedge \neg\alpha))$, which is equivalent to $(\neg\beta \wedge q\mathsf{S}a) \wedge \neg\beta \wedge \neg(\alpha\mathsf{U}\beta)$. Hence

$$\begin{aligned} \models_{PLTL} (q\mathsf{S}a \wedge \neg(\alpha\mathsf{U}\beta)) \Leftrightarrow \quad & [(q \wedge \neg\beta\mathsf{S}a) \wedge \neg\beta \wedge \neg(\alpha\mathsf{U}\beta)] \vee \\ & [\neg\beta \wedge \neg\alpha \wedge (\neg\beta \wedge q\mathsf{S}a)] \vee \\ & (q\mathsf{S}\neg\beta \wedge \neg\alpha \wedge q \wedge (q \wedge \neg\beta\mathsf{S}a)) \end{aligned}$$

Proof of the key lemma (3): $(q \vee (\alpha \mathsf{U} \beta) \mathsf{S} a)$:

$$\models_{PLTL} \neg(q \vee (\alpha \mathsf{U} \beta) \mathsf{S} a) \Leftrightarrow \boxminus \neg a \vee \underbrace{(\neg a \mathsf{S} \neg a \wedge \neg q \wedge \neg(\alpha \mathsf{U} \beta))}_{F}$$

and F is an instance of $(Q \mathsf{U} A \wedge \neg(\alpha \mathsf{U} \beta))$, already considered as case (2).

(4): $(q \vee \neg(\alpha \mathsf{U} \beta) \mathsf{S} a)$:

A direct chech shows that

$$\begin{aligned}(q \vee \neg(\alpha \mathsf{U} \beta) \mathsf{S} a) \Leftrightarrow \quad & (\neg a \wedge [(\neg a \wedge \alpha \mathsf{S} \neg q \wedge \neg a) \Rightarrow \neg \beta] \mathsf{S} a) \wedge \\ & ((\neg a \wedge \alpha \mathsf{S} \neg q \wedge \neg a) \Rightarrow \neg[\beta \vee (\alpha \wedge (\alpha \mathsf{U} \beta))]).\end{aligned}$$

Proof of the key lemma (5): $(q \vee (\alpha \mathsf{U} \beta)\mathsf{S}a \wedge (\alpha \mathsf{U} \beta))$:

$$(q \vee (\alpha \mathsf{U} \beta)\mathsf{S}a \wedge (\alpha \mathsf{U} \beta)) \Leftrightarrow \quad (\alpha \mathsf{S} a) \wedge [\beta \vee (\alpha \wedge (\alpha \mathsf{U} \beta))] \vee \left(\begin{array}{l} (\beta \vee \alpha \vee \neg(\neg\beta \mathsf{S} \neg q)\mathsf{S}\beta \wedge (\alpha \mathsf{S} a)) \wedge \\ \{[\beta \vee (\alpha \wedge (\alpha \mathsf{U} \beta))] \vee \neg(\neg \beta \mathsf{S} \neg q)\} \end{array} \right)$$

(6): $(q \vee (\alpha \mathsf{U} \beta)\mathsf{S}a \wedge \neg(\alpha \mathsf{U} \beta))$:

$(q \vee (\alpha \mathsf{U} \beta)\mathsf{S}a \wedge \neg(\alpha \mathsf{U} \beta))$ is equivalent to

$$[(q \wedge \neg\beta \mathsf{S} a) \wedge \neg\beta \wedge \neg(\alpha \wedge (\alpha \mathsf{U} \beta))] \vee$$
$$(q \vee (\alpha \mathsf{U} \beta)\mathsf{S}\neg\alpha \wedge \neg\beta \wedge (q \vee (\alpha \mathsf{U} \beta)) \wedge (q \wedge \neg\beta \mathsf{S} a)).$$

which can be given the required form using the equivalences (3) and (5).

Proof of the key lemma (8): $(q \vee \neg(\alpha \mathsf{U} \beta)\mathsf{S}a \wedge \neg(\alpha \mathsf{U} \beta))$

By $\models_{PLTL} \neg(A\mathsf{S}B) \Leftrightarrow \boxminus\neg B \vee (A \wedge \neg B\mathsf{S}\neg B \wedge \neg A)$ we have

$$\begin{aligned} \models_{PLTL} \neg(q \vee y\mathsf{S}a \wedge x) \quad \Leftrightarrow \quad & \boxminus(\neg a \vee \neg x)\vee \\ & (\neg a \vee \neg x\mathsf{S}\neg q \wedge \neg y \wedge \neg a)\vee \\ & (\neg a \vee \neg x\mathsf{S}\neg q \wedge \neg y \wedge \neg x), \end{aligned}$$

For $x \doteq y \doteq \neg(\alpha \mathsf{U} \beta)$, we derive

$$\begin{aligned} \models_{PLTL} (q \vee \neg(\alpha \mathsf{U} \beta)\mathsf{S}a \wedge \neg(\alpha \mathsf{U} \beta)) \Leftrightarrow \quad & \boxminus(\neg a \vee (\alpha \mathsf{U} \beta))\vee \\ & (\neg a \vee (\alpha \mathsf{U} \beta)\mathsf{S}\neg q \wedge (\alpha \mathsf{U} \beta) \wedge \neg a)\vee \\ & (\neg a \vee (\alpha \mathsf{U} \beta)\mathsf{S}\neg q \wedge (\alpha \mathsf{U} \beta)). \end{aligned}$$

The middle disjunctive member of this formula can be excluded. The first disjunctive member is, by definition, $\neg(\top\mathsf{S}a \wedge \neg(\alpha \mathsf{U} \beta))$ and can be given the required form using case (2). The second can be handled using case (5).

Proof of the key lemma (7): $(q \vee \neg(\alpha \mathsf{U} \beta) \mathsf{S} a \wedge (\alpha \mathsf{U} \beta))$

$$(q \vee \neg(\alpha \mathsf{U} \beta) \mathsf{S} a \wedge (\alpha \mathsf{U} \beta)) \Leftrightarrow$$
$$(q \vee \neg(\alpha \mathsf{U} \beta) \mathsf{S} \beta \wedge (q \vee \neg(\alpha \mathsf{U} \beta)) \wedge (\alpha \wedge q \mathsf{S} a)) \vee$$
$$[(\alpha \wedge q \mathsf{S} a) \wedge \beta] \vee$$
$$[(\alpha \wedge q \mathsf{S} a) \wedge (\alpha \mathsf{U} \beta)]$$

By distributivity, the first disjunctive member is equivalent to

$$(q \vee \neg(\alpha \mathsf{U} \beta) \mathsf{S} \beta \wedge q \wedge (\alpha \wedge q \mathsf{S} a)) \vee$$
$$(q \vee \neg(\alpha \mathsf{U} \beta) \mathsf{S} \beta \wedge (\alpha \wedge q \mathsf{S} a)) \wedge \neg(\alpha \mathsf{U} \beta)$$

and can be given the required form using cases (4) and (8).

Sources for the proof of the key lemma

This proof:

Dov Gabbay, Ian Hodkinson and Mark Reynolds, *Temporal Logic: Mathematical Foundations and Computational Aspects. Volume I*, OUP, 1994

For another proof of the key lemma see:

E. Clarke and B.-H. Schlingloff, Model Checking, Chapter 24 of *Handbook of Automated Reasoning*, A. Robinson and A. Voronkov (eds), Elsevier, 2001.

also available at:

`http://www2.informatik.hu-berlin.de/~hs/Publikationen/2000_Handbook-of-Automated-Reasoning_Clarke-Schlingloff_Model-Checking.ps`

Proof of Gabbay's separation theorem: more lemmata

Lemma 3 Let α and β be purely propositional and the only (.U.)-subformula of θ be $(\alpha \mathsf{U} \beta)$. Then θ is equivalent to a boolean combination of $(\alpha \mathsf{U} \beta)$ and past formulas.

Proof: Induction on the nesting depth of the occurrences of $(\alpha \mathsf{U} \beta)$ in (.S.)-subformulas of θ. $\dashv$

Lemma 4 Let α_i and β_i be purely propositional, $i = 1, \ldots, n$. Let the only (.U.)-subformulas of θ be $(\alpha_1 \mathsf{U} \beta_1)$, ..., $(\alpha_n \mathsf{U} \beta_n)$. Then θ is equivalent to a boolean combination of $(\alpha_1 \mathsf{U} \beta_1)$, $\ldots (\alpha_n \mathsf{U} \beta_n)$ and past formulas.

Proof: Induction on n. Apply the previous lemma to

$$\theta' \rightleftharpoons [u_i/(\alpha_i \mathsf{U} \beta_i) : i = 2, \ldots, n]\theta$$

to obtain a b. c. of $(\alpha_1 \mathsf{U} \beta_1)$ and past formulas containing $u_2, \ldots, u_n$. $\dashv$

Gabbay's separation theorem: last lemma and proof of the theorem

Lemma 5 Formulas θ with no occurrences of (.S.) in the scope of (.U.) are separable.

Proof: Obtain θ' from θ by substituting $(a_i \mathsf{U} b_i)$ for the occurrences of $(\alpha_i \mathsf{U} \beta_i)$ which are not in the scope of (.U.) themselves. Separate θ' first and then replace a_i, b_i by α_i, β_i. $\dashv$

Exchanging (.S.) and (.U.) preserves the validity of all the lemmata.

Proof: [of the separation theorem] Induction on the alternating depth of the nesting of (.U.) and (.S.) in the given formula φ. φ is either separated or

> φ has subformulas ψ of the form $(\alpha \mathsf{U} \beta)$ $((\alpha \mathsf{S} \beta))$ where α and β are past (future) and at least one of them has an occurrence of (.S.) (.U.).

In the latter case alternation depth can be decreased by replacing the ψs with equivalent separated formulas. $\dashv$

Elimination of $(.\mathsf{S}.)$ **in** $PLTL$

Theorem 2 Let φ be a $PLTL$ formula. Then there exists a future $PLTL$ formula ψ such that for all models σ

$$\sigma, 0 \models \varphi \Leftrightarrow \psi.$$

Proof: Obtain ψ by replacing all the $(.\mathsf{S}.)$-subformulas by $\perp$ in a separated equivalent of φ. $\dashv$

Expressive completeness of $PLTL$

$\sigma : \omega \rightarrow \mathbf{L}$ can be viewed as a model for the monadic first order theory of the linear order $\langle \omega, < \rangle$ with a unary predicate symbols P for every $p \in \mathbf{L}$:

$$P^{\sigma}(n) \leftrightarrow p \in \sigma(n) \text{ for all } n < \omega.$$

A $PLTL$ formula φ defines the unary predicate $\sigma, n \models \varphi$ on n. This predicate is definable in the first order theory via the standard translation ST:

$$\mathsf{ST}(\bot) \rightleftharpoons \bot, \qquad \mathsf{ST}(p) \rightleftharpoons P(n), \qquad \mathsf{ST}(\varphi \Rightarrow \psi) \rightleftharpoons \mathsf{ST}(\varphi) \Rightarrow \mathsf{ST}(\psi)$$

$$\mathsf{ST}((\varphi \mathsf{U} \psi)) \rightleftharpoons \exists k(n < k \wedge [k/n]\mathsf{ST}(\psi) \wedge \forall i(n < i \wedge i < k \Rightarrow [i/n]\mathsf{ST}(\varphi)))$$

$$\mathsf{ST}((\varphi \mathsf{S} \psi)) \rightleftharpoons \exists k(k < n \wedge [k/n]\mathsf{ST}(\psi) \wedge \forall i(k < i \wedge i < n \Rightarrow [i/n]\mathsf{ST}(\varphi)))$$

where k and i do not occur $\mathsf{ST}(\varphi)$, $\mathsf{ST}(\psi)$.

Q: Is every f.o.-defined unary predicate definable in $PLTL$ too?

Expressive completeness of $PLTL$ (Hans Kamp, 1968)

Theorem 3 Every f.o. definable unary predicate is definable in $PLTL$.

Definition 1 A temporal connective $\#$ of arity k is f.o. definable, if there is a f.o. formula $\alpha_\#$ with just one free variable t and possibly having occurrences of $P_1, \ldots, P_k$ such that

$$\sigma, n \models \#p_1 \ldots p_k \leftrightarrow \langle \omega, < \rangle, \lambda i.(\sigma, i \models p_1), \ldots, \lambda i.(\sigma, i \models p_k), n \models \alpha_\#.$$

ST can be defined for such connectives $\#$ in the obvious way.

Corollary 2 All f.o.-definable connectives can be regarded as derived in $PLTL$ with (.S.) and (.U.) as the basic connectives.

Proof: Since $\alpha_\#(n, P_1, \ldots, P_k)$ is equivalent to some temporal formula $\varphi[\alpha_\#]$ written with (.S.) and (.U.) using $p_1, \ldots, p_k$, we can define $\#$ by the clause

$$\#(p_1, \ldots, p_k) \rightleftharpoons \varphi[\alpha_\#]. \dashv$$

Expressive completeness of $PLTL$: proof

predicate formula $\alpha(t) \to$ temporal formula $\varphi[\alpha]$ s.t.

$$\sigma, i \models \varphi[\alpha] \leftrightarrow \langle \omega, < \rangle, \lambda n.(p_1 \in \sigma_n), \ldots, \lambda n.(p_k \in \sigma_n), i \models \alpha$$

Induction on the $\exists$-height $d_\exists(\alpha)$ of α.

$$\varphi[t < t] \rightleftharpoons \bot, \quad \varphi[P_i(t)] \rightleftharpoons p_i, \quad \varphi[\alpha_1 \Rightarrow \alpha_2] \rightleftharpoons \varphi[\alpha_1] \Rightarrow \varphi[\alpha_2] \tag{1}$$

Let α be $\exists x \beta$. We can assume $t \notin BV(\beta)$. Then, by the equivalences

$$\beta \Leftrightarrow P_i(t) \wedge [\top / P_i(t)]\beta \vee \neg P_i(t) \wedge [\bot / P_i(t)]\beta, \; i = 1, \ldots, k,$$

β is equivalent to a $\bigvee_i \delta_i \wedge \beta_i$ where δ_i are $\exists$-free, $x \notin FV(\delta_i)$, $d_\exists(\beta_i) \leq d_\exists(\beta)$, and the only occurrences of t in β_i have the forms $t < y$ and $y < t$ for some $y \in BV(\beta_i) \cup \{x\}$. We only need to handle $\exists x \beta_i$, because

$$\models \exists x(\delta_i \wedge \beta_i) \Leftrightarrow \delta_i \wedge \exists x \beta_i.$$

Expressive completeness of $PLTL$: Proof

$\exists x \beta$; t occurs in β only in $y < t$, $t < y$

We introduce fresh predicate symbols $R_{t<.}$, $R_{.<t}$. Let

$$\beta' \rightleftharpoons [R_{t<.}(y)/t < y, R_{.<t}(y)/y < t]\beta$$

Then

$$\langle \omega, < \rangle, \lambda n.(p_1 \in \sigma_n), \ldots, \lambda n.(p_k \in \sigma_n), \lambda n.(i < n), \lambda n.(n < i), i \models \exists x \beta \Leftrightarrow \exists x \beta'$$

$FV(\beta') = \{x\}$ and $d_\exists(\beta') < d_\exists(\alpha)$. By the induction hypothesis there exists a temporal $\varphi[\beta']$ such that for $\sigma' : \omega \to \mathcal{P}(\{p_1, \ldots, p_k, r_{t<.}, r_{.<t}\})$

$\sigma', i \models \varphi[\beta']$ iff

$$\langle \omega, < \rangle, \lambda n.(p_1 \in \sigma'_n), \ldots, \lambda n.(p_k \in \sigma'_n), \lambda n.(r_{t<.} \in \sigma'_n), \lambda n.(r_{.<t} \in \sigma'_n), i \models \beta'$$

We put $\varphi[\exists x \beta'] \rightleftharpoons \diamondminus\varphi[\beta'] \vee \varphi[\beta'] \vee \Diamond\varphi[\beta']$

Expressive completeness of $PLTL$: Proof

We have $\sigma', i \models \varphi[\exists x \beta']$ iff

$\langle \omega, < \rangle, \lambda n.(p_1 \in \sigma'_n), \ldots, \lambda n.(p_k \in \sigma'_n), \lambda n.(r_{t<.} \in \sigma'_n), \lambda n.(r_{.<t} \in \sigma'_n), i \models \exists x \beta'$

and

$\langle \omega, < \rangle, \lambda n.(p_1 \in \sigma'_n), \ldots, \lambda n.(p_k \in \sigma'_n), \lambda n.(i < n), \lambda n.(n < i), i \models \exists x \beta' \Leftrightarrow \exists x \beta.$

Hence

$$\langle \omega, < \rangle, \lambda n.(p_1 \in \sigma'_n), \ldots, \lambda n.(p_k \in \sigma'_n), \lambda n.(i < n), \lambda n.(n < i), i \models \exists x \beta$$

is equivalent to

$$\sigma', i \models \varphi[\exists x \beta'], \qquad (\varphi[\exists x \beta'] \text{ is } \diamondminus \varphi[\beta'] \vee \varphi[\beta'] \vee \Diamond \varphi[\beta'])$$

provided that for all n we have $r_{t<.} \in \sigma'_n \leftrightarrow i < n$, $r_{.<t} \in \sigma'_n \leftrightarrow n < i$.

Expressive completeness of $PLTL$: Proof

The only remaining problem is that $r_{t<.}, r_{.<t} \in \mathrm{Var}(\varphi[\exists x \beta'])$.

ψ - a separated equivalent to $\varphi[\exists x \beta']$

Let ψ' is the result of applying the substitutions

$[\bot/r_{t<.}, \bot/r_{.<t}]$ to the variables in ψ not in the scope of $(.\mathrm{S}.)$ and $(.\mathrm{U}.)$,

$[\bot/r_{t<.}, \top/r_{.<t}]$ to the $(.\mathrm{S}.)$-subformulas of ψ

and

$[\top/r_{t<.}, \bot/r_{.<t}]$ to the $(.\mathrm{U}.)$-subformulas of ψ, respectively.

If $r_{t<.} \in \sigma'_n \leftrightarrow i < n$ and $r_{.<t} \in \sigma'_n \leftrightarrow n < i$ for all n, then $\sigma', i \models \psi \Leftrightarrow \psi'$.

Finally, $\sigma', i \models \psi'$ iff $\sigma, i \models \psi'$, because $r_{t<.}, r_{.<t} \notin \mathrm{Var}(\psi')$. Hence

$$\langle \omega, < \rangle, \lambda n.(p_1 \in \sigma_n), \ldots, \lambda n.(p_k \in \sigma_n), i \models \exists x \beta \text{ iff } \sigma, i \models \psi'$$

and we can define $\varphi[\exists x \beta]$ as ψ'.

Interval temporal logic: a more expressive linear temporal logic

$PLTL$ is as expressive as the MFO theory of $\langle \omega, < \rangle$.

MSO theory of $\langle \omega, < \rangle$ is decidable too. It is captured by

automata on infinite words

(ω-)regular expressions, whose general form is

$$\bigcup_i L_i \cdot M_i^\omega.$$

where L_i and M_i denote regular expressions and

$$L^\omega = \{\alpha_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n \cdot \ldots : \alpha_i \in L \text{ for all } i < \omega\}.$$

Interval Temporal Logic (ITL, Moszkowski, 1985).

Two variants: finite or infinite intervals of time.

ITL on finite intervals

$$\varphi ::= \bot \mid p \mid \varphi \Rightarrow \varphi \mid \circ\varphi \mid (\varphi;\varphi) \mid \varphi^*.$$

Models finite sequences $\sigma \in (\mathcal{P}(\mathbf{L}))^+$.

$|\sigma|$ - the length of σ minus 1. $\sigma = \sigma_0\sigma_1 \ldots \sigma_{|\sigma|}$.

$\sigma \not\models \bot$		
$\sigma \models p$	iff	$p \in \sigma_0$
$\sigma \models \varphi \Rightarrow \psi$	iff	$\sigma \not\models \varphi$ or $\sigma \models \psi$
$\sigma \models \circ\varphi$	iff	$\lvert\sigma\rvert > 0$ and $\sigma_1 \ldots \sigma_{\lvert\sigma\rvert} \models \varphi$
$\sigma \models (\varphi;\psi)$	iff	there exists an $i \in \{0, \ldots, \lvert\sigma\rvert\}$ such that
		$\sigma_0 \ldots \sigma_i \models \varphi$ and $\sigma_i \ldots \sigma_{\lvert\sigma\rvert} \models \psi$
$\sigma \models \varphi^*$	iff	either $\lvert\sigma\rvert = 0$ or there exists an $n < \omega$
		and a finite sequence $0 = i_0 < \ldots < i_n = \lvert\sigma\rvert$
		such that $\sigma_{i_{k-1}} \ldots \sigma_{i_k} \models \varphi$ for $k = 1, \ldots, n$.

Derived constructs in ITL

$$
\begin{array}{lcl}
\mathsf{empty} & \rightleftharpoons & \neg \circ \top \\
\mathsf{skip} & \rightleftharpoons & \circ \mathsf{empty} \\
\Diamond \varphi & \rightleftharpoons & (\top ; \varphi) \\
\Box \varphi & \rightleftharpoons & \neg \Diamond \neg \varphi \\
\varphi^{+} & \rightleftharpoons & (\varphi ; \varphi^{*})
\end{array}
$$

Guarded normal form in ITL

Exercise 3 Prove that every ITL formula has an equivalent one of the form

$$\xi \wedge \mathsf{empty} \vee \bigvee_i \alpha_i \wedge \circ\psi_i$$

where ξ and the α_is have no occurrences of temporal operators and the α_is form a full system.

Fact 1 Given an arbitrary formula φ, there exists a finite set of formulas X such that $\varphi \in X$ and a system of purely propositional formulas ξ_ψ, $\psi \in X$ and $\alpha_{\psi,\chi}$, $\psi, \chi \in X$ such that $\{\alpha_{\psi,\chi} : \chi \in X\}$ is a full system for each $\psi \in X$ and

$$\models_{ITL} \psi \Leftrightarrow \xi_\chi \wedge \mathsf{empty} \vee \bigvee_{\chi \in X} \alpha_{\psi,\chi} \wedge \circ\chi. \qquad (2)$$

Propositional quantification in ITL

$\sigma \models \exists p \varphi$ iff there exists a $\sigma' \in (\mathcal{P}(\mathbf{L}))^{|\sigma|+1}$ such that

$\sigma'_i \setminus \{p\} = \sigma_i \setminus \{p\}$ for all $i = 0, \ldots, |\sigma|$ and $\sigma' \models \varphi$

Theorem 4 Propositional existential quantification is definable in ITL, that is, every formula of the form $\exists p \varphi$ is equivalent to a quantifier-free formula.

Propositional quantifier elimination in ITL

We assume that φ is quantifier-free. Let X be as in Fact 1. Then for $\psi \in X$

$$\models_{ITL} \exists p\psi \Leftrightarrow ([\bot/p]\xi_\chi \vee [\top/p]\xi_\chi) \wedge \mathsf{empty} \vee \bigvee_{\chi \in X} ([\bot/p]\alpha_{\psi,\chi} \vee [\top/p]\alpha_{\psi,\chi}) \wedge \circ \exists p\chi.$$

This is equivalent to

$$\exists p\psi \Leftrightarrow \eta_\chi \wedge \mathsf{empty} \vee \bigvee_{\chi \in X} (\beta_{\psi,\chi} \wedge \mathsf{skip}; \exists p\chi).$$

which is a system of equations wrt the unknowns $\exists p\chi$.

$\sigma \models \beta_{\psi,\chi} \wedge \mathsf{skip}$ does not depend on $\sigma_{|\sigma|}$.

If δ_1, δ_2 have this property, then so do $\delta_1 \vee \delta_2$, $(\delta_1; \delta_2)$ and δ_1^*.

Propositional quantifier elimination in ITL

Assume a system of equations

$$\exists p\psi \Leftrightarrow \gamma_\chi \vee \bigvee_{\chi \in X} (\delta_{\psi,\chi}; \exists p\chi). \quad (3)$$

where $\delta_{\psi,\chi}$ are s.t. $\sigma \models \delta_{\psi,\chi}$ does not depend on $\sigma_{|\sigma|}$.

Then we can solve the system wrt any chosen $\exists p\psi$:

$$\exists p\psi \Leftrightarrow (\delta^*_{\psi,\psi}; \gamma_\chi) \vee \bigvee_{\chi \in X \setminus \{\psi\}} (\delta^*_{\psi,\psi}; (\delta_{\psi,\chi}; \exists p\chi)).$$

Substituting this formula for $\exists p\psi$ elsewhere in the system and using that

$$\models_{ITL} (\alpha; \beta_1 \vee \beta_2) \Leftrightarrow (\alpha; \beta)_1 \vee (\alpha; \beta)_2,$$

we obtain system of the same form with fewer equations. Finally we reach a defining formula for $\exists p\varphi$.

Note that $\neg$ and $\wedge$ occur only in the purely propositional subformulas of the quantifier-free formula for $\exists p\varphi$. The price for this is the heavy use of $(.)^*$.

Infinite intervals

The clauses for $\models_{ITL}$ on infinite intervals differs only for $(.;.)$ and $(.)^*$:

$\sigma \models (\varphi;\psi)$	iff	either $\sigma \models \varphi$ or
		there exists an $i < \omega$ such that
		$\sigma_0 \ldots \sigma_i \models \varphi$ and $\sigma_i \ldots \models \psi$
$\sigma \models \varphi^*$	iff	there exists a finite sequence $0 = i_0 < \ldots < i_n$
		s. t. $\sigma_{i_{k-1}} \ldots \sigma_{i_k} \models \varphi$ for $k = 1, \ldots, n,$ and $\sigma_{i_n} \ldots \models \varphi$

The definition for $(\varphi;\psi)$ allows the class of the infinite intervals to be defined by the constant

$$\mathsf{inf} \rightleftharpoons (\top;\bot).$$

The End