Process-Algebraic Interpretations of Positive Linear and Relevant Logics

Mads Dam*
Dept. of Computer Science
University of Edinburgh, U.K.

Abstract

We investigate the use of positive linear and relevant logics to provide logical accounts of static process structure, and combinations of relevance and modality to account also for dynamic behaviour. A general notion of model is introduced, based on which three examples are given, using Milner's synchronous process calculus SCCS. The structure of models is enriched by prefixing operators to cover also dynamic behaviour. Logically dynamic behaviour is captured by adding past and future modal operators. The resulting logic is given sound and complete axiomatisations and shown to conservatively extend the positive fragment of linear logic. Finally the induced interpretations of formulas on process terms are characterised, and axiomatisations are given which are sound and complete with respect to validity in the process-based interpretations. The completeness proofs are based on rewriting and provide procedures for deciding validity and consistency of formulas with respect to the process-based interpretations.

1 Introduction

In this paper we study interpretations of positive, propositional fragments of linear and relevant logics in terms of process algebras. The basic idea is similar in spirit to Urquhart's semilattice interpretation of relevant logics [28]: Parallel composition furnishes a binary operation $\times$ and relative to an element x the implication is interpreted as the operation that transforms properties by left multiplication with x. That is,

$$x \models \phi \to \psi \text{ iff for all } y, \text{ if } y \models \phi \text{ then } x \times y \models \psi \quad (1)$$

*Supported by Danish Technical Research Council, grant 16-3809.E, and SERC grant GR/F 32219. New address: Swedish Institute of Computer Science, Box 1263, S-164 28 Kista, Sweden

Thus $\rightarrow$ expresses a relativisation of properties to properties of parallel contexts. As parallel composition is usually assumed to be commutative the restriction to left-multiplication is harmless. With this definition the implication can be used as a general handle to address the difficult problem of deriving compositional theories for concurrency. One rule for parallel composition is sufficient, namely:

$$\frac{x \models \phi \rightarrow \psi \quad y \models \phi}{x \times y \models \psi}$$

The problem of compositionality has thus been transformed into the problem of verifying implicative properties. As a property-transformer this implication has numerous well-known relatives in Computer Science. The classical example is the weakest preconditions of Dijstra [9]. In concurrency close relatives are the weakest inner environments of Larsen [17] and the doubly relativised turnstiles of Stirling [27].

The relation to relevant and linear logics arise in the following way: Parallel composition is usually assumed to be associative as well as commutative, and to possess an identity 1. For instance in CCS (Milner [20]) the identity is NIL; in SCCS (Milner [19]) it is 1; and in theoretical CSP (Brookes et al [6]) it is RUN for $\times$ the operator $||$ and STOP for $\times$ the operator $|||$. With this structure the implication of (1) corresponds naturally to the consequence relation $\phi_1, \ldots, \phi_n \models \psi$ that holds whenever $x_1 \models \phi_1, \ldots, x_n \models \phi_n$ implies $x_1 \times \cdots \times x_n \models \psi$ where the empty product is 1. Letting Φ, Ψ range over finite strings of formulas observe then that the following structural rules are validated:

$$\text{Reflexivity:} \quad \phi \models \phi$$

$$\text{Permutation:} \quad \frac{\Phi \models \phi}{\Psi \models \phi} \quad (\Psi \text{ a permutation of } \Phi)$$

$$\text{Cut:} \quad \frac{\Phi \models \phi \quad \Psi_1, \phi, \Psi_2 \models \psi}{\Psi_1, \Phi, \Psi_2 \models \psi}$$

Moreover with the intended interpretation of $\times$ as parallel composition the following structural rules will in general fail:

$$\text{Contraction:} \quad \frac{\Phi, \phi, \phi \models \psi}{\Phi, \phi \models \psi}$$

$$\text{Weakening:} \quad \frac{\Phi \models \psi}{\Phi, \phi \models \psi}$$

so the consequence relation is indeed linear in the sense of Girard [13].

Related observations have been made in the context of Petri nets by a number of authors (Brown [7], Gehlot and Gunter [12], Marti-Oliet and Meseguer [18], Winskel and Engberg [11]). Abadi and Plotkin [1] uses an implication related to ours to account for the assumption-guarantee principle for safety properties (c.f.

Pnueli [25]). Other ways of relating linear logic to concurrency have also been tried: Abramsky and Vickers [3] uses quantales, a topological variant of linear logic, to account for notions of process testing; and Abramsky and Jagadeesan [2] uses dataflow networks to interpret proofs in linear logic.

1.1 Outline of Paper

With the interpretation (1) the intensional (or multiplicative, in Girard's terminology [13]) fragment expresses purely structural properties of processes and is thus by itself of little interest. Other connectives are needed to capture also the dynamic properties of processes. Our aim in the present paper is to explore ways in which such extensions can be made while both

1. obtaining close connections to linear and relevant logics, and
2. giving concrete computational justifications for the choice of models and connectives.

The computational setting which we take as basic is that of process calculi such as CCS and SCCS [20, 19]. Processes are terms given computational meaning by an operational semantics in the style of Plotkin [24]. Formulas, as in e.g. Hennessy-Milner logic [15], denote sets of processes expressing their computational capabilities. Thus formulas can suitably be viewed as process specifications, and typical process verification problems include:

a. Given specification ϕ, does there exist a process satisfying ϕ?

b. Given specification ϕ and process p, does p satisfy ϕ?

This interpretation of processes and properties is, however, far too concrete and syntactic to support a really tight connection to linear and relevant logics. A more liberal approach is to consider instead models based on algebras with a structure akin to that of processes up to a suitable notion of semantical equivalence. This is the approach taken in the first part of this paper. We take as our point of departure a notion of model for positive linear and relevant logics whose underlying frame is a semilattice-ordered monoid. The intention is to relate the monoid operation to parallel composition and the semilattice operation to some form of choice operator. Logically this notion of model is a generalisation of Urquhart's semilattice model for relevant logics capable of capturing a wide range of positive linear and relevant logics in a uniform way. We give three examples of frames based on fragments of the synchronous calculus SCCS, two appropriate to positive linear logic, and one appropriate to the positive fragment of the relevant system **R**. The semantical equivalences used are simulation and bisimulation equivalence (c.f. Hennessy, Milner [15]), and the testing equivalence of De Nicola and Hennessy [21].

Building frames based on process calculi shows soundness of the axiom systems concerned. The first part of the paper addresses two main questions:

1. How can the general notion of frame be extended to cover also dynamic behaviour, and how can the logic be extended to reflect this?
2. In particular, can 1. be answered in such a manner that completeness is obtainable too?

The answers we propose are based on extensions of frames by unary operators, akin to the prefixing operators of CCS and SCCS, and a constant 0 for deadlock or divergence. Two equational classes of frames are considered, one for which potentiality for deadlock/divergence is ignored, and one where it is viewed as catastrophic. The first case is appropriate to safety properties, and the second to liveness properties. Computationally, these classes are motivated by containing as initial members two of the frames considered earlier based on SCCS with a corresponding version of testing equivalence. Or in other words: The equations determining the class of frames concerned gives a sound and complete axiomatisation of processes up to a form of testing equivalence. Logically this added frame structure is reflected by forwards and backwards modalities $|a>$ and $<a|$. We give an axiomatisation of these modalities which is shown to be sound and complete with respect to both classes of frames, and which is moreover shown to be conservative over linear logic.

Of particular computational interest, however, are the interpretations induced on process terms proper. That is, the interpretations on terms p induced by $p \models \phi$ iff $[p]_{\simeq} \models \phi$ where $\simeq$ is the semantical equivalence concerned. These interpretations form the topic of the second main part of the paper. It is important to obtain a characterisation of the process-based interpretations in purely operational/syntactical terms, since it is this characterisation which gives direct computational meaning to formulas, analogous, for instance, to the way transition systems give computational meaning to formulas in Hennessy-Milner logic. We obtain such a characterisation and show the usual logical characterisation result that

$$p \simeq q \text{ iff for all formulas } \phi,\ p \models \phi \text{ iff } q \models \phi. \tag{2}$$

The completeness results for arbitrary frames obtained in the first part of the paper do not apply to the process-based models. Thus they are insufficient for answering problems such as a. and b. above which motivated our work from the outset. One reason for the failure of completeness is the extraordinary expressive power of formulas when only the concrete process-based interpretations are considered. Using the modal operators an extensional falsehood constant $\perp$ denoting the empty set is definable. Then $\phi \to \perp$ denotes the inconsistency of ϕ ($p \models \phi \to \perp$ iff no q exists for which $q \models \phi$), and similarly $(\phi \to \perp) \to \perp$ denotes the consistency of ϕ ($p \models (\phi \to \perp) \to \perp$ iff for some q, $q \models \phi$). The problems this expressive power gives rise to do not appear particular to the modalities considered in the present paper: It is hard to think of a process specification language which is closed under extensional conjunction and does not have the power of expressing an unsatisfiable property. One problem is that the Henkin-style approach used in the earlier completeness proofs becomes difficult to use. In effect a syntactical characterisation of consistent and inconsistent formulas seems to be called for, and an attractive alternative approach is therefore to use rewriting techniques. Sets of new axiom schemas are given by which formulas can be rewritten into normal forms. As consistent normal forms can easily be given models completeness follows. Moreover, since the rewriting procedure is effective, and properties of normal form are easily determined, byproducts of the completeness proofs are procedures to decide for instance consistency and inconsistency of formulas.

1.2 Fusion or Implication

The main thrust of our work, in the tradition of relevance logic, is to take the implication and its interaction with extensional and modal connectives as the issues of primary interest. An alternative, more algebraically oriented approach is to emphasize instead the operator of fusion, or intensional conjunction, $\circ$, associated to the implication by the adjunction

$$\vdash \phi \rightarrow (\psi \rightarrow \gamma) \text{ iff } \vdash \phi \circ \psi \rightarrow \gamma. \tag{3}$$

Here $\vdash$ denotes provability in the axiom system under consideration. One reason for adopting this approach is that when arbitrary infinite disjunctions are available then $\rightarrow$ is derivable by

$$\phi \rightarrow \psi = \bigvee \{\gamma \mid\vdash \gamma \circ \phi \rightarrow \psi\}. \tag{4}$$

This is the view taken, for instance, in quantale-based models (c.f. Abramsky and Vickers [3]). The use of infinite disjunctions in (4) is, however, essential, and does not apply in the present setting of finitary unquantified propositional logic. The addition of infinite disjunctions or other higher-order mechanisms are extensions that may well prove valuable (c.f. Engberg and Winskel [11] for an example where propositional fixed point operators are considered briefly), but the increase in expressive power would be very substantial indeed, and such extensions are therefore left for future consideration.

One example of a process logic which uses the fusion operator is the compositional model checker of Winskel [29]. There $\circ$ ($\otimes$ in [29]) is introduced with the interpretation $p \models \phi_1 \circ \phi_2$ iff $\exists p_1, p_2$ such that $p = p_1 \times p_2$, $p_1 \models \phi_1$ and $p_2 \models \phi_2$. With this interpretation $\circ$ is restricted to occurring only in antecedents of ground implications, as otherwise the language would be able to distinguish processes according to their static structure only, something which most process equivalences do not allow. This restriction can be lifted by quotienting with respect to a suitable semantical equivalence $\simeq$ so that

$$p \models \phi_1 \circ \phi_2 \text{ iff there are } p_1, p_2 \text{ such that } p \simeq p_1 \times p_2,\ p_1 \models \phi_1 \text{ and } p_2 \models \phi_2. \tag{5}$$

This reference to $\simeq$ is, however, in some respects unfortunate: First it requires models to be explicitly parametrised by $\simeq$. However, one would, and indeed should, expect that $\simeq$ is determined by the logic in the sense of (2). Secondly, and more seriously, with the interpretation (5) $\circ$ lacks a direct computational interpretation in contrast to the other connectives we consider. Nonetheless the more algebraic perspective that the fusion lends itself to is valuable in several respects, to justify our notions of model and satisfaction, to justify our choice of connectives by using adjunctions as in (3), and to throw light on the axiomatisations considered.

The paper is structured as follows: In section 2 we introduce our general model for positive linear and relevant logics, obtain soundness and completeness results for the positive fragments of linear logic as well as the relevant system **R**, and justify our notion of model in terms of quantales. Examples of models based on processes are given in sections 3 and 4. In section 5 synchronous algebras, extending general frames by action operators, are introduced. Representation theorems for their initial algebras are proved, and it is shown how these representations provide processes with a fully abstract denotational semantics. In section 6 linear logic is extended by operators to reflect the additional structure of synchronous algebras. Soundness, completeness and conservative extension results are obtained, and the relations to corresponding extensions of quantales by operators are discussed. From section 7 onwards attention is focused on the process-based models. In section 7 the interpretations induced on process terms are characterised, and it is shown how using these characterisations the logics induce the expected semantical equivalences on terms. The remaining part addresses the problem of completely axiomatising validity of formulas with respect to the process-based interpretations only. In section 8 the axiomatisations are introduced, and their soundness proved, and sections 9 and 10 contain the proofs of completeness and decidability. Finally, in section 11 possible extensions and future work is discussed.

2 Models for Positive Relevant Logics

In this section we develop a notion of model for positive fragments of linear logic with a structure resembling the static structure of process calculi such as CCS and SCCS. Syntactically, the language of *positive formulas* is generated by the abstract syntax

$$\phi ::= X \mid \mathrm{t} \mid \phi \to \phi \mid \phi \circ \phi \mid \phi \wedge \phi \mid \phi \vee \phi$$

where X ranges over atomic propositions. The intensional connectives are the (intensional) truthhood constant t, the implication $\to$, and the operation $\circ$ known variously as fusion, intensional conjunction, tensor, or times. The extensional connectives are $\wedge$ and $\vee$. We generally assume $\to$ to have least binding power. In linear logic terminology, t corresponds to the constant 1, $\circ$ to $\otimes$, $\to$ to linear implication, and $\wedge$ and $\vee$ to the additive "with" (&) and "plus" ($\oplus$) respectively.

2.1 Semantics

For the semantics it is well known (c.f. Urquhart [28], Dunn [10]) that the standard set-theoretic interpretation of $\wedge$ as intersection and $\vee$ as union is problematic in the context of relevant logics. The semantics of (Routley and Meyer [26]) remedies this by introducing a ternary relation R on elements of models, replacing the interpretation (1) of section 1 by

$$x \models \phi \rightarrow \psi \text{ iff for all } y, z, \text{ if } y \models \phi \text{ and } R(x, y, z) \text{ then } z \models \psi \tag{6}$$

The ternary relation can be understood by reading $R(x, y, z)$ as "the combination of the pieces of information x and y $(\cdots)$ is a piece of information in z" [10]. Thus both ideas of intensional combination of information and of information content are involved. We propose separating these notions, using the monoid structure to account for the first, and a semilattice structure to account for the second. This allows us to easily capture also logics such as linear logic for which distributivity of $\wedge$ over $\vee$ fails, something for which the ternary relation model is not well equipped. In terms of processes our intention is to relate the monoid operation to parallel composition and the semilattice operation to process-algebraic choice operators.

Definition 2.1 (Frame, Model). A *frame* is a structure $F = (S, \sqcap, \times, 1)$ where

1. $1 \in S$,
2. $(S, \sqcap)$ is a semilattice,
3. $(S, \times, 1)$ is a commutative monoid,
4. $\times$ distributes over $\sqcap$. That is, $x \times (y \sqcap z) = (x \times y) \sqcap (x \times z)$ for all $x, y, z \in S$.

A set $B \subseteq S$ is a *filter*, if for all $x, y \in S$, $x, y \in B$ iff $x \sqcap y \in B$. A *model* (based on F) is a pair $M = (F, V)$ where F is a frame and V is a valuation which for each propositional letter X gives a filter $V(X)$.

The partial ordering $\leq$ on models is derived in the usual way: $x \leq y$ iff $x \sqcap y = x$. Our usage of the term filter is slightly nonstandard in that filters are usually assumed to be neither empty nor improper. Note that B is a filter iff

1. $x \in B$ and $x \leq y$ implies $y \in B$, and
2. $x, y \in B$ implies $x \sqcap y \in B$.

The filter property of valuations is quite natural if $\sqcap$ is understood as expressing intersection of information contents: More information entails more atomic properties should hold, and for any two elements their common information is sufficient to establish their common atomic properties. The distributivity of $\times$ over $\sqcap$ can be understood in similar terms.

Definition 2.2 (Satisfaction). The relation of satisfaction, $x \models_M \phi$, is defined in the following way:

1. $x \models_M X$ iff $x \in V(X)$,
2. $x \models_M \mathrm{t}$ iff $1 \leq x$,
3. $x \models_M \phi \to \psi$ iff for all $y \in S$, $y \models_M \phi$ only if $x \times y \models_M \psi$.
4. $x \models_M \phi \circ \psi$ iff there are $x_1, x_2 \in S$ such that $x_1 \times x_2 \leq x$, $x_1 \models_M \phi$ and $x_2 \models_M \psi$.
5. $x \models_M \phi \wedge \psi$ iff $x \models_M \phi$ and $x \models_M \psi$,
6. $x \models_M \phi \vee \psi$ iff $x \models_M \phi$ or $x \models_M \psi$ or there are $x_1, x_2 \in S$ such that $x_1 \sqcap x_2 \leq x$, $x_1 \models_M \phi$ and $x_2 \models_M \psi$.

Let $\mathcal{M}$ be a class of models. A formula ϕ is *$\mathcal{M}$-valid*, if $1 \models_M \phi$ for all $M \in \mathcal{M}$. If $\mathcal{M}$ is the class of all models, ϕ is said to be *universally valid*.

We usually omit indexing of $\models$ by M when M is understood from the context. The filter property for atomic propositions extends to the full language. This property is used extensively in the proof of soundness below.

Proposition 2.3 (The Filter Property) *For all ϕ, $\{x \in S_M \mid x \models_M \phi\}$ is a filter.*

PROOF: An easy structural induction. For $\to$, suppose first that $x \models \phi \to \psi$ and $x \leq y$. To check $y \models \phi \to \psi$ let $z \models \phi$. Then $x \times z \models \psi$ so by monotonicity of $\times$ and the induction hypothesis also $y \times z \models \psi$. So $y \models \phi \to \psi$. Conversely if $x, y \models \phi \to \psi$ and $z \models \phi$ then $x \times z, y \times z \models \psi$ so also $(x \times z) \sqcap (y \times z) = (x \sqcap y) \times z \models \psi$. Thus $x \sqcap y \models \phi \to \psi$ as desired.

For $\circ$, if $x \models \phi \circ \psi$ and $x \leq y$ then $y \models \phi \circ \psi$ is immediate. Conversely let $x, y \models \phi \circ \psi$. Then we find x_1, x_2, y_1, y_2 such that $x_1, y_1 \models \phi$, $x_2, y_2 \models \psi$, $x_1 \times x_2 \leq x$ and $y_1 \times y_2 \leq y$. By the induction hypothesis, $x_1 \sqcap y_1 \models \phi$ and $x_2 \sqcap y_2 \models \psi$. Moreover $(x_1 \sqcap y_1) \times (x_2 \sqcap y_2) \leq x \sqcap y$ whence $x \sqcap y \models \phi \circ \psi$.

For $\vee$ suppose first that $x \models \phi \vee \psi$ and $x \leq y$. If $x \models \phi$ or $x \models \psi$ then $y \models \phi \vee \psi$ by the induction hypothesis, and if there are x_1, x_2 such that $x_1 \sqcap x_2 \leq x$, $x_1 \models \phi$ and $x_2 \models \psi$ then $x_1 \sqcap x_2 \leq y$ so $y \models \phi \vee \psi$. Conversely suppose that $x, y \models \phi \vee \psi$. If $x, y \models \phi$ or $x, y \models \psi$ then $x \sqcap y \models \phi \vee \psi$ by the induction hypothesis. If $x \models \phi$ and $y \models \psi$, say, then immediately $x \sqcap y \models \phi \vee \psi$. If $x \models \phi$ and $y_1, y_2 \leq y$, $y_1 \models \phi$ and $y_2 \models \psi$ then $x \sqcap y_1 \models \phi$ by the induction hypothesis, so indeed $x \sqcap y \models \phi \vee \psi$. The other cases are similar.

The remaining cases are easy exercises. □

2.2 Axiomatisation

The appropriate logic for axiomatising universal validity is the positive fragment of linear logic axiomatised by the following Hilbert-type system (Avron [5]):

Axioms	**I**	$\phi \to \phi$
	B	$(\psi \to \gamma) \to ((\phi \to \psi) \to (\phi \to \gamma))$
	C	$(\phi \to (\psi \to \gamma)) \to (\psi \to (\phi \to \gamma))$
	$\wedge$**-Intro**	$(\phi \to \psi) \wedge (\phi \to \gamma) \to (\phi \to \psi \wedge \gamma)$
	$\wedge$**-Elim1**	$\phi \wedge \psi \to \phi$
	$\wedge$**-Elim2**	$\phi \wedge \psi \to \psi$
	$\vee$**-Intro1**	$\phi \to \phi \vee \psi$
	$\vee$**-Intro2**	$\psi \to \phi \vee \psi$
	$\vee$**-Elim**	$(\phi \to \gamma) \wedge (\psi \to \gamma) \to (\phi \vee \psi \to \gamma)$
	t1	t
	t2	$\mathrm{t} \to (\phi \to \phi)$
	$\circ$**1**	$\phi \to (\psi \to (\phi \circ \psi))$
	$\circ$**2**	$(\phi \to (\psi \to \gamma)) \to ((\phi \circ \psi) \to \gamma)$
Rules	**Detachment**	$\dfrac{\phi \quad \phi \to \psi}{\psi}$
	Adjunction	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}$

I is known also as reflexivity, **B** as transitivity, and **C** as permutation. Let $\vdash_{\mathbf{LL}^+} \phi$ if ϕ is provable in this system.

Theorem 2.4 (Soundness and Completeness, $\mathbf{LL}^+$) $\vdash_{\mathbf{LL}^+} \phi$ *iff* ϕ *is universally valid.*

Proof: Soundness is proved as usual by showing the axioms valid and the rules validity preserving. Completeness is proved by a modification of the Henkin-style construction standard in relevance logic (c.f. Dunn [10]). Let an $\mathbf{LL}^+$-*theory* be any set T of formulas for which

1. $\phi \in T$ and $\vdash_{\mathbf{LL}^+} \phi \to \psi$ implies $\psi \in T$, and
2. $\phi, \psi \in T$ implies $\phi \wedge \psi \in T$.

We then define a canonical model $M(\mathbf{LL}^+)$ by letting S be the set of all $\mathbf{LL}^+$-theories, $\sqcap$ intersection, 1 the set of all $\mathbf{LL}^+$-theorems, and defining the multiplication $\times$ and the valuation V by

$$\begin{aligned} T_1 \times T_2 &= \{\psi \mid \exists \phi \in T_2 . \phi \to \psi \in T_1\} \\ V(X) &= \{T \mid T \text{ an } \mathbf{LL}^+\text{-theory and } X \in T\} \end{aligned}$$

Clearly $\sqcap$, 1 and V are well-defined. For $\times$ suppose that $\psi \in T_1 \times T_2$ and that $\vdash_{\mathbf{LL}^+} \psi \to \gamma$. Then there is some $\phi \in T_2$ such that $\phi \to \psi \in T_1$. By **B**, **C** and detachment, $\phi \to \gamma \in T_1$ too so $\gamma \in T_1 \times T_2$. Secondly if $\psi_1, \psi_2 \in T_1 \times T_2$ then there are $\phi_1, \phi_2 \in T_2$ such that $\phi_1 \to \psi_1, \phi_2 \to \psi_2 \in T_1$. Then $\phi_1 \wedge \phi_2 \in T_2$. By $\wedge$-Elim (1 and 2), **B** and detachment, we obtain $\phi_1 \wedge \phi_2 \to \psi_1, \phi_1 \wedge \phi_2 \to \psi_2 \in T_1$, so by $\wedge$-Intro, $\phi_1 \wedge \phi_2 \to \psi_1 \wedge \psi_2 \in T_1$ too, so $\psi_1 \wedge \psi_2 \in T_1 \times T_2$ as desired. Note that in terms of $\circ$,

$$T_1 \times T_2 = \{\gamma \mid \exists \phi \in T_1, \psi \in T_2.\ \vdash_{\mathbf{LL}^+} \phi \circ \psi \to \gamma\}.$$

To check the monoid properties we first prove commutativity. For this it suffices to show $\vdash_{\mathbf{LL}^+} \phi \to ((\phi \to \psi) \to \psi)$, so that if $\phi \in T$ then $(\phi \to \psi) \to \psi \in T$ too. But $\vdash_{\mathbf{LL}^+} (\phi \to \psi) \to (\phi \to \psi)$ so the result follows by **C** and detachment. For the identity of 1 assume first that $\psi \in 1 \times T$. Then there is a $\phi \in T$ such that $\phi \to \psi \in 1$. But then $\vdash_{\mathbf{LL}^+} \phi \to \psi$ so $\psi \in T$ as desired. Conversely if $\phi \in T$ then by **I** also $\phi \in 1 \times T$. For associativity of $\times$ assume that $\gamma \in T_1 \times (T_2 \times T_3)$. Then there is a $\psi \in T_2 \times T_3$ such that $\psi \to \gamma \in T_1$, and thus a $\phi \in T_3$ such that $\phi \to \psi \in T_2$. By **B**, $(\phi \to \psi) \to (\phi \to \gamma) \in T_1$ so $\phi \to \gamma \in T_1 \times T_2$, and thus $\gamma \in (T_1 \times T_2) \times T_3$.

It remains to show $\times$ distributive. The containment $T_1 \times (T_2 \sqcap T_3) \subseteq (T_1 \times T_2) \sqcap (T_1 \times T_3)$ is clear. For the converse containment let $\psi \in T_1 \times T_2$ and $\psi \in T_1 \times T_3$. Then there are $\phi_2 \in T_2$ and $\phi_3 \in T_3$ such that $\phi_2 \to \psi, \phi_3 \to \psi \in T_1$. By $\vee$-Intro, $\phi_2 \vee \phi_3 \in T_2 \sqcap T_3$, and by $\vee$-Elim, $\phi_2 \vee \phi_3 \to \psi \in T_1$, giving the result.

We have thus shown the canonical model indeed to be a model. The proof is then completed by showing that $\phi \in T$ iff $T \models_{M(\mathbf{LL}^+)} \phi$ using induction in the structure of ϕ. For atomic propositions and $\wedge$ the result is immediate. For t, if $\mathrm{t} \in T$ and $\phi \in 1$, i.e. $\vdash_{\mathbf{LL}^+} \phi$ then $\vdash_{\mathbf{LL}^+} \mathrm{t} \to \phi$ by t2, **C** and detachment, so $\phi \in T$. Thus $T \models \mathrm{t}$. Conversely if $1 \subseteq T$ then $\mathrm{t} \in T$ by t1.

For $\vee$ assume that $\phi \vee \psi \in T$. Let $T_1 = \{\gamma \mid \vdash_{\mathbf{LL}^+} \phi \to \gamma\}$ and $T_2 = \{\gamma \mid \vdash_{\mathbf{LL}^+} \psi \to \gamma\}$. Then $T_1 \sqcap T_2 = \{\gamma \mid \vdash_{\mathbf{LL}^+} \phi \vee \psi \to \gamma\}$. Hence $T_1 \sqcap T_2 \leq T$. But then $T \models \phi \vee \psi$ as by the induction hypothesis $T_1 \models \phi$ and $T_2 \models \psi$. Conversely assume that $T \models \phi \vee \psi$. If $T \models \phi$ or $T \models \psi$ we are done by the induction hypothesis, so let instead $T_1 \sqcap T_2 \leq T$, $T_1 \models \phi$ and $T_2 \models \psi$. By the induction hypothesis $\phi \in T_1$ and $\psi \in T_2$, so $\phi \vee \psi \in T_1 \sqcap T_2$ whence $\phi \vee \psi \in T$ too.

For $\to$ let $\phi \to \psi \in T$ and $T_1 \models \phi$ and we must show $T \times T_1 \models \psi$. By the induction hypothesis, $\phi \in T_1$, so $\psi \in T \times T_1$ and the result follows by the induction hypothesis. For the converse direction let $T \models \phi \to \psi$. Let $T_1 = \{\gamma \mid \vdash_{\mathbf{LL}^+} \phi \to \gamma\}$. Then $T_1 \models \phi$ by the induction hypothesis, so $T \times T_1 \models \psi$. Thus $\psi \in T \times T_1$ by the induction hypothesis, and it follows that there is some $\gamma \in T_1$ such that $\gamma \to \psi \in T$. But then $\vdash_{\mathbf{LL}^+} \phi \to \gamma$ so $\phi \to \psi \in T$ too as desired.

Finally for $\circ$ suppose first that $\phi \circ \psi \in T$. Let $T_1 = \{\gamma \mid \vdash_{\mathbf{LL}^+} \phi \to \gamma\}$ and $T_2 = \{\gamma \mid \vdash_{\mathbf{LL}^+} \psi \to \gamma\}$. Then $T_1 \times T_2 \leq T$. For if $\gamma \in T_1 \times T_2$ then there is some γ' such that $\vdash_{\mathbf{LL}^+} \psi \to \gamma'$ and $\vdash_{\mathbf{LL}^+} \phi \to (\gamma' \to \gamma)$. But then by **B** and **C**, $\vdash_{\mathbf{LL}^+} \phi \to (\psi \to \gamma)$ such that $\vdash_{\mathbf{LL}^+} (\phi \circ \psi) \to \gamma$ by $\circ$**2**, and then $\gamma \in T$ as desired. By the induction hypothesis, $T_1 \models \phi$ and $T_2 \models \psi$, so we obtain $T \models \phi \circ \psi$. Conversely if $T \models \phi \circ \psi$ we find T_1, T_2 such that $T_1 \models \phi$, $T_2 \models \psi$, and $T_1 \times T_2 \leq T$. By the induction hypothesis, $\phi \in T_1$ and $\psi \in T_2$. By $\circ\mathbf{1}$, $\psi \rightarrow (\phi \circ \psi) \in T_1$. Thus $\phi \circ \psi \in T_1 \times T_2$ and we are done. □

It is not hard to verify that Theorem 2.4 applies equally to the $\circ$-free fragment of $\mathbf{LL}^+$. Moreover, as $\mathbf{LL}^+$-theories include the empty theory it follows from the proof of Theorem 2.4 that $\mathbf{LL}^+$ is sound and complete with respect to models that contain an element 0 which is zero for both $\sqcap$ and $\times$. Other well-known relevance logics, both stronger and weaker than $\mathbf{LL}^+$, are obtained by corresponding variations on frame-conditions and axiomatisations (c.f. Dam [8]). An important example is the positive fragment, $\mathbf{R}^+$, of the standard relevant system $\mathbf{R}$ (c.f. Dunn [10]). This system is axiomatised by adding to the axioms for $\mathbf{LL}^+$ the two axioms:

S	$(\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \gamma))$
Distribution	$(\phi \vee \psi) \wedge \gamma \rightarrow (\phi \wedge \gamma) \vee (\psi \wedge \gamma)$

For the semantics an $\mathbf{R}^+$-*frame* is a frame F with the following two properties:

1. For all $x \in S_F$, $x \times x \leq x$,
2. Whenever $x \sqcap y \leq z$, $x \not\leq z$ and $y \not\leq z$ then there are $x' \geq x$ and $y' \geq y$ such that $x' \sqcap y' = z$.

Condition 1 is referred to as *semi-idempotency*. Condition 2 is very close to the standard notion of distributivity in semilattices: Whenever $x \sqcap y \leq z$ then there are $x' \geq x$ and $y' \geq y$ such that $x' \sqcap y' = z$, regardless of whether $x \leq z$ or $y \leq z$ or not. For unital semilattices (semilattices with a unit $\top$ for $\sqcap$), or more generally for semilattices in which each pair of elements has an upper bound (that is, for all x, y there is some z such that $x \leq z$ and $y \leq z$), the definitions coincide. An $\mathbf{R}^+$-*model* is a model which is based on an $\mathbf{R}^+$-frame.

Theorem 2.5 (Soundness and Completeness, $\mathbf{R}^+$) *$\vdash_{\mathbf{R}^+} \phi$ iff ϕ is $\mathcal{M}$-valid where $\mathcal{M}$ is the class of all $\mathbf{R}^+$-models.*

PROOF: The soundness of **S** and Distribution is proved as usual. For completeness all that is needed is to check that the canonical model $M(\mathbf{R}^+)$ constructed as in the proof of Theorem 2.4 validates the two extra model conditions.

For semi-idempotency let T be an $\mathbf{R}^+$-theory and $\psi \in T \times T$. Then there is some $\phi \in T$ such that $\phi \rightarrow \psi \in T$ too. The following derivation shows that $\vdash_{\mathbf{R}^+} (\phi \rightarrow \psi) \wedge \phi \rightarrow \psi$ which is sufficient to establish the result:

1.	$(\phi \rightarrow \psi) \wedge \phi \rightarrow (\phi \rightarrow \psi)$	$\wedge$-Elim1
2.	$((\phi \rightarrow \psi) \wedge \phi \rightarrow \phi) \rightarrow ((\phi \rightarrow \psi) \wedge \phi \rightarrow \psi)$	1, by **S**
3.	$(\phi \rightarrow \psi) \wedge \phi \rightarrow \psi$	2, by $\wedge$-Elim2

For distributivity let $T_1 \sqcap T_2 \leq T$, $T_1 \not\leq T$ and $T_2 \not\leq T$. Let $T'_i = \{\gamma \mid \exists\psi \in T_i, \psi' \in T$ such that $\vdash_{\mathbf{R}^+} \psi \wedge \psi' \rightarrow \gamma\}$, $i \in \{1,2\}$. Clearly T'_i is a $\mathbf{R}^+$-theory; it is the least $\mathbf{R}^+$-theory containing $T_i \cup T$. We must show that $T'_1 \sqcap T_2 = T$. The verification of $T \subseteq T'_1 \sqcap T_2$ is entirely straightforward. For $T'_1 \sqcap T_2 \subseteq T$ let $\phi \in T'_1 \sqcap T'_2$. Then there are $\phi_1 \in T'_1$ and $\phi_2 \in T'_2$ such that $\vdash_{\mathbf{R}^+} \phi_1 \vee \phi_2 \rightarrow \phi$. Let $\phi_i \in T'_i$, $i \in \{1,2\}$. We then find some $\psi_i \in T_i$ and $\psi'_i \in T$ such that $\vdash_{\mathbf{R}^+} \psi_i \wedge \psi'_i \rightarrow \phi_i$. Let $\psi' = \psi'_1 \wedge \psi'_2$. It follows that $\vdash_{\mathbf{R}^+} (\psi_1 \wedge \psi') \vee (\psi_2 \wedge \psi') \rightarrow \phi$. But then by Distribution also $\vdash_{\mathbf{R}^+} (\psi_1 \vee \psi_2) \wedge \psi' \rightarrow \phi$. But both $\psi_1 \vee \psi_2$ and ψ' are in T so $\phi \in T$ too. □

2.3 Quantales, and Algebraic Models

The presentation of section 2.2 takes implication as primitive and derive fusion by axioms ∘**1** and ∘**2**. Alternatively fusion can be taken as primary. This is the approach taken in the algebraic models of Dunn (c.f. [10]) or in those based on quantales (c.f. [3]). Here the algebraic models serve mainly to justify our notion of model and the relation of satisfaction. In an algebraic setting an equational presentation is more appropriate than the Hilbert-type presentation of section 2.2.

Definition 2.6 (Quantale). A *quantale* is a structure $(Q, \circ_q, \mathrm{t}_q)$ for which

1. Q is a complete lattice,
2. $(Q, \circ_q, \mathrm{t}_q)$ is a commutative monoid, and
3. $\circ_q$ distributes over arbitrary joins, i.e. $u \circ_q (\bigvee_i v_i) = \bigvee_i (u \circ_q v_i)$.

In quantales the implication can be defined by

$$u \rightarrow_q v = \bigvee\{w \mid w \circ_q u \leq v\} \tag{7}$$

where the partial ordering $\leq$ is derived in the usual way by $u \leq v$ iff $u \wedge v = u$. If only finite joins are available $\rightarrow$ is not generally definable and Q is then required to possess a right adjoint $\rightarrow_q$ for $\circ_q$, i.e. an operation $\rightarrow_q$ satisfying

$$u \leq v \rightarrow_q w \text{ iff } u \circ_q v \leq w. \tag{8}$$

This property is important in that it provides a characterisation of the implication in terms of fusion, and vice versa. In the terminology of Dunn [10] it amounts to Q being *residuated*.

By means of the quantale structure together with (7) quantales provide algebraic models for linear and relevant logics in the obvious way: In any quantale Q, an interpretation $[\![X]\!] \in Q$ of the propositional letters X is extended uniquely to an interpretation $[\![\phi]\!] \in Q$ of arbitrary formulas, such that $[\![\phi]\!]$ respects formula structure.

This interpretation is sound and complete with respect to the axiomatisation of section 2.2 in the sense that $\vdash_{\mathbf{LL}^+} \phi$ iff ϕ is *valid*, $\mathrm{t}_q \leq [\![\phi]\!]$, in all interpretations in all quantales Q. This can be seen either directly, or by exploiting the tight connection between quantales and the models of section 2.1. Given a frame F the *filter completion* of F is the quantale $\mathrm{qu}(F)$ consisting of all filters in F with $\bigvee\{B_i\}_{i\in I} = \{x \mid \exists x_1, \ldots, x_n \in \bigcup\{B_i\}_{i\in I}.\ x_1 \sqcap \cdots \sqcap x_n \leq x\}$, $B_1 \circ_q B_2 = \{z \mid \exists x \in B_1, y \in B_2.\ x \times y \leq z\}$, and $\mathrm{t}_q = \{x \mid 1 \leq x\}$. A straightforward inductive argument verifies that the relation $x \in B$ satisfies all the conditions 2.2.2–6. Indeed this property can be taken to justify Definition 2.2 itself. The construction of the filter completion $\mathrm{fr}(Q)$ of a quantale Q, on the other hand, is essentially that given in the completeness part of Theorem 2.4. The frame $\mathrm{fr}(Q)$ consists of all filters T of Q with $\sqcap = \cap$, $T_1 \times T_2 = \{w \mid \exists u \in T_1, v \in T_2.\ u \circ_q v \leq w\}$, and $1 = \{u \mid \mathrm{t}_q \leq u\}$. Note that filters in quantales correspond to theories as defined in the proof of Theorem 2.4. Furthermore $T \models u$ according to 2.2.2–6 iff $u \in T$. Soundness and completeness of the quantale based interpretation then follows simply by observing that$\vdash_{\mathbf{LL}^+} \phi$ iff $\vdash_{\mathbf{LL}^+} \mathrm{t} \rightarrow \phi$ iff $\mathrm{t}_q = \uparrow\uparrow\mathrm{t} = [\![\mathrm{t}]\!] \leq \uparrow\uparrow\phi = [\![\phi]\!]$ where $\uparrow x = \{y \mid x \leq y\}$ is the upper closure of x.

Quantale-based models are easily adapted to $\mathbf{R}^+$ by assuming that Q is distributive as a lattice and satisfies $x \leq x \cdot x$ (c.f. [10]).

3 Synchronous processes as models, I

In this section we give two examples of models based on a fragment of Milner's SCCS [19] under simulation and bisimulation equivalence (c.f. Hennessy and Milner [15]). The fragment involved contains synchronous parallel composition ($\times$) together with choice ($+$), prefixing ($a.\bot$) and a unit process (1). Terms $p \in P^+$ in this fragment are given by the following abstract syntax:

$$p ::= 1 \mid a.p \mid p + p \mid p \times p$$

where a ranges over a set L of *labels* with a binary operation $\cdot$ of label multiplication defined on it. Various assumptions may be made on the properties of the label structure $(L, \cdot)$. Here we assume it to form a commutative monoid with unit e; later, as in [19], we assume also inverses such that it forms an abelian group. The operational semantics of process terms is given by the transition relation $\stackrel{a}{\rightarrow}$ determined by the following axioms and rules:

$$1 \stackrel{e}{\rightarrow} 1 \qquad\qquad a.p \stackrel{a}{\rightarrow} p$$

$$\frac{p \stackrel{a}{\rightarrow} p'}{p + q \stackrel{a}{\rightarrow} p'} \qquad \frac{q \stackrel{a}{\rightarrow} p'}{p + q \stackrel{a}{\rightarrow} p'} \qquad \frac{p \stackrel{a}{\rightarrow} p' \quad q \stackrel{b}{\rightarrow} q'}{p \times q \stackrel{a \cdot b}{\rightarrow} p' \times q'}$$

Models are constructed from terms by quotienting under suitable behavioral congruence relations.

Definition 3.1 (Simulation, Bisimulation) A binary relation R on process terms is a *simulation* [1], if pRq implies

1. whenever $q \xrightarrow{a} q'$ then $p \xrightarrow{a} p'$ and $p'Rq'$ for some term p'.

If whenever pRq then (1) holds and in addition its converse

2. whenever $p \xrightarrow{a} p'$ then $q \xrightarrow{a} q'$ and $p'Rq'$ for some term q',

then R is a *bisimulation*. If there is a simulation (bisimulation) R such that pRq then p *simulates* q, $p \sqsubseteq_s q$ (p and q are *bisimulation equivalent*, $p \simeq_b q$). If $p \sqsubseteq_s q$ and $q \sqsubseteq_s p$ then p and q are *simulation equivalent*, $p \simeq_s q$.

It is well known that bisimulation equivalence is strictly finer than simulation equivalence [15]. It is not difficult to verify that $\sqsubseteq_s$ is a precongruence and $\simeq_b$ a congruence with respect to the operations on terms. For $\simeq$ one of $\simeq_s$, $\simeq_b$ we can then form the quotient structure $P^+/\simeq$ in the obvious way by letting

$$\begin{aligned} [p]_\simeq \sqcap [q]_\simeq &= [p+q]_\simeq \\ [p]_\simeq \times [q]_\simeq &= [p \times q]_\simeq \\ 1 &= [1]_\simeq \end{aligned}$$

Theorem 3.2 *1. $P^+/\simeq_b$ is a frame.*

2. $P^+/\simeq_s$ is an $\mathbf{R}^+$-frame when label multiplication is idempotent.

PROOF: Most of the checks involved are standard. The only exceptions are semi-idempotency and distributivity for $P^+/\simeq_s$. For semi-idempotency it suffices to show that $\{(p \times p, p) \mid p \in P^+\}$ is a simulation: If $p \xrightarrow{a} p'$ then $p \times p \xrightarrow{a} p' \times p'$ by idempotency of label multiplication.

For distributivity note first that $\sqsubseteq_s$ coincides with the induced semilattice ordering, for $p + q \sqsubseteq_s p$ holds always, and $p \sqsubseteq_s p + q$ iff $p \sqsubseteq_s q$. Let then $\mathrm{init}(p) = \{a \mid \exists p'.p \xrightarrow{a} p'\}$ and $p/a = \{p' \mid p \xrightarrow{a} p'\}$. Assume that $p + q \sqsubseteq_s r$, $p \not\sqsubseteq_s r$ and $q \not\sqsubseteq_s r$. Then

1. either $\mathrm{init}(r) \not\subseteq \mathrm{init}(p)$ or $\mathrm{init}(r) \subseteq \mathrm{init}(p)$ and there is some $a \in \mathrm{init}(r)$ and $r' \in r/a$ such that for all $p' \in p/a$, $p' \not\sqsubseteq_s r'$, and

2. the same for q.

Then $\mathrm{init}(r) \cap \mathrm{init}(p) \neq \emptyset$ and $\mathrm{init}(r) \cap \mathrm{init}(q) \neq \emptyset$, for $\mathrm{init}(r) \neq \emptyset$ and if for instance $\mathrm{init}(r) \cap \mathrm{init}(p) = \emptyset$ then $q \sqsubseteq_s r$. By the semilattice properties of $+$ we can use the $\sum$-notation for finite, nonempty sums. Let now for $a \in \mathrm{init}(p) \cap \mathrm{init}(r)$

$$p_a = \sum \{a.r' \mid r' \in r/a \text{ and for some } p'' \in p/a,\ p'' \sqsubseteq_s r'\}$$

[1] In fact, according to [15] a *reverse* simulation.

and then $p' = \sum_{a \in \mathrm{init}(p) \cap \mathrm{init}(r)} p_a$. Define q' similarly. Clearly the sums involved are finite. Furthermore assume that for all $a \in \mathrm{init}(p) \cap \mathrm{init}(r)$ and for all $r' \in r/a$ there is no $p'' \in p/a$ such that $p'' \sqsubseteq_s r'$. Then, as $p + q \sqsubseteq_s r$, $\mathrm{init}(p) \cap \mathrm{init}(r) \subseteq \mathrm{init}(q)$ and for all $r' \in r/a$ there is some $q' \in q/a$ s.t. $q' \sqsubseteq_s r'$. Moreover, whenever $a \in \mathrm{init}(r) \setminus \mathrm{init}(p)$, $a \in \mathrm{init}(q)$ and the same holds, as $p + q \sqsubseteq_s r$. But then $q \sqsubseteq_s r$—a contradiction. Hence the sums are also nonempty, and p', q' are well-defined. Clearly $p \sqsubseteq_s p'$ and $q \sqsubseteq_s q'$. Also $p' + q' \simeq_s r$, for if $r \xrightarrow{a} r'$ then $p' + q' \xrightarrow{a} r'$ and if $p' + q' \xrightarrow{a} r'$ then $r \xrightarrow{a} r'$. □

In the context of process algebra the assumption of idempotency of label multiplication of Theorem 3.2.2 is often realistic: It is appropriate for instance for multiway synchronisation.

The frame structures of $P^+/\simeq_b$ and $P^+/\simeq_s$ gives rise to natural interpretations of positive formulas as in section 2. Given a valuation V into any of those frames these interpretations induce corresponding interpretations directly on the process terms themselves, by

$$p \models \phi \text{ iff } [p]_{\simeq} \models \phi \tag{9}$$

where $\simeq$ is either $\simeq_b$ or $\simeq_s$. An important issue is if modalities can be added to account for dynamic behaviour in the style of Hennessy-Milner logic [15]. Basic as it is to our semantical framework it is essential that any such extension does not violate the filter property. For the case of simulation we can add modalities $[a]\phi$ with the interpretation:

$$p \models [a]\phi \text{ iff for all } p' \text{ such that } p \xrightarrow{a} p',\ p' \models \phi.$$

In order to use (9) to extend satisfaction to the quotient structure $P^+/\simeq_s$ we must first of all make sure that if $p \simeq_s q$ and $p \models \phi$ then $q \models \phi$ too, where ϕ may involve modalities $[a]$. In fact it turns out to be easier to check the filter property directly. That is, if $p \simeq_s p + q$ (or, equivalently, $p \sqsubseteq_s q$) and $p \models \phi$ then $q \models \phi$ too, and if $p, q \models \phi$ then $p + q \models \phi$.

For bisimulation we can with a little care add also the dual operator $\mathord{<}a\mathord{>}$ with the interpretation

$$p \models \mathord{<}a\mathord{>}\phi \text{ iff for some } p',\ p \xrightarrow{a} p' \text{ and } p' \models \phi.$$

Let a *restricted formula* be any formula ϕ with the property that all occurrences of $\mathord{<}a\mathord{>}$ in ϕ is within the scope of some $[a']$ in ϕ. Here the filter property is checked in two steps: First we show for unrestricted ϕ that if $p \simeq_b q$ and $p \models \phi$ then $q \models \phi$ too. Secondly we show the filter property for restricted ϕ only: If $p \simeq_b p + q$ and $p \models \phi$ then also $q \models \phi$, and if $p, q \models \phi$ then $p + q \models \phi$. The detailed checks for both simulation and bisimulation are straightforward and left to the reader.

4 Synchronous processes as models, II

The connection to linear and relevant logics established by results such as Theorem 3.2 is a very weak one: They only establish soundness of the induced interpretations. In this section we introduce an example for which completeness can be established too. Thus this gives a technically precise sense in which linear logic is *exactly* the logic of static process structure.

The operational setting is a variation on that of the previous section. Instead of the sum-operator $+$ we allow the formation of a finite set P of process terms as a process term itself. The intended meaning of set formation is as an internal, or uncontrollable choice operator in contrast to the controllable choice involved in $+$, and we derive the deadlock constant 0 by $0 \triangleq \emptyset$ and the binary internal choice operator $\oplus$ by $p \oplus q \triangleq \{p, q\}$ (c.f. Hennessy [14]). Concerning the label structure we adopt from this point onwards the assumption of SCCS that $(L, \cdot)$ forms an abelian group with unit e and a^{-1} the inverse of a. The set of process terms thus obtained is denoted by $P^{\oplus}$. A structured operational semantics in the style of CCS and SCCS can be given (c.f. Dam [8]). Here, however, we prefer a style akin to that of Hennessy and Plotkin [16]. The relations p may Λ and p must Λ where Λ is a finite and nonempty set of labels, and the successor operations p after a, are defined inductively as follows:

may:

$$\frac{e \in \Lambda}{1 \text{ may } \Lambda} \qquad \frac{a \in \Lambda}{a.p \text{ may } \Lambda}$$

$$\frac{p \text{ may } \Lambda \quad p \in P}{P \text{ may } \Lambda} \qquad \frac{p \text{ may } \Lambda_1 \qquad q \text{ may } \Lambda_2}{p \times q \text{ may } \{a \cdot b \mid a \in \Lambda_1, b \in \Lambda_2\}}$$

must:

$$\frac{e \in \Lambda}{1 \text{ must } \Lambda} \qquad \frac{a \in \Lambda}{a.p \text{ must } \Lambda}$$

$$\frac{p_1 \text{ must } \Lambda_1 \quad \cdots \quad p_n \text{ must } \Lambda_n}{\{p_1, \ldots, p_n\} \text{ must } \Lambda_1 \cup \cdots \cup \Lambda_n} \qquad \frac{p \text{ must } \Lambda_1 \qquad q \text{ must } \Lambda_2}{p \times q \text{ must } \{a \cdot b \mid a \in \Lambda_1, b \in \Lambda_2\}}$$

after:

$$1 \text{ after } a = \begin{cases} \{1\} & \text{if } a = e \\ \emptyset & \text{otherwise} \end{cases}$$

$$a.p \text{ after } b = \begin{cases} \{p\} & \text{if } a = b \\ \emptyset & \text{otherwise} \end{cases}$$

$$P \text{ after } a = \bigcup_{p \in P} p \text{ after } a$$

$$p \times q \text{ after } a = \bigcup_{(a_1, a_2): a_1 \cdot a_2 = a} \{p' \times q' \mid p' \in p \text{ after } a_1, q' \in q \text{ after } a_2\}$$

For notational convenience we derive the relation "can" by p can a iff p may $\{a\}$ and the predicate "live" by p live iff p must Λ for some (nonempty) set Λ. The following properties of the basic operational notions are easily established.

Proposition 4.1 *1. p may Λ iff p can a for some $a \in \Lambda$,*

2. p can a iff p after $a \neq \emptyset$,

3. p must Λ iff p live and $\{a \mid p \text{ can } a\} \subseteq \Lambda$.

4. If (p after a) live for some $a \in L$ then p live

PROOF: By structural induction. □

Two behavioral preorders on processes are considered. The first, $\sqsubseteq_1$, is a safety preorder, and the second, $\sqsubseteq_2$, is a liveness preorder. The essential difference between the two is the way they treat the deadlock constant 0. The safety preorder is inverse language containment. It ignores the potentiality for deadlock thus identifying the process terms p and $p \oplus 0$. The liveness preorder, on the other hand, views deadlock as catastrophic, identifying 0 and $p \oplus 0$.

Definition 4.2 (Behavioral Preorders $\sqsubseteq_1$ and $\sqsubseteq_2$)

1. The preorder $\sqsubseteq_1$ on process terms is the largest (under containment) for which $p \sqsubseteq_1 q$ implies

   a. for all labels a, if q can a then p can a and q after $a \sqsubseteq_1 p$ after a.

2. The preorder $\sqsubseteq_2$ is the largest for which $p \sqsubseteq_2 q$ implies

   a. for all Λ, if p must Λ then q must Λ, and

   b. for all a, if p live and q can a then p can a and p after $a \sqsubseteq_2 q$ after a.

3. For $i \in \{1, 2\}$, $p \simeq_i q$ iff $p \sqsubseteq_i q$ and $q \sqsubseteq_i p$.

Both preorders can be characterised as testing preorders along the lines of De Nicola and Hennessy [21]. Interpret 0 as the divergent process, usually denoted Ω, and $\oplus$ as the CCS internal choice operator derived by $p \oplus q = \tau.p + \tau.q$. With this interpretation, $\sqsubseteq_1$ can be seen to coincide with the inverse of the "may"-preorder of [21] and $\sqsubseteq_2$ with the "must"-preorder (Dam [8]). Relating to the equivalences and preorders of Section 3 it is well known that in general $\simeq_b$ is strictly finer than both $\simeq_1$ and $\simeq_2$, while $\sqsubseteq_s$ is strictly finer than $\sqsubseteq_1$, and $\simeq_2$ and $\simeq_s$ are incomparable (c.f. [21]). An alternative interpretation is to view $\oplus$ as a general choice operator such as the CCS +, and the preorders $\sqsubseteq_i$ as trace preorders. Note that for the present restricted language conditions 4.2.2.a and b can be replaced by the single condition

c. if p live then

i. q live, and

ii. for all a, if q can a then p can a and p after $a \sqsubseteq_2 q$ after a.

This follows from Proposition 4.1.3. It is not hard to verify that both $\sqsubseteq_1$ and $\sqsubseteq_2$ are precongruences with respect to the operations on terms, and the quotient structures $P^{\oplus}/\simeq_i$ are then formed as in section 3 by associating to $\sqcap$ the internal choice operator $\oplus$.

Theorem 4.3 *For $i \in \{1, 2\}$, $P^{\oplus}/\simeq_i$ is a frame.*

PROOF: A consequence of the Algebraic Characterisation Theorem 5.6 below. $\square$

5 Synchronous algebras

In this section we extend the notion of frame to account more fully for the static and dynamic behaviour of processes, and arrive at the following equational presentation of processes:

Definition 5.1 (Synchronous Algebras) A *synchronous algebra* (over a given label group L) is a structure $A = (S, \sqcap, 0, \times, 1, \tilde{\cdot})$ where

1. $\tilde{\cdot}$ is a group homomorphism which to each $a \in L$ associates a unary operator $\tilde{a}.\bot$ on S,
2. $(S, \sqcap, \times, 1)$ is a frame, and
3. the following equations hold for all $x, y \in S$ and labels $a, b \in L$:

   a. $x \times 0 = 0$

   b. $\tilde{a}.(x \sqcap y) = (\tilde{a}.x) \sqcap (\tilde{a}.y)$

   c. $(\tilde{a}.x) \times (\tilde{b}.y) = (\widetilde{a \cdot b}).(x \times y)$

   d. $\tilde{e}.1 = 1$

If in addition 0 is greatest with respect to the induced semilattice ordering $\leq$ then A is a *safety*, or *type 1 synchronous algebra*, and if 0 is least with respect to $\leq$ then A is a *liveness*, or *type 2 synchronous algebra.*

Thus safety and liveness algebras are only distinguished on the way they treat 0. The homomorphism property of $\tilde{\cdot}$ ensures that the operators $\tilde{a}$ are equipped with an abelian group structure reflecting that of L: using the same notation for the operations in both groups, $\widetilde{a \cdot b} = \tilde{a} \cdot \tilde{b}$ and $\widetilde{a^{-1}} = \tilde{a}^{-1}$. For our purpose it is harmless to identify the label a with the operator $\tilde{a}$, thus generally writing $a.x$ in place of $\tilde{a}.x$.

5.1 The Initial Safety and Liveness Algebras

It is not hard to verify that $P^{\oplus}/\simeq_i$ forms a type i synchronous algebra for both $i=1$ and $i=2$. We go on to show that safety algebras characterise processes under $\sqsubseteq_1$, and that liveness algebras similarly characterise processes under $\sqsubseteq_2$. First representation theorems for the initial algebras are proved. These are used in section 5.2 to provide fully abstract semantics for processes. In view of the uncontrollable nature of $\oplus$ it is natural to expect members of the initial algebras to be represented as appropriately closed sets of strings of labels.

Definition 5.2 (Paths, Normal Paths) Assume that L and $\{0,1\}$ are disjoint.

1. A *path* σ is a member of $L^* \cdot \{0,1\}$. A path σ is *normal* if $e1$ is not a suffix of σ.

2. If $\sigma = \alpha j$, $\alpha \in L^*$ and $j \in \{0,1\}$, then $\mathrm{pre}(\sigma) = \alpha$ and $\mathrm{suf}(\sigma) = j$.

3. Normal paths are ordered by $\sigma_1 \leq \sigma_2$ iff either

   a. $\sigma_1 = \sigma_2$,

   b. $\mathrm{suf}(\sigma_1) = \mathrm{suf}(\sigma_2) = 0$ and $\mathrm{pre}(\sigma_1)$ is a prefix of $\mathrm{pre}(\sigma_2)$, or

   c. $\mathrm{suf}(\sigma_1) = 0$, $\mathrm{suf}(\sigma_2) = 1$, and $\mathrm{pre}(\sigma_1)$ is a prefix of $\mathrm{pre}(\sigma_2)(e^n)$ for some $n \geq 0$.

A set of paths Σ is *normal* if all $\sigma \in \Sigma$ are normal. Below Σ is assumed to range over normal sets. For the initial safety algebra elements are represented by downwards closed normal sets, and for the initial liveness algebra by upwards closed normal sets. A set Σ is *downwards* or *1-closed* if whenever $\sigma \in \Sigma$ and $\sigma' \leq \sigma$ then $\sigma' \in \Sigma$. Dually Σ is *upwards* or *2-closed* if whenever $\sigma \in \Sigma$ and $\sigma \leq \sigma'$ then $\sigma' \in \Sigma$. For $i \in \{1,2\}$ the i-closure of a set Σ is denoted $\mathrm{cl}_i(\Sigma)$. If $\Sigma = \mathrm{cl}_i(\Sigma')$ for some finite set Σ' then Σ is *i-finitely generated* (i-f.g.). If Σ is i-f.g. then there is a least set Σ' generating Σ. The representations of the initial algebras are built using nonempty, closed, and f.g. sets Σ.

Next the operations on paths and normal sets are defined. Path prefixing is defined by $a.\sigma = a\sigma$ whenever either $a \neq e$ or $\sigma \neq 1$, and $e.1 = 1$. Multiplication $\times$ of paths is defined inductively by letting 0 be zero and 1 be unit for $\times$, and then $a_1\sigma_1 \times a_2\sigma_2 = (a_1 \cdot a_2).(\sigma_1 \times \sigma_2)$. The constants and operations on sets are given by

$$0_1 = \{0\},\ 0_2 = \{\sigma \mid \sigma \text{ a normal path}\}$$

$$1_1 = \{1\} \cup \{e^n 0 \mid n \in \omega\},\ 1_2 = \{1\}$$

$$a.(\Sigma)_1 = \{a.\sigma \mid \sigma \in \Sigma\} \cup \{0\},\ a.(\Sigma)_2 = \{a.\sigma \mid \sigma \in \Sigma\}$$

$$\Sigma_1 \oplus_i \Sigma_2 = \Sigma_1 \cup \Sigma_2$$

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}$$

Let then D_i, $i \in \{1,2\}$, be the algebra obtained by taking the set of all i-f.g., i-closed and nonempty normal sets Σ together with the constants and operations as just defined. Note that the induced semilattice ordering is $\supseteq$ for both $i = 1$ and $i = 2$. In the safety case this corresponds to the converse of the well-known Hoare-ordering: $\Sigma_1 \supseteq \Sigma_2$ iff for all $\sigma_2 \in \Sigma_2$ there is a $\sigma_1 \in \Sigma_1$ such that $\sigma_2 \leq \sigma_1$; and in the liveness case to the Smyth-ordering: $\Sigma_1 \supseteq \Sigma_2$ iff for all $\sigma_2 \in \Sigma_2$ there is a $\sigma_1 \in \Sigma_1$ such that $\sigma_1 \leq \sigma_2$.

Theorem 5.3 (Representation of Initial Algebras) *For $i \in \{1,2\}$, D_i is (up to isomorphism) the initial type i synchronous algebra.*

PROOF: It is very easy to verify for both $i = 1$ and $i = 2$ that indeed D_i is a type i synchronous algebra. To prove the result we then need for every type i synchronous algebra A to establish a unique homomorphism $f : D_i \to A$.

First, let gen_i denote the operator that given each i-closed set Σ gives its least generating set. The following equations hold:

1. $a.(\Sigma)_i = \text{cl}_i\{a.\sigma \mid \sigma \in \text{gen}_i(\Sigma)\}$
2. $\Sigma_1 \oplus_i \Sigma_2 = \text{cl}_i(\text{gen}_i(\Sigma_1) \cup \text{gen}_i(\Sigma_2))$
3. $\Sigma_1 \times \Sigma_2 = \text{cl}_i\{\sigma_1 \times \sigma_2 \mid \sigma_1 \in \text{gen}_i(\Sigma_1), \sigma_2 \in \text{gen}_i(\Sigma_2)\}$

Note that any map $f : D_i \to A$ determines a map $f^\dagger$ from finite, nonempty sets of normal paths to A, defined by

$$\begin{aligned} f^\dagger(\{\sigma_1, \ldots, \sigma_n\}) &= f(\text{cl}_i\{\sigma_1, \ldots, \sigma_n\}) \\ &= f(\text{cl}_i\{\sigma_1\} \oplus_i \cdots \oplus_i \text{cl}_i\{\sigma_n\}) \end{aligned}$$

for $n \geq 1$. Further, f is a homomorphism iff $f^\dagger$ satisfies

i. $f^\dagger\{\sigma_1, \ldots, \sigma_n\} = f^\dagger\{\sigma_1\} \oplus_A \cdots \oplus_A f^\dagger\{\sigma_n\}$, $n \geq 1$,

ii. $f^\dagger\{0\} = 0_A$,

iii. $f^\dagger\{1\} = 1_A$,

iv. $f^\dagger\{a\sigma\} = a.(f^\dagger\{\sigma\})_A$,

and any such $f^\dagger$ determines f. The only-if direction is straightforward, and clearly conditions i.–iv. defines $f^\dagger$, so if f is a homomorphism it is also unique. It remains to check existence. Note that $f^\dagger$ has the properties

a. $f^\dagger\{a.\sigma\} = a.(f^\dagger\{\sigma\})_A$,

b. $f^\dagger\{\Sigma\} = \sum_A\{f^\dagger\{\sigma\} \mid \sigma \in \Sigma\}$, for Σ finite,

c. $f^\dagger\{\sigma_1 \cdot \sigma_2\} = f^\dagger\{\sigma_1\} \times_A f^\dagger\{\sigma_2\}$.

In b. $\sum$ denotes the finite internal sum operator. There is now little difficulty in verifying the homomorphism properties of f. First $f(0_i) = f^\dagger\{0\} = 0_A$, and $f(1_i) = f^\dagger\{1\} = 1_A$. Next

$$\begin{aligned} f(a.(\Sigma)_i) &= f^\dagger(\mathrm{gen}_i(a.(\Sigma)_i)) \\ &= f^\dagger\{a.\sigma \mid \sigma \in \mathrm{gen}_i(\Sigma)\} \quad \text{(by 1.)} \\ &= \textstyle\sum_A\{f^\dagger\{a.\sigma\} \mid \sigma \in \mathrm{gen}_i(\Sigma)\} \quad \text{(by b.)} \\ &= \textstyle\sum_A\{a.(f^\dagger\{\sigma\})_A \mid \sigma \in \mathrm{gen}_i(\Sigma)\} \quad \text{(by a.)} \\ &= a.(\textstyle\sum_A\{f^\dagger\{\sigma\} \mid \sigma \in \mathrm{gen}_i(\Sigma)\})_A \quad \text{(by equational reasoning)} \\ &= a.(f^\dagger(\mathrm{gen}_i(\Sigma)))_A \quad \text{(by b.)} \\ &= a.(f(\Sigma))_A \end{aligned}$$

For the internal sum operator:

$$\begin{aligned} f(\Sigma_1 \oplus_i \Sigma_2) &= f^\dagger(\mathrm{gen}_i(\Sigma_1 \oplus_i \Sigma_2)) \\ &= f^\dagger(\mathrm{gen}_i(\Sigma_1) \cup \mathrm{gen}_i(\Sigma_2)) \quad \text{(by 2.)} \\ &= f^\dagger(\mathrm{gen}_i(\Sigma_1)) \oplus_A f^\dagger(\mathrm{gen}_i(\Sigma_2)) \quad \text{(by b.)} \\ &= f(\Sigma_1) \oplus_A f(\Sigma_2) \end{aligned}$$

Finally for parallel composition:

$$\begin{aligned} f(\Sigma_1 \times \Sigma_2) &= f^\dagger(\mathrm{gen}_i(\Sigma_1 \times \Sigma_2)) \\ &= f^\dagger(\mathrm{gen}_i(\Sigma_1) \times \mathrm{gen}_i(\Sigma_2)) \quad \text{(by 3.)} \\ &= \textstyle\sum_A\{f^\dagger\{\sigma_1 \cdot \sigma_2\} \mid \sigma_1 \in \mathrm{gen}_i(\Sigma_1), \sigma_2 \in \mathrm{gen}_i(\Sigma_2)\} \quad \text{(by b.)} \\ &= \textstyle\sum_A\{f^\dagger\{\sigma_1\} \times_A f^\dagger\{\sigma_2\} \mid \sigma_1 \in \mathrm{gen}_i(\Sigma_1), \sigma_2 \in \mathrm{gen}_i(\Sigma_2)\} \quad \text{(by c.)} \\ &= (f^\dagger(\mathrm{gen}_i(\Sigma_1))) \times_A (f^\dagger(\mathrm{gen}_i(\Sigma_2))) \quad \text{(by equational reasoning)} \\ &= f(\Sigma_1) \times_A f(\Sigma_2). \end{aligned}$$

The check that f is monotone is straightforward. We have thus established the homomorphism property of f, and the proof is complete. □

5.2 The Algebraic Characterisation Theorem

As $P^\oplus$, up to the use of sets in term formation, is a term algebra there are unique homomorphisms $[\![\cdot]\!]_i$ from $P^\oplus$ to D_i for $i \in \{1, 2\}$. These homomorphisms are used to produce isomorphisms between $P^\oplus / \simeq_i$ and D_i thus establishing the Algebraic Characterisation Theorem below. For this purpose it suffices to show that $[\![\cdot]\!]_i$ is *fully abstract*, meaning that $p \sqsubseteq_i q$ iff $[\![p]\!]_i \supseteq [\![q]\!]_i$. To prove full abstraction the operational structure of processes is mimicked using the representations D_i. For a set Σ define

1. Σ may Λ iff $\sigma = a.\sigma'$ for some $\sigma \in \Sigma$, $a \in \Lambda$, and path σ',

2. Σ must Λ iff for all $\sigma \in \Sigma$ there is some $a \in \Lambda$ and path σ' such that $\sigma = a.\sigma'$,

3. σ after $a = \{\sigma' \mid \exists \sigma \in \Sigma.\ \sigma = a.\sigma'\}$.

This operational structure can be characterised in purely algebraic terms. In an arbitrary synchronous algebra the operation "after" can be taken to satisfy

$$(x \leq a.(x \text{ after } a) \text{ and } (x \text{ after } a) \leq y) \text{ iff } x \leq a.y. \tag{10}$$

In algebras with arbitrary infima the "after"-operation can be defined by

$$x \text{ after } a = \sum \{z \mid x \leq a.z\}. \tag{11}$$

It is not hard to verify that (10) is satisfied with "after" defined in this way. The relation "may" can then be characterised by the condition

$$x \text{ may } \Lambda \text{ iff } x \leq a.(x \text{ after } a) \text{ for some } a \in \Lambda, \tag{12}$$

and "must" can be characterised by

$$x \text{ must } \Lambda \text{ iff } \sum \{a.(x \text{ after } a) \mid a \in \Lambda\} \leq x. \tag{13}$$

It is an easy exercise to verify that the relations "may", "must", and "after" as defined by 1.–3. indeed satisfies (10)–(13). The following lemma relates the operational structure of terms and that of their representations.

Lemma 5.4 *1.* p may Λ *iff* $[\![p]\!]_1$ may Λ

2. If p can a *then* $[\![p \text{ after } a]\!]_1 = [\![p]\!]_1$ after a

3. p must Λ *iff* $[\![p]\!]_2$ must Λ

4. If p live *and* p can a *then* $[\![p \text{ after } a]\!]_2 = [\![p]\!]_2$ after a

PROOF: All four statements are proved by an essentially straightforward structural induction. For instance for 4 assume that p can a and that p live. Thus $p \neq 1$. For the remaining cases we calculate:

$$\begin{aligned} [\![1 \text{ after } a]\!]_2 &= [\![1]\!]_2 && \text{as } p \text{ can } a \text{ iff } a = 1 \\ &= [\![1]\!]_2 \text{ after } a \end{aligned}$$

$$\begin{aligned} [\![b.p \text{ after } a]\!]_2 &= [\![p]\!]_2 && \text{as } b.p \text{ can } a \text{ iff } a = b \\ &= [\![b.p]\!]_2 \text{ after } a \end{aligned}$$

$$\begin{aligned} [\![p \oplus q \text{ after } a]\!]_2 &= [\![(p \text{ after } a) \cup (q \text{ after } a)]\!]_2 \\ &= [\![p \text{ after } a]\!]_2 \cup [\![q \text{ after } a]\!]_2 \end{aligned}$$

$$
\begin{array}{rl}
\llbracket p \text{ after } a \rrbracket_2 & \\
= & \llbracket \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{p'_1 \times p'_2 \mid p'_1 \in p_1 \text{ after } a_1, p'_2 \in p_2 \text{ after } a_2\} \rrbracket_2 \\
= & \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{\llbracket p'_1 \times p'_2 \rrbracket_2 \mid p'_1 \in p_1 \text{ after } a_1, p'_2 \in p_2 \text{ after } a_2\}_2 \\
= & \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{\llbracket p'_1 \rrbracket_2 \times \llbracket p'_2 \rrbracket_2 \mid p'_1 \in p_1 \text{ after } a_1, p'_2 \in p_2 \text{ after } a_2\}_2 \\
= & \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{\llbracket p_1 \text{ after } a_1 \rrbracket_2 \times \llbracket p_2 \text{ after } a_2 \rrbracket_2\} \\
= & \cup_{(a_1,a_2):a_1 \cdot a_2 = a} \{\llbracket p_1 \rrbracket_2 \text{ after } a_1 \times \llbracket p_2 \rrbracket_2 \text{ after } a_2\} \\
= & \llbracket p \rrbracket_2 \text{ after } a
\end{array}
$$

□

It thus remains to prove that the behavioral preorders $\sqsubseteq_i$ on terms induce the appropriate ordering $\subseteq$ on the representations. This is done in two steps, using the "may", "must", and "after" relations on D_i to induce orderings $\sqsubseteq_i$ on D_i as in Definition 4.2.

Lemma 5.5 (Full Abstraction) *For $i \in \{1,2\}$, $p \sqsubseteq_i q$ iff $\llbracket p \rrbracket_i \supseteq \llbracket q \rrbracket_i$,*

PROOF: Note first that $p \sqsubseteq_i q$ iff $\llbracket p \rrbracket_i \sqsubseteq_i \llbracket q \rrbracket_i$ by Lemma 5.4. It thus remains to show that $\Sigma_1 \sqsubseteq_i \Sigma_2$ iff $\Sigma_1 \supseteq \Sigma_2$ where Σ_1, Σ_2 are i-closed. The proofs for $i = 1$ and $i = 2$ are very similar and we prove here only the case for $i = 2$. So suppose $\Sigma_1 \sqsubseteq_2 \Sigma_2$ and that $\sigma_2 \in \Sigma_2$. If $\sigma_2 = 0$ then Σ_2 is not live so neither is Σ_1 whence $\sigma_2 \in \Sigma_1$. Suppose $\sigma_2 = 1$. If $1 \notin \Sigma_1$ then either Σ_1 can e fails or else there is a maximal n such that (Σ_1 after e^n) can e. Here the "after"-operation is extended to finite strings in the obvious way by Σ after $(a_1 \cdots a_n) = ((\cdots(\Sigma \text{ after } a_1)\cdots) \text{ after } a_n)$. The first case ($\Sigma_1$ can e fails) leads to a contradiction whether Σ_1 live or not. For the second case, if for some $m \leq n$, Σ_1 after e^m is not live then $1 \in \Sigma_1$ as Σ_1 is 2-closed. Otherwise Σ_1 after $e^{n+1} \sqsubseteq_2 \Sigma_2$ after e^{n+1} but (Σ_2 after e^{n+1}) can e which fails for Σ_1, a contradiction. Suppose finally that $\sigma_2 = a\sigma'_2$. If Σ_1 is not live then $\sigma_2 \in \Sigma_1$. If Σ_1 is live, as Σ_2 can a then Σ_1 can a and $\sigma'_2 \in \Sigma_1$ after a by the induction hypothesis. But then $\sigma_2 \in \Sigma_1$ as desired. The converse implication is a straightforward check that the conditions of Def. 4.2.2 are satisfied. □

Corollary 5.6 (Algebraic Characterisation Theorem) *For $i \in \{1,2\}$, $P^{\oplus}/\simeq_i$ is (up to isomorphism) the initial type i synchronous algebra with $\sqsubseteq_i / \simeq_i$ the induced ordering.*

PROOF: By the Full Abstraction Lemma. □

6 A modal linear logic of processes

In this section the language of positive formulas is extended by indexed future modalities $|a>$ and past modalities $<a|$. The interpretations of these connectives are associated to the prefixing operators in a way mirroring the way the interpretations of implication and fusion are associated to parallel composition. Our choice of connectives allows a simple and elegant logical account of the structure

of synchronous algebras, in particular the interplay between the static operations of multiplication and internal choice, and prefixing, expressing the dynamic capabilities of processes.

6.1 Semantics

A synchronous algebra A is extended to a *model* $M = (A, V)$ by, as in Definition 2.1, adjoining a valuation V for which $V(X)$ is a filter in A for each propositional letter X. The relation of satisfaction is then defined by adding to the conditions of section 2 the following two conditions for the modal operators:

$$x \models_M |a{>}\phi \quad \text{iff} \quad \text{there is a } y \in S_M \text{ such that } a.y \leq x \text{ and } y \models_M \phi, \tag{14}$$

$$x \models_M {<}a|\phi \quad \text{iff} \quad a.x \models_M \phi. \tag{15}$$

Intuitively, $|a{>}$ and ${<}a|$ can be thought of as specialised forwards, respectively backwards nexttime modalities. The reverse modality can alternatively be characterised by the satisfaction condition

$$x \models_M {<}a|\phi \text{ iff there is a } y \in S_M \text{ such that } y \text{ can } a,\ y \text{ after } a \leq x, \text{ and } y \models \phi \tag{16}$$

reflecting (10) of section 5, and the forwards modality can be characterised as a left adjoint for the reverse. More concrete characterisations for the forwards modality with respect to just the initial algebra interpretations are given in section 7 below. These characterisations are important as they provide more concrete intuitions as to the meaning of the forwards modalities than are warranted by just the general algebraically based interpretation of (14). Note that the filter property extends to the full language. This property is needed to establish (16). For the future modalities, $x, y \models |a{>}\phi$ iff there are x', y' such that $a.x' \leq x$, $a.y' \leq y$ and $x', y' \models \phi$ iff there are x', y' such that $a.x' \sqcap a.y' = a.(x' \sqcap y') \leq x \sqcap y$ and $x' \sqcap y' \models \phi$ (by the induction hypothesis) iff $x \sqcap y \models |a{>}\phi$. The past modalities are similar.

6.2 Axiomatisation

To axiomatise validity with respect to the class of all safety and liveness models respectively $\mathbf{LL}^+$ is extended by the following axioms and rules concerning the modal operators.

Axioms	$|a{>}$-$\vee$	$|a{>}(\phi \vee \psi) \rightarrow |a{>}\phi \vee |a{>}\psi$
	${<}a|$-$\wedge$	${<}a|\phi \wedge {<}a|\psi \rightarrow {<}a|(\phi \wedge \psi)$
	$\rightarrow$-${<}a|$-$|a{>}$	$\phi \rightarrow {<}a||a{>}\phi$
	$|a{>}$-${<}a|$-$\rightarrow$	$|a{>}{<}a|\phi \rightarrow \phi$
	a-b-**synchronisation**	${<}a|(|b{>}\phi \rightarrow \psi) \leftrightarrow (\phi \rightarrow {<}a \cdot b|\psi)$
Rules	$|e{>}$-**necessitation**	$\dfrac{\phi}{|e{>}\phi}$

$$\langle e|\text{-necessitation} \qquad \frac{\phi}{\langle e|\phi}$$

$$|a\rangle\text{-monotonicity} \qquad \frac{\phi \rightarrow \psi}{|a\rangle\phi \rightarrow |a\rangle\psi}$$

$$\langle a|\text{-monotonicity} \qquad \frac{\phi \rightarrow \psi}{\langle a|\phi \rightarrow \langle a|\psi}$$

Write $\vdash_{\mathbf{PL}} \phi$ if ϕ is provable in this extension of $\mathbf{LL}^+$. Of the new axioms and rules most are entirely straightforward. The axiom $|a\rangle$-$\vee$ expresses the existential nature of the future modality and similarly the axiom $\langle a|$-$\wedge$ expresses the universal nature of the past modality. The rules express the expected necessitation and monotonicity properties; thus the distributivity of $|a\rangle$ over $\vee$ and $\langle a|$ over $\wedge$ is derivable. The axioms $\rightarrow$-$\langle a|$-$|a\rangle$ and $|a\rangle$-$\langle a|$-$\rightarrow$ are less obvious; they express a degree of duality between the future and past modalities. Finally the axiom a-b-synchronisation is the axiom that captures the dynamic properties of parallel composition. We note a few theorems of **PL** for future reference.

Proposition 6.1 (Theorems of **PL**)

1. $\vdash_{\mathbf{PL}} |a\rangle(\phi \vee \psi) \leftrightarrow |a\rangle\phi \vee |a\rangle\psi$
2. $\vdash_{\mathbf{PL}} \langle a|(\phi \wedge \psi) \leftrightarrow \langle a|\phi \wedge \langle a|\psi$
3. $\vdash_{\mathbf{PL}} |a^{-1} \cdot b\rangle(\phi \rightarrow \psi) \rightarrow (|a\rangle\phi \rightarrow |b\rangle\psi)$

PROOF: For 1 and 2 use $|a\rangle$-$\vee$ and $\langle a|$-$\wedge$ for one direction, and for the other the monotonicity rules together with the axioms for $\wedge$ and $\vee$. The following derivation establishes 3.

1.	$\psi \rightarrow \langle b\|\|b\rangle\psi$	by $\rightarrow$-$\langle b\|$-$\|b\rangle$
2.	$(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \langle b\|\|b\rangle\psi)$	1, by transitivity, detachment
3.	$(\phi \rightarrow \langle b\|\|b\rangle\psi) \rightarrow \langle a^{-1} \cdot b\|(\|a\rangle\phi \rightarrow \|b\rangle\psi)$	by $a^{-1} \cdot b$-a-synchronisation
4.	$(\phi \rightarrow \psi) \rightarrow \langle a^{-1} \cdot b\|(\|a\rangle\phi \rightarrow \|b\rangle\psi)$	2,3, by transitivity, detachment
5.	$\|a^{-1} \cdot b\rangle(\phi \rightarrow \psi) \rightarrow \|a^{-1} \cdot b\rangle\langle a^{-1} \cdot b\|(\|a\rangle\phi \rightarrow \|b\rangle\psi)$	4, by $\|a^{-1} \cdot b\rangle$-monotonicity
6.	$\|a^{-1} \cdot b\rangle(\phi \rightarrow \psi) \rightarrow (\|a\rangle\phi \rightarrow \|b\rangle\psi)$	5, by $\|a^{-1} \cdot b\rangle$-$\langle a^{-1} \cdot b\|$-$\rightarrow$, transitivity, detachment

□

As the satisfaction conditions do not refer to the constant 0 and as in the absence of 0, safety and liveness algebras are each others duals, it is not surprising that soundness for safety algebras entails soundness for liveness algebras as well. For completeness this is slightly more subtle as in this case an interpretation for 0 must be provided.

Theorem 6.2 (Soundness and Completeness, **PL**) *The following statements are equivalent:*

1. $\vdash_{\mathbf{PL}} \phi$,
2. ϕ *is* $\mathcal{M}$*-valid where* $\mathcal{M}$ *is the class of all models based on safety algebras,*
3. ϕ *is* $\mathcal{M}$*-valid where* $\mathcal{M}$ *is the class of all models based on liveness algebras.*

PROOF: The proof extends the corresponding proof for $\mathbf{LL}^+$. Soundness is proved as usual. For instance for $|a>$-$\vee$ assume that $x \models |a>(\phi \vee \psi)$. Then there is an x' such that $x' \models \phi \vee \psi$ and $a.x' \leq x$. If $x' \models \phi$ or $x' \models \psi$ then we are done. Otherwise let $x_1' \sqcap x_2' \leq x'$, $x_1' \models \phi$ and $x_2' \models \psi$. Then $a.x_1' \models |a>\phi$ and $a.x_2' \models |a>\psi$ so $(a.x_1') \sqcap (a.x_2') \models |a>\phi \vee |a>\psi$ and then $x \models |a>\phi \vee |a>\psi$ by the filter property, as $(a.x_1') \sqcap (a.x_2') = a.(x_1' \sqcap x_2') \leq a.x' \leq x$. As another example consider a-b-synchronisation. Suppose $x \models <a|(|b>\phi \rightarrow \psi)$. Then $a.x \models |b>\phi \rightarrow \psi$. Let $y \models \phi$ and we must show $x \times y \models <a \cdot b|\psi$. Now $b.y \models |b>\phi$ so $(a.x) \times (b.y) = (a \cdot b).(x \times y) \models \psi$. Thus $x \times y \models <a \cdot b|\psi$ as desired. Soundness of the converse implication, and of the remaining axioms and rules is established in a similar manner.

Completeness, safety algebras. A canonical model construction is given, based on the completeness proof for positive linear logic, Theorem 2.4. Similar to $\mathbf{LL}^+$-theories, **PL**-*theories* are sets of formulas closed under implications provable in **PL**, and adjunction. Moreover, in the case of safety algebras, **PL**-theories are required to be nonempty. The valuation V and operations $\sqcap$ and $\times$ are unchanged. The constant 0 is the set of all **PL** formulas, and 1 is the set of all **PL** theorems. Finally prefixing is defined by $a.T = \{\phi \mid <a|\phi \in T\}$. It is not hard to check that the constants and operations are well-defined. Clearly 1 and 0 are nonempty **PL**-theories, and by the proof of Theorem 2.4 $\sqcap$ and $\times$ map **PL**-theories to **PL**-theories. To see they also preserve nonemptiness suppose $\phi \in T_1$ and $\psi \in T_2$. Then $\phi \vee \psi \in T_1 \sqcap T_2$. For $\times$ note that

$$\vdash_{\mathbf{LL}^+} \phi \rightarrow (\psi \rightarrow ((\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow \gamma))$$

so that $\psi \rightarrow ((\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow \gamma) \in T_1$ whence $(\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow \gamma \in T_1 \times T_2$. To verify the well-definedness of prefixing suppose $\vdash_{\mathbf{PL}} \phi \rightarrow \psi$ and $\phi \in a.T$. Then $<a|\phi \in T$ so by $<a|$-monotonicity, $<a|\psi \in T$ too. Hence $\psi \in a.T$ as desired. Also if $\phi, \psi \in a.T$ then $<a|\phi, <a|\psi \in T$ so $<a|\phi \wedge <a|\psi \in T$, and then by $<a|$-$\wedge$, $<a|(\phi \wedge \psi) \in T$ as well. Hence $\phi \wedge \psi \in a.T$. For nonemptiness suppose that $\phi \in T$. Then by $\rightarrow$-$<a|$-$|a>$ also $<a||a>\phi \in T$, so $|a>\phi \in a.T$, and we have completed the well-definedness check.

To check that the canonical structure forms a safety algebra we know from the completeness proof for $\mathbf{LL}^+$ that it forms a frame. In addition equations (i)–(iv), Definition 5.1 must be checked. Trivially $0 \times T \subseteq 0$. For the other direction let ϕ be an arbitrary formula. As T is nonempty (!) we can find some $\psi \in T$.

Then $\psi \rightarrow \phi \in 0$ so that $\phi \in 0 \times T$. So it only remains to check the properties relating to prefixing. For equation (ii) we obtain $\phi \in a.(T_1 \sqcap T_2)$ iff $|a{>}\phi \in T_1$ and $|a{>}\phi \in T_2$ (by the above observation) iff $\phi \in (a.T_1) \sqcap (a.T_2)$. For equation (iii) assume first that $\psi \in (a.T_1) \times (b.T_2)$. Then for some $\phi \in b.T_2$, $\phi \rightarrow \psi \in a.T_1$. Then $|a{>}(\phi \rightarrow \psi) \in T_2$. By $|b{>}$-${<}b|$-$\rightarrow$ transitivity and detachment we obtain $\vdash_{\mathbf{PL}} (\phi \rightarrow \psi) \rightarrow (|b{>}{<}b|\phi \rightarrow \psi)$, so $|b{>}{<}b|\phi \rightarrow \psi \in a.T_1$, and then ${<}a|(|b{>}{<}b|\phi \rightarrow \psi) \in T_1$. Then by a-b-synchronisation, ${<}a|\phi \rightarrow {<}a \cdot b|\psi \in T_1$ as well, thus ${<}a \cdot b|\psi \in T_1 \times T_2$. But then $\psi \in (a \cdot b).(T_1 \times T_2)$ as desired. For the converse inclusion, assume that this holds, thus ${<}a \cdot b|\psi \in T_1 \times T_2$. Then for some $\phi \in T_2$ does $\phi \rightarrow {<}a \cdot b|\psi \in T_1$, and then by a-b-synchronisation, ${<}a|(|b{>}\phi \rightarrow \psi) \in T_1$ as well, so that $|b{>}\phi \rightarrow \psi \in a.T_1$. Also $|b{>}\phi \in b.T_2$ as we saw above and thus $\psi \in (a.T_1) \times (b.T_2)$ as needed. Equation (iv) is left as an easy exercise.

Finally we need to check that $\phi \in T$ iff $T \models \phi$. This part of the proof is common to both the safety and the liveness case. The proof is by induction in the structure of ϕ, and for all connectives except the modal ones the proof is identical to the corresponding part in the proof of completeness for $\mathbf{LL}^+$, Theorem 2.4. For the modal connectives:

$\phi = |a{>}\phi'$. If $T \models \phi$ then there is some nonempty $\mathbf{PL}$-theory T' s.t. $T' \models \phi'$ and $a.T' \subseteq T$. By the induction hypothesis $\phi' \in T'$ thus $|a{>}\phi' \in a.T'$ by the above observation and then $|a{>}\phi' \in T$. Conversely, if $|a{>}\phi' \in T$ then $a.\mathrm{th}\{\phi'\} \subseteq T$ where $\mathrm{th}\{\phi'\}$ is the least $\mathbf{PL}$-theory containing the set $\{\phi'\}$. By the induction hypothesis, $\mathrm{th}\{\phi'\} \models \phi'$ so $T \models |a{>}\phi'$.

$\phi = {<}a|\phi'$. If $T \models \phi$ then $a.T \models \phi'$ and by the induction hypothesis, $\phi' \in a.T$ whence ${<}a|\phi' \in T$. Conversely, if ${<}a|\phi' \in T$ then $\phi' \in a.T$ and by the induction hypothesis $a.T \models \phi'$—i.e. $T \models {<}a|\phi'$.

The proof for the safety case is then complete, for if $\not\vdash_{\mathbf{PL}} \phi$ then $\phi \notin 1$, thus $1 \not\models \phi$.

Completeness, liveness algebras. This part of proof is a simple adaptation of the completeness proof for safety algebras. Here we can take $0 = \emptyset$ and proceed as above. It suffices to note that the required properties of 0 holds in this case. □

We can now show **PL** to be a conservative extension of $\mathbf{LL}^+$ by embedding general models as in section 2 into models based on liveness algebras in a way that preserves satisfaction.

Theorem 6.3 **PL** *is a conservative extension of* $\mathbf{LL}^+$.

Proof: If $\not\vdash_{\mathbf{LL}^+} \phi$ for some positive formula ϕ then we find a general model M such that $1_M \not\models \phi$, by 2.4. Moreover, as we noted, M may be assumed to contain an element 0 which is zero for both $\sqcap$ and $\times$. We can turn M into a model M' based on a liveness algebra by defining $a.x = x$ for all $a \in L$. Then it is a simple induction to verify that for all elements x, $x \models_M \phi$ iff $x \models_{M'} \phi$ for all positive formulas ϕ. But then $1_{M'} \not\models \phi$ so $\not\vdash_{\mathbf{PL}} \phi$ by Theorem 6.2, and we are done. □

6.3 Synchronous Quantales

In analogy to the quantale-based interpretation of $\mathbf{LL}^+$ of section 2.3 in this section we develop synchronous quantales as algebraic correlates of **PL**.

Definition 6.4 (Synchronous quantale). A *synchronous quantale* is a structure $(Q, \circ_q, \mathrm{t}_q, |\cdot{>}_q)$ where

1. $|\cdot{>}_q$ is a group homomorphism which to each $a \in L$ associates a unary operator $|a{>}_q$ on Q,
2. $(Q, \circ_q, \mathrm{t}_q)$ is a quantale,
3. $|a{>}_q$ distributes over arbitrary joins, i.e. $|a{>}_q(\bigvee_i u_i) = \bigvee_i \{|a{>}_q u_i\}$,
4. $|a \cdot b{>}_q(u \circ_q v) = (|a{>}_q u) \circ_q (|b{>}_q v)$,
5. $|e{>}_q \mathrm{t}_q = \mathrm{t}_q$.

Reflecting the adjunction of fusion and implication in quantales, in synchronous quantales the reverse modality ${<}a|_q$ can be characterised as a right adjoint for $|a{>}_q$. That is, in analogy with 8, ${<}a|_q$ is a right adjoint for $|a{>}_q$:

$$u \leq {<}a|_q v \text{ iff } |a{>}_q u \leq v, \tag{17}$$

and using infinite joins ${<}a|_q$ can be defined by

$${<}a|_q u = \bigvee \{v \mid |a{>}_q v \leq u\}. \tag{18}$$

The notions of interpretation and validity with respect to synchronous quantales follow those of section 2.3 entirely. For a synchronous algebra A the *filter completion* of A is the synchronous quantale $\mathrm{qu}(A)$ with $\bigvee$, $\circ_q$ and t_q defined as in section 2.3, and $|a{>}_q B = \{x \mid \exists y \in B.\ a.y \leq x\}$. The verification that $\mathrm{qu}(A)$ is indeed a synchronous quantale, and that the relation $x \in B$ satisfies conditions 2.2.2–6 as well as (14) and (15) is left to the reader. Conversely, following the proof of the completeness theorem 6.2, the filter completion $\mathrm{fr}(Q)$ of a synchronous quantale Q comes in two variants, according to whether a safety or a liveness algebra is being constructed. Thus for the safety case $\mathrm{fr}_1(Q)$ consists of all nonempty filters T of Q with $0 = Q$ and $a.T = \{u \mid {<}a|_q u \in T\} = \{u \mid \exists v \in T.\ a.u \leq v\}$. For the liveness case $\mathrm{fr}_2(Q)$ consist of all filters of Q with $0 = \emptyset$ and $a.\bot$ as in $\mathrm{fr}_1(Q)$. In both cases it is easy to check that $\mathrm{fr}_1(Q)$ and $\mathrm{fr}_2(Q)$ are both well-defined, and that $T \models u$ iff $u \in T$. Soundness and completeness with respect to the synchronous quantale interpretation then follows as in section 2.3.

7 The process-based interpretations

While the semantics of formulas of section 6 is given in terms of general synchronous algebras, as in section 3 it is the induced interpretations on the process terms themselves defined by

$$p \models_i \phi \text{ iff } [p]_{\simeq_i} \models \phi$$

that are ultimately of real computational interest. In this section we begin investigating these interpretations further. It is shown, in particular, that these induced interpretations characterise the corresponding behavioral preorders on processes in the sense that $p \sqsubseteq_i q$ iff for all ϕ, if $p \models_i \phi$ then $q \models_i \phi$. For this to make sense we must require that only *closed* formulas, formulas without occurrences of atomic propositions, are considered. This is similar to the situation in e.g. Hennessy-Milner logic. To regain sufficient expressive power we then have to extend the language of positive modal formulas by adding a constant $\underline{0}$ whose interpretation is tied to the process constant 0 just as the interpretation of t is tied to the process constant 1. That is, for general models, $x \models_M \underline{0}$ iff $0 \leq x$. Note that for safety algebras $\underline{0}$ denotes the singleton set $\{0\}$, and for liveness algebras $\underline{0}$ is the extensional truthhood constant. For a discussion of the problems involved in extending the soundness and completeness results of the preceding section to the extended language see Dam [8].

We first consider the interpretation of extended closed formulas in terms of the initial safety and liveness algebras. Note that the satisfaction conditions for conjunction, linear implication, and past modalities are given in purely structural terms, i.e. they do not refer to the ordering $\leq$ corresponding for the initial algebras to the behavioral preorders on terms. Hence no characterisation of the initial algebra interpretations is needed in these cases.

Proposition 7.1 (Initial Safety Algebra Interpretation) *Let* $\Sigma \in D_1$*.*

1. $\Sigma \models \mathrm{t}$ *iff* Σ *may* Λ *implies* $e \in \Lambda$*, and* Σ *may* $\{e\}$ *implies* Σ *after* $e \models \mathrm{t}$*,*
2. $\Sigma \models \phi \circ \psi$ *iff there are* $\Sigma_1, \Sigma_2 \in D_1$ *such that* $\Sigma_1 \models \phi$*,* $\Sigma_2 \models \psi$ *and for all* $\sigma \in \Sigma$ *there are* $\sigma_1 \in \Sigma_1$ *and* $\sigma_2 \in \Sigma_2$ *such that* $\sigma = \sigma_1 \times \sigma_2$*,*
3. $\Sigma \models \phi \vee \psi$ *iff for all* $\sigma \in \Sigma$*,* $cl_1\{\sigma\} \models \phi$ *or* $cl_1\{\sigma\} \models \psi$*,*
4. $\Sigma \models |a{>}\phi$ *iff*
    a. $\Sigma' \models \phi$ *for some* $\Sigma' \in D_1$*,*
    b. Σ *may* Λ *implies* $a \in \Lambda$*, and* Σ *may* $\{a\}$ *implies* Σ *after* $a \models \phi$*,*
5. $\Sigma \models \underline{0}$ *iff there is no* Λ *for which* Σ *may* Λ*.*

PROOF: 1. $\Sigma \models \mathrm{t}$ iff $\Sigma \subseteq \mathrm{cl}_1(1)$ iff for all $\sigma \in \Sigma$, $\sigma \leq 1$, iff Σ may Λ implies $e \in \Lambda$ and Σ may $\{e\}$ implies Σ after $e = \Sigma \models \mathrm{t}$.
2. Immediate by the definitions.
3. Assume $\Sigma \models \phi \vee \psi$ and let $\sigma \in \Sigma$. Then $\Sigma \supseteq \mathrm{cl}_1\{\sigma\}$, so by the filter property also $\mathrm{cl}_1\{\sigma\} \models \phi \vee \psi$. Either $\mathrm{cl}_1\{\sigma\} \models \phi$ or $\mathrm{cl}_1\{\sigma\} \models \psi$ in which case we are done, or there are $\Sigma_1, \Sigma_2 \in D_1$ such that $\Sigma_1 \models \phi$, $\Sigma_2 \models \psi$ and $\Sigma_1 \cup \Sigma_2 \supseteq \mathrm{cl}_1\{\sigma\}$. But $\mathrm{cl}_1\{\sigma\}$ is *coprime* with respect to $\subseteq$, that is, whenever $\Sigma_1 \sqcap \Sigma_2 \subseteq \mathrm{cl}_1\{\sigma\}$ then either $\Sigma_1 \subseteq \mathrm{cl}_1\{\sigma\}$ or $\Sigma_2 \subseteq \mathrm{cl}_1\{\sigma\}$. Hence by the filter property also in this case either $\mathrm{cl}_1\{\sigma\} \models \phi$ or $\mathrm{cl}_1\{\sigma\} \models \psi$. For the converse direction assume that for all $\sigma \in \Sigma$, either $\mathrm{cl}_1\{\sigma\} \models \phi$ or $\mathrm{cl}_1\{\sigma\} \models \psi$. As Σ is generated by a finite set Σ', Σ can be written as a finite union $\cup\{\mathrm{cl}_1(\sigma) \mid \sigma \in \Sigma'\}$. As each $\mathrm{cl}_1\{\sigma\} \models \phi \vee \psi$ by the filter property also $\Sigma \models \phi \vee \psi$.
4. Similar to 1.
5. $\Sigma \models \underline{0}$ iff $\mathrm{cl}_1(0) \supseteq \Sigma$ iff $\Sigma = \{0\}$ iff for no Λ, Σ may Λ. □

Note that 7.1.2 is somewhat unsatisfactory in that references to $\leq$ are hidden in the use of path equality. We return to this issue below. The elements of the form $\mathrm{cl}_i\{\sigma\}$, $i \in \{1, 2\}$, are exactly those elements of D_i that are coprime with respect to $\subseteq$. The statement of Proposition 7.1.3 can consequently be read as

$$\Sigma \models \phi \vee \psi \text{ iff for all coprime } \Sigma' \subseteq \Sigma,\ \Sigma' \models \phi \text{ or } \Sigma' \models \psi.$$

Thus the interpretation of disjunction with respect to the initial safety algebra is seen to be related to the interpretation of disjunction in Beth models for propositional intuitionistic logic, and to that of Allwein and Dunn's recent Kripke models for linear logic [4]. Similar comments applies to the initial liveness algebra interpretation:

Proposition 7.2 (Initial Liveness Algebra Interpretation) *Let* $\Sigma \in D_2$*.*

1. $\Sigma \models \mathrm{t}$ *iff* Σ *must* $\{e\}$ *and* Σ *after* $e \models \mathrm{t}$,
2. $\Sigma \models \phi \circ \psi$ *iff there are* $\Sigma_1, \Sigma_2 \in D_2$ *such that* $\Sigma_1 \models \phi$*,* $\Sigma_2 \models \psi$ *and for all* $\sigma \in \Sigma$ *there are* $\sigma_1 \in \Sigma_1$ *and* $\sigma_2 \in \Sigma_2$ *such that* $\sigma = \sigma_1 \times \sigma_2$*,*
3. $\Sigma \models \phi \vee \psi$ *iff for all* $\sigma \in \Sigma$*,* $cl_2\{\sigma\} \models \phi$ *or* $cl_2\{\sigma\} \models \psi$*,*
4. $\Sigma \models |a{>}\phi$ *iff* Σ *must* $\{a\}$ *and* Σ *after* $a \models \phi$*,*
5. $\Sigma \models \underline{0}$ *(always).*

PROOF: Similar to the proof of Proposition 7.1. □

Using Lemma 5.4 it is straightforward to derive from these two propositions equivalent, operationally determined satisfaction conditions directly on process terms. In addition to the relations may, must, and after, a syntactic characterisation of the coprime elements is needed. The appropriate notion is that of a *trace*: a process term built using only 0, 1, prefixing and $\times$. The set traces(p) of traces of p is defined in the obvious way by

$$\text{traces}(0) = \{0\},$$

$$\text{traces}(1) = \{1\},$$

$$\text{traces}(a.p) = \{a.q \mid q \in \text{traces}(p)\},$$

$$\text{traces}(p_1 \oplus p_2) = \text{traces}(p_1) \cup \text{traces}(p_2),$$

$$\text{traces}(p_1 \times p_2) = \{q_1 \times q_2 \mid q_1 \in \text{traces}(p_1), q_2 \in \text{traces}(p_2)\}.$$

The satisfaction conditions on terms derived from Propositions 7.1.2 and 3 and 7.2.2 and 3 are then the following:

$$p \models_i \phi \circ \psi \quad \text{iff} \quad \text{there are } p_1, p_2 \text{ such that } p_1 \models_i \phi,\ p_2 \models_i \psi, \text{ and for all traces } q \text{ of } p \text{ there are traces } q_1 \text{ of } p_1 \text{ and } q_2 \text{ of } p_2 \text{ such that } q \simeq_i q_1 \times q_2, \tag{19}$$

$$p \models_i \phi \vee \psi \quad \text{iff} \quad \text{for all traces } q \text{ of } p,\ q \models_i \phi \text{ or } q \models_i \psi. \tag{20}$$

Note that in contrast to the case for the other connectives, for fusion (19) has not yet succeeded in eliminating references to the behavioral equivalence relations $\simeq_i$ entirely. This is certainly possible, but only, it appears, by replacing these references by normal forms, or the semantical mappings $[\![\cdot]\!]_i$. This inelegance is one reason why we prefer the implication to the fusion when operational interpretations of the **PL** (and indeed $\mathbf{LL}^+$) connectives are concerned.

We then show that the initial algebra interpretations induce the appropriate orderings. Each member $\Sigma \in D_i$, $i \in \{1, 2\}$, is generated by a least, finite set Σ'. The *characteristic formula*, $\text{cf}(\Sigma)$, of Σ is then determined as the disjunction of the representation $\text{cf}(\sigma)$ of each member σ of Σ where 0 is represented as $\underline{0}$, 1 as t, and prefixing as the future modality.

Lemma 7.3 *For $i \in \{1, 2\}$ and $\Sigma_1, \Sigma_2 \in D_i$ the following statements are equivalent:*

1. $\Sigma_2 \models cf(\Sigma_1)$.
2. $\Sigma_1 \supseteq \Sigma_2$.
3. *For all extended, closed ϕ, if $\Sigma_1 \models \phi$ then $\Sigma_2 \models \phi$.*

PROOF: 1 iff 2. Assume that $\Sigma_2 \models \text{cf}(\Sigma_1)$. By an argument similar to that of the proof of Proposition 7.1.3 this holds iff $\text{cl}_i\{\sigma_2\} \models \text{cf}(\Sigma_1)$ whenever σ_2 is a member of the least generating subset of Σ_2. This is the case iff $\text{cl}_i\{\sigma_2\} \models \text{cf}\{\sigma_1\}$ for some member σ_1 of the least generating subset of Σ_1. We then just need to show that $\text{cl}_i\{\sigma_2\} \models \text{cf}\{\sigma_1\}$ iff $\sigma_2 \leq \sigma_1$ for the case $i = 1$, and $\text{cl}_i\{\sigma_2\} \models \text{cf}\{\sigma_1\}$ iff $\sigma_1 \leq \sigma_2$ for $i = 2$. This is shown by easy induction in the length of σ_1.
2 implies 3. By the filter property.
3 implies 2. By the implication 2 to 1 it follows that $\Sigma_1 \models \text{cf}(\Sigma_1)$. Assuming 3 we obtain $\Sigma_2 \models \text{cf}(\Sigma_1)$, so $\Sigma_1 \supseteq \Sigma_2$ by the implication 1 to 2. □

The Logical Characterisation Theorem now follows as an easy corollary.

Corollary 7.4 (Logical Characterisation Theorem) *For* $i \in \{1,2\}$, $p \sqsubseteq_i q$ *iff for all extended, closed* ϕ, *if* $p \models_i \phi$ *then* $q \models_i \phi$.

PROOF: By Lemma 7.3, Lemma 5.5, and Propositions 7.1 and 7.2. □

8 Axiomatisation

In the remaining part of the paper we give procedures for deciding validity of formulas with respect to the process-based interpretations. That is, procedures that, given an extended closed formula ϕ, decides if ϕ is *i-valid*, meaning that the unit process satisfies ϕ under the D_i interpretation, for $i \in \{1,2\}$. The procedures use a rewriting-based approach. We add a number of new axiom schemas which are used to rewrite arbitrary extended closed formulas into a normal form. Thus sound and complete axiomatisations are obtained as byproducts using this approach. For the initial safety algebra, in particular, soundness depends on the underlying label group being infinite. In the present section these axiomatisations are presented and their soundness proved.

We consider only the $\circ$-free fragment here. The primary reason is that the double induction used in the proof of completeness below means that the length of the proof increases with the square of the number of logical connectives. We see no essential problems, however, in extending the results to cover $\circ$ as well.

Note first that with respect to the initial algebra interpretations the extensional falsehood constant $\bot$, and consequently also an "intuitionistic" negation $\neg$ and the extensional truthhood constant $\top$, can be derived: Let $\bot \triangleq <a||b>\underline{0}$ for some fixed $a, b \in L$ such that $a \neq b$, $\neg\phi \triangleq \phi \rightarrow \bot$, and $\top \triangleq \neg\bot$. To see that this is reasonable note that in both D_1 and D_2, if $a.\Sigma_1 \leq b.\Sigma_2$ then $a = b$. Hence for neither of the two interpretations can there be a Σ for which $\Sigma \models <a||b>\phi$ when $a \neq b$. Thus for the initial algebra interpretations $\bot$ expresses the empty set. This has extremely curious consequences for the expressive power of the derived negation. For instance $\neg\phi$ expresses the nonexistence of a $\Sigma \in D_i$ for which $\Sigma \models \phi$. When this is the case we say that ϕ is *i-unsatisfiable*. As a consequence $\neg\neg\phi$ expresses *i-satisfiability*, or consistency: $\Sigma \models \phi$ for some $\Sigma \in D_i$. Given this expressive power the Henkin-style approach used earlier appears untenable: Theories must have the global property that $\phi \in T$ for some theory T if and only if $\neg\neg\phi \in T'$ for all theories T'. This motivates our rewriting-based approach in which the satisfiable and unsatisfiable can be given direct syntactical characterisations.

The extensional truth- and falsehood constants are governed by the expected axioms:

$\bot$-**Elim**	$\bot \rightarrow \phi$
$\top$-**Intro**	$\phi \rightarrow \top$

Let $\mathbf{PL}^-$ be $\mathbf{PL}$ minus the axioms ∘1 and ∘2 governing ∘, and let $\mathbf{PL}^-_{\top,-}$ be $\mathbf{PL}^-$ augmented with the axioms ⊥-**Elim** and ⊤-**Intro**. The following Proposition summarises some theorems and derived rules of $\mathbf{PL}^-_{\top,-}$:

Proposition 8.1 (Theorems of $\mathbf{PL}^-_{\top,-}$)

1. $\vdash_{\mathbf{PL}^-_{\top,-}} (\phi \to \neg\psi) \to (\psi \to \neg\phi)$
2. $\vdash_{\mathbf{PL}^-_{\top,-}} \phi \to \neg\neg\phi$
3. $\vdash_{\mathbf{PL}^-_{\top,-}} \neg\neg\neg\phi \to \neg\phi$
4. $\vdash_{\mathbf{PL}^-_{\top,-}} \neg\phi \to (\psi \to \neg\phi)$
5. $$\frac{\vdash_{\mathbf{PL}^-_{\top,-}} \top \to \phi \quad \vdash_{\mathbf{PL}^-_{\top,-}} (\phi \wedge \psi) \to \gamma}{\vdash_{\mathbf{PL}^-_{\top,-}} \psi \to \gamma}$$

Proof: 1: An instance of C. 2: Use 1 and I. 3: Use 1, 2 and B. 4: Use ⊥-Elim, B and C. 5: By the ∧-axioms and transitivity we obtain $\vdash_{\mathbf{PL}^-_{\top,-}} (\top \wedge \psi) \to \gamma$, and then by the ∧-axioms, transitivity and ⊤-Intro, $\vdash_{\mathbf{PL}^-_{\top,-}} \psi \to \gamma$. □

The extended logics $\mathbf{PL}^-(D_1)$ and $\mathbf{PL}^-(D_2)$ are determined by adding the following axioms to the axioms and rules of $\mathbf{PL}^-_{\top,-}$. We shall make no attempt to justify each of these axioms intuitively. Each axiom reflects some property which holds in the initial algebra interpretation concerned but which fails to hold in general. A simple example is the Distribution axiom below which is a direct consequence of Propositions 7.1 and 7.2. The axioms common to both $\mathbf{PL}^-(D_1)$ and $\mathbf{PL}^-(D_2)$ are the following:

Distribution	$\phi \wedge (\psi \vee \gamma) \to (\phi \wedge \psi) \vee (\phi \wedge \gamma)$
$\vert a\rangle$-∧	$\vert a{>}\phi \wedge \vert a{>}\psi \to \vert a{>}(\phi \wedge \psi)$
$<a\vert$-∨	$<a\vert(\phi \vee \psi) \to <a\vert\phi \vee <a\vert\psi$
$<a\vert$-$\vert a>$-→	$<a\vert\vert a{>}\phi \to \phi$
$\neg\vert a>$	$\neg\phi \leftrightarrow \neg\vert a{>}\phi$
→-∨	$(\vert a{>}\phi \to \bigvee_{b\in\Lambda} \vert b{>}\psi_b) \to \bigvee_{b\in\Lambda}(\vert a{>}\phi \to \vert b{>}\psi_b)$

where in →-∨ Λ ranges over finite, nonempty subsets of L. The additional axioms for $\mathbf{PL}^-(D_1)$ are the following six:

s1	$\neg\neg\underline{0}$
s2	$(\neg\neg\phi) \to (\underline{0} \to \phi)$
s3	$\vert a{>}\phi \wedge \vert b{>}\psi \to \underline{0}$ (provided $a \neq b$)
s4	$(\neg\neg\phi \wedge \neg\neg\psi \wedge (\vert a{>}\phi \to \vert b{>}\psi)) \to \vert a^{-1}\cdot b{>}(\phi \to \psi)$
s5	$(\top \to \bigvee_{a\in\Lambda} \vert a{>}\phi_a) \leftrightarrow (\underline{0} \wedge (\bigvee_{a\in\Lambda} \neg\neg\phi_a))$
s6	$(\neg\neg\phi \wedge (\vert a{>}\phi \to \underline{0})) \to \underline{0}$

For $\mathbf{PL}^{-}(D_2)$ the following four axioms are added instead:

l1	$\top \to \underline{0}$
l2	$(\phi \wedge \|a{>}\top) \to \|a{>}{<}a\|\phi$
l3	$(\neg\neg\phi \wedge (\|a{>}\phi \to \|b{>}\psi)) \to \|a^{-1} \cdot b{>}(\phi \to \psi)$
l4	$\neg(\top \to \bigvee_{a \in \Lambda} \|a{>}\phi_a)$

We note a number of theorems of $\mathbf{PL}^{-}(D_1)$ and $\mathbf{PL}^{-}(D_2)$ for later use.

Proposition 8.2 (Theorems of $\mathbf{PL}^{-}(D_1)$ and $\mathbf{PL}^{-}(D_2)$)
Let $\vdash_{\mathbf{PL}^{-}(D)} \phi$ *if* $\vdash_{\mathbf{PL}^{-}(D_i)} \phi$ *for both* $i = 1$ *and* $i = 2$.

1. $\vdash_{\mathbf{PL}^{-}(D)} |a{>}(\phi \wedge \psi) \leftrightarrow (|a{>}\phi) \wedge (|a{>}\psi)$
2. $\vdash_{\mathbf{PL}^{-}(D)} {<}a|(\phi \vee \psi) \leftrightarrow ({<}a|\phi) \vee ({<}a|\psi)$
3. $\vdash_{\mathbf{PL}^{-}(D)} \phi \leftrightarrow {<}a||a{>}\phi$
4. $\vdash_{\mathbf{PL}^{-}(D)} \mathrm{t} \leftrightarrow {<}e|\mathrm{t}$
5. $\vdash_{\mathbf{PL}^{-}(D)} \neg\phi \to \neg{<}a|\phi$
6. $\vdash_{\mathbf{PL}^{-}(D_1)} (\neg\neg\phi) \leftrightarrow (\underline{0} \to \phi)$
7. $\vdash_{\mathbf{PL}^{-}(D_1)} \neg{<}a|\underline{0}$
8. $\vdash_{\mathbf{PL}^{-}(D)} \neg{<}a||b{>}\phi$, *provided* $a \neq b$
9. $\vdash_{\mathbf{PL}^{-}(D_2)} \neg(|a{>}\phi \wedge |b{>}\psi)$, *provided* $a \neq b$

PROOF: 1: $\leftarrow$ by $|a{>}$-$\wedge$, $\to$ by the $\wedge$-axioms, transitivity and $|a{>}$-monotonicity. 2: Similar. 3: By $\to$-${<}a|$-$|a{>}$ and ${<}a|$-$|a{>}$-$\to$. 4: By ${<}e|$-necessitation and t1, $\vdash_{\mathbf{PL}^{-}(D)} {<}e|\mathrm{t}$, so $\vdash_{\mathbf{PL}^{-}(D)} \mathrm{t} \to {<}e|\mathrm{t}$ by t2. By similar reasoning $\vdash_{\mathbf{PL}^{-}(D)} \mathrm{t} \to |e{>}\mathrm{t}$. Then by ${<}e|$-monotonicity, transitivity and ${<}e|$-$|e{>}$-$\to$, $\vdash_{\mathbf{PL}^{-}(D)} {<}e|\mathrm{t} \to \mathrm{t}$. 5: By $|a{>}$-${<}a|$-$\to$, transitivity and permutation, $\vdash_{\mathbf{PL}^{-}(D)} (\neg\phi) \to (\neg|a{>}{<}a|\phi)$, so by $\neg|a{>}$ and transitivity the result obtains. 6: $\to$ by s2. $\leftarrow$ by transitivity, permutation and s1. 7: Let $b \neq a$. An outline of the proof follows:

1.	$\neg\|b{>}\underline{0} \to \neg{<}b\|\|b{>}\underline{0}$	by 5
2.	$\neg\neg{<}b\|\|b{>}\underline{0} \to \neg\neg\|b{>}\underline{0}$	by standard reasoning
3.	$(\underline{0} \to {<}b\|\|b{>}\underline{0}) \to (\underline{0} \to \|b{>}\underline{0})$	by 7
4.	$\underline{0} \to \|b{>}\underline{0}$	by $\to$-${<}b\|$-$\|b{>}$
5.	$\neg{<}a\|\underline{0}$	by ${<}a\|$-monotonicity and def. $\neg$

8: Let $a \neq b$. For $\mathbf{PL}^{-}(D_1)$ first, by s3, ${<}a|$-monotonicity and distribution of ${<}a|$ over $\wedge$, $\vdash_{\mathbf{PL}^{-}(D_1)} {<}a||a{>}\top \wedge {<}a||b{>}\phi \to {<}a|\underline{0}$. Then by $\to$-${<}a|$-$|a{>}$, ${<}a|$-$|a{>}$-$\to$ and standard reasoning, $\vdash_{\mathbf{PL}^{-}(D_1)} {<}a||b{>}\phi \to {<}a|\underline{0}$. But by 7 it then follows that $\vdash_{\mathbf{PL}^{-}(D_1)} \neg{<}a||b{>}\phi$. 9: The proof is outlined as follows:

1.	$\|a>\phi \wedge \|b>\psi \rightarrow \|a>\phi \wedge \|b>\psi \wedge \|a>\top$	by standard reasoning
2.	$\|a>\phi \wedge \|b>\psi \rightarrow \|a><a\|(\|a>\phi \wedge \|b>\psi)$	by l2
3.	$\|a>\phi \wedge \|b>\psi \rightarrow \|a>(<a\|\|a>\phi \wedge <a\|\|b>\psi)$	by distribution of $<a\|$ over $\wedge$
4.	$\|a>\phi \wedge \|b>\psi \rightarrow \|a>(<a\|\|a>\phi \wedge \bot)$	by 6, assuming $a \neq b$
5.	$\|a>\phi \wedge \|b>\psi \rightarrow \|a>\bot$	by standard reasoning
6.	$\neg(\|a>\phi \wedge \|b>\psi)$	by $\neg\|a>$

□

Theorem 8.3 (Soundness of $\mathbf{PL}^-(D_1)$ and $\mathbf{PL}^-(D_2)$) *For all extended, closed formulas ϕ, if $\vdash_{\mathbf{PL}^-(D_i)} \phi$ then $1_i \models \phi$.*

Proof: The proof is largely routine, and relies on Propositions 7.1 and 7.2. A representative collection of cases is proved below.

$|a>$-$\wedge$. If $\Sigma \models |a>\phi$ and $\Sigma \models |a>\psi$ then either $i = 1$ and $\Sigma = 0_1$, in which case $\Sigma \models |a>(\phi \wedge \psi)$, or else $\Sigma = a.\Sigma'$ and Σ after $a \models \phi \wedge \psi$ by 7.1.3 and 7.2.3.

$\rightarrow$-$\vee$. Suppose $\Sigma \models |a>\phi \rightarrow \bigvee_{b\in\Lambda} |b>\psi_b$. First if there is no Σ' such that $\Sigma \models |a>\phi$ then we are done, so assume not. Let $\Lambda' = \{c \mid \exists\sigma.c.\sigma \in \Sigma\}$. If $\Lambda' = \emptyset$ then in case $i = 1$ $\Sigma = 0_1$, and $\Sigma \models |a>\phi \rightarrow |b>\psi_b$ for all $b \in \Lambda$. In case $i = 2$ we obtain a contradiction. So $\Lambda' \neq \emptyset$. For each $c \in \Lambda'$ let $\Sigma_c = \mathrm{cl}_i\{\sigma \in \Sigma \mid \exists\sigma'.\sigma = c.\sigma'\}$. Then $\Sigma = \bigcup_{c\in\Lambda'} \Sigma_c$. It suffices to show that for each $c \in \Lambda'$ there is a $b \in \Lambda$ such that $\Sigma_c \models |a>\phi \rightarrow |b>\psi_b$. Fix $c \in \Lambda'$. Then $\Sigma_c \models |a>\phi \rightarrow \bigvee_{b\in\Lambda} |b>\psi_b$ by the filter property. Let $\Sigma_1 \models |a>\phi$. We can assume that Σ_1 has the form $a.\Sigma_1'$. Hence $\Sigma_c \times \Sigma_1 \models \bigvee_{b\in\Lambda} |b>\psi_b$, so $c \cdot a \in \Lambda$ and $\Sigma_c \times \Sigma_1 \models |c \cdot a>\psi_{c\cdot a}$. But the chosen disjunct was independent of Σ_1, so we have verified that $\Sigma_c \models |a>\phi \rightarrow |c \cdot a>\psi_{c\cdot a}$.

s4. Suppose ϕ and ψ are both satisfiable in D_1 and that $\Sigma \models |a>\phi \rightarrow |b>\psi$. If $\Sigma = 0_1$ the proof is easily completed, so suppose not. If $\Sigma \neq (a^{-1} \cdot b).\Sigma'$ for some Σ' then there is some σ in the least generating set of Σ such that $\sigma = c.\sigma'$ for some σ' and $c \neq a^{-1} \cdot b$. Then $\mathrm{cl}_1\{\sigma\} \not\models |b>\psi$, a contradiction. So indeed $\Sigma = (a^{-1} \cdot b).\Sigma'$ and whenever $\Sigma_1 \models \phi$ then $\Sigma' \times \Sigma_1 \models \psi$, that is, $\Sigma \models |a^{-1} \cdot b>(\phi \rightarrow \psi)$.

s5. Suppose $\Sigma \models \top \rightarrow \bigvee_{a\in\Lambda} |a>\phi_a$ and suppose for a contradiction that $\Sigma \neq 0_1$. Then for some b and σ, $b.\sigma \in \Sigma$. Pick any $c \notin \Lambda$. As L is infinite such a c exists. Now $(b^{-1} \cdot c).0_1 \models \top$ so $\Sigma \times (b^{-1} \cdot c).0_1 \models \bigvee_{a\in\Lambda} |a>\phi_a$. But then $\mathrm{cl}_1\{b.\sigma\} \times \mathrm{cl}_1\{(b^{-1} \cdot c).0\} = \mathrm{cl}_1\{c.0\} \models |a>\phi_a$ by 7.1.2, but this is a contradiction by 7.1.3. For the second conjunct observe that $0_1 \models \bigvee_{a\in\Lambda} |a>\phi_a$ so $0_1 \models |a>\phi_a$ for some $a \in \Lambda$ by 7.1.2 and then ϕ_a is 1-satisfiable by 7.1.3. For the converse implication, if ϕ_a is 1-satisfiable then for any Σ, $0_1 \times \Sigma = 0_1 \models \bigvee_{a\in\Lambda} |a>\phi_a$ by 7.1.2 and 3, so $0_1 \models \top \rightarrow \bigvee_{a\in\Lambda} |a>\phi_a$.

l2. If $\Sigma \models \phi$ and $\Sigma \models |a>\top$ then $\Sigma = a.\Sigma'$ for some Σ'. But then $\Sigma \models |a><a|\phi$.

l4. If $\Sigma \models \top \rightarrow \bigvee_{a\in\Lambda} |a>\phi_a$ then $\Sigma \times 0_2 \models \bigvee_{a\in\Lambda} |a>\phi_a$, a contradiction. □

For finite label groups soundness of $\mathbf{PL}^-(D_1)$, axiom s5 in particular, fails in general. The problem is the implication $(\top \rightarrow \bigvee_{a\in\Lambda} |a>\phi_a) \rightarrow \underline{0}$. For if indeed L

is finite then $\{a.0 \mid a \in L\} \cup \{0\} \models \top \rightarrow \bigvee_{a \in L} |a\rangle\top$ while $\{a.0 \mid a \in L\} \cup \{0\} \not\models \underline{0}$. The problem of devising a sound and complete axiomatisation for the case of finite label groups appears a difficult one and remains open.

9 Completeness and decidability

In this section the completeness of $\mathbf{PL}^-(D_1)$ and $\mathbf{PL}^-(D_2)$ is proved, and it is shown how the proof determines decision procedures for the properties of i-validity and i-satisfiability. The proof has three ingredients. First a suitable notion of normal form is introduced. Secondly the i-valid and i-satisfiable normal forms are characterised in syntactic terms. Thirdly, and finally, we show that each formula is provably equivalent to a formula in normal form.

Definition 9.1 (Normal Form, Satisfiable Normal Form) The set satNF of *satisfiable normal forms* is defined inductively by

1. t, $\top$, $\underline{0} \in$ satNF,
2. if for each $a \in \Lambda$, $\phi_a \in$ satNF then $\bigvee_{a \in \Lambda} |a\rangle\phi_a \in$ satNF.

The set NF of *normal forms* is NF $=$ satNF $\cup \{\bot\}$.

Proposition 9.2 *Let $i \in \{1, 2\}$. The following statements are equivalent.*

1. $\phi \in$ satNF
2. $\vdash_{\mathbf{PL}^-(D_i)} \neg\neg\phi$
3. $\Sigma \models \phi$ *for some* $\Sigma \in D_i$

PROOF: 1 iff 2. By soundness $\not\vdash_{\mathbf{PL}^-(D_i)} \neg\neg\bot$ so we need just check $\vdash_{\mathbf{PL}^-(D_i)} \neg\neg\phi$ whenever $\phi \in$ satNF. An easy structural induction suffices to establish this. For t and $\top$ use I or $\top$-Intro, t1 and 8.1.2. For $\underline{0}$ and $i = 1$ use I and 8.2.6, and for $i = 2$ use l1. For $\bigvee_{a \in \Lambda} |a\rangle\phi_a \in$ satNF use the induction hypothesis and $\neg|a\rangle$.
3 implies 1. If $\Sigma \models \phi$ then $\phi \in$ satNF.
2 implies 3. If $\vdash_{\mathbf{PL}^-(D_i)} \neg\neg\phi$ then $1_i \models \neg\neg\phi$ by soundness. □

Next the valid normal forms are characterised. In order to be able to define valNF uniformly we assume here that $\underline{0}$ in the case $i = 2$ is defined by $\underline{0} \triangleq \top$.

Definition 9.3 (Valid Normal Forms) The set valNF of *valid normal forms* is defined inductively by

1. t, $\top \in$ valNF,
2. if $\bigvee_{a \in \Lambda} |a\rangle\phi_a \in$ satNF, $e \in \Lambda$, and $\phi_e \in$ valNF then $\bigvee_{a \in \Lambda} |a\rangle\phi_a \in$ valNF.

Proposition 9.4 *Let $i \in \{1,2\}$ and $\phi \in NF$. The following statements are equivalent.*

1. $\phi \in$ valNF
2. $\vdash_{\mathbf{PL}^-(D_i)} \phi$
3. $1_i \models \phi$

PROOF: 1 iff 2. An easy structural induction in ϕ. Note $\not\vdash_{\mathbf{PL}^-(D_i)} \bot$, $\not\vdash_{\mathbf{PL}^-(D_i)} \underline{0}$ for $i = 1$, $\vdash_{\mathbf{PL}^-(D_i)}$ t, and $\vdash_{\mathbf{PL}^-(D_i)} \top$. Let $\phi = \bigvee_{a \in \Lambda} |a{>}\phi_a \in$ satNF and $\vdash_{\mathbf{PL}^-(D_i)} \phi$. Then $1_i \models \phi$ by soundness, so $e \in \Lambda$ and $\vdash_{\mathbf{PL}^-(D_i)} \phi_e$. By the induction hypothesis $\phi_e \in$ valNF so $\phi \in$ valNF as well. If conversely $\phi \in$ valNF and $\vdash_{\mathbf{PL}^-(D_i)} \phi_e$ then $\vdash_{\mathbf{PL}^-(D_i)} \phi$ by the induction hypothesis, $|e{>}$-necessitation and $\vee$-Intro.
2 implies 3. By soundness.
3 implies 1. An easy structural induction in ϕ. □

The largest single step of the completeness proof is the normalisation theorem below. At first glance it might seem surprising that as sparse a vocabulary as the constants plus $\vee$ and $|a{>}$ suffices to express the whole language. On the other hand we have already seen that in both D_1 and D_2 all occurrences of $\times$ are eliminable in favour of operators 0, 1, $\oplus$ and prefixing only.

Theorem 9.5 (Normalisation) *Let $i \in \{1,2\}$. There is an effective procedure which given any extended closed ϕ produces a $\phi' \in$NF such that $\vdash_{\mathbf{PL}^-(D_i)} \phi \leftrightarrow \phi'$.*

PROOF: See section 10. □

Corollary 9.6 (Completeness of $\mathbf{PL}^-(D_i)$) *Let $i \in \{1,2\}$ and $1_i \models \phi$ for ϕ an extended closed formula. Then $\vdash_{\mathbf{PL}^-(D_i)} \phi$.*

PROOF: By soundness, the Normalisation Theorem and Proposition 9.4. □

In a similar fashion the decidability of the properties of i-validity and i-satisfiability is easily seen.

10 Proof of the Normalisation Theorem

The proof proceeds by cases and induction in the modal depth of formulas. Abbreviate $\vdash_{\mathbf{PL}^-(D_i)} \phi \leftrightarrow \phi'$ by $\phi \equiv_i \psi$, or just $\equiv$ when i is understood from the context. We prove the slightly more general statement that for each extended closed ϕ there is an effectively computable $\phi' \in$ NF of a modal depth not exceeding that of ϕ such that $\phi \equiv_i \phi'$.

10.1 The safety case

Let first $\phi = \phi_1 \to \phi_2$, and assume that $\phi_1, \phi_2 \in \mathrm{NF}$. We proceed cases on ϕ_1 and when necessary also ϕ_2.

a. $\phi_1 = \bot$. Then $\phi \equiv \top$.

b. $\phi_1 = \top$. We proceed by cases on ϕ_2.

   i. $\phi_2 = \bot$. Then $\phi \equiv \bot$.

   ii. $\phi_2 = \top$. Then $\phi \equiv \top$.

   iii. $\phi_2 = \underline{0}$. Here $\phi \equiv \underline{0}$.

   iv. $\phi_2 = \mathrm{t}$. Then $\phi \equiv \underline{0}$ by s5.

   v. $\phi_2 = \bigvee_{a\in\Lambda} |a{>}\phi_a$. Here $\phi \equiv \underline{0}$, again by s5.

c. $\phi_1 = \underline{0}$. If $\phi_2 = \bot$ then $\phi \equiv \bot$. Otherwise $\phi_2 \in \mathrm{satNF}$ so $\vdash_{\mathbf{PL}^-(D_1)} \phi$ by Proposition 9.2, so $\phi \equiv \top$ by $\top$-Intro and Propositions 8.2.6 and 8.1.4.

d. $\phi_1 = \mathrm{t}$. Here $\phi \equiv \mathrm{t}$.

e. $\phi_1 = \bigvee_{a\in\Lambda_1} |a{>}\psi_a$. We proceed again by cases on ϕ_2.

   i. $\phi_2 = \bot$. Then $\phi \equiv \bot$.

   ii. $\phi_2 = \top$. Here $\phi \equiv \top$.

   iii. $\phi_2 = \underline{0}$. Note first that $\vdash_{\mathbf{PL}^-(D_1)} \underline{0} \to \phi$. For $\vdash_{\mathbf{PL}^-(D_1)} \phi_1 \to \neg\neg\underline{0}$, giving the result by s2 and permutation. For the converse implication $\vdash_{\mathbf{PL}^-(D_1)} \phi \to \bigwedge_{a\in\Lambda_1}(|a{>}\psi_a \to \underline{0})$ by standard reasoning. Secondly $\vdash_{\mathbf{PL}^-(D_1)} \neg\neg\psi_a$ for all $a \in \Lambda_1$ by Proposition 9.2 so by s6 and Proposition 8.1.5 $\vdash_{\mathbf{PL}^-(D_1)} \bigwedge_{a\in\Lambda_1}(|a{>}\psi_a \to \underline{0}) \to \underline{0}$ and we're done.

   iv. $\phi_2 = \mathrm{t}$. By standard reasoning we first obtain $\phi \equiv \bigwedge_{a\in\Lambda_1}(|a{>}\psi_a \to |e{>}\mathrm{t})$. As $\vdash_{\mathbf{PL}^-(D_1)} \neg\neg|a{>}\psi_a$ for each $a \in \Lambda_1$ and $\vdash_{\mathbf{PL}^-(D_1)} \neg\neg|e{>}\mathrm{t}$ then by s4 $\phi \equiv \bigwedge_{a\in\Lambda_1} |a^{-1}{>}(\psi_a \to \mathrm{t})$. By the induction hypothesis we find for each $a \in \Lambda_1$ a $\gamma_a \in \mathrm{NF}$ such that $\psi_a \to \mathrm{t} \equiv \gamma_a$ and thus $\phi \equiv \bigwedge_{a\in\Lambda_1} |a^{-1}{>}\gamma_a$. If some γ_a is $\bot$ then $\phi \equiv \bot$ by $\neg|a^{-1}{>}$. Otherwise $\vdash_{\mathbf{PL}^-(D_1)} \underline{0} \to \bigwedge_{a\in\Lambda_1} |a^{-1}{>}\gamma_a$ by Proposition 9.2, 8.2.6 and $\wedge$-Intro, so if Λ_1 has size greater than 1 then by s3, $\phi \equiv \underline{0}$. Otherwise let $\Lambda_1 = \{a\}$ and we obtain $\phi \equiv |a^{-1}{>}\gamma_a$.

   v. $\phi_2 = \bigvee_{b\in\Lambda_2} |b{>}\gamma_b$. By standard reasoning, $\phi \equiv \bigwedge_{a\in\Lambda_1} \bigvee_{b\in\Lambda_2}(|a{>}\psi_a \to |b{>}\gamma_b)$, and as in case (iv) we obtain $\phi \equiv \bigwedge_{a\in\Lambda_1} \bigvee_{b\in\Lambda_2} |a^{-1}\cdot b{>}(\psi_a \to \gamma_b)$. So by the induction hypothesis we find for each pair a, b a $\delta_{a,b} \in \mathrm{NF}$ such that $\delta_{a,b} \equiv \psi_a \to \gamma_b$, and then $\phi \equiv \bigwedge_{a\in\Lambda_1} \bigvee_{b\in\Lambda_2} |a^{-1}\cdot b{>}\delta_{a,b}$. Using Distribution

$$\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2} \bigwedge_{a\in\Lambda_1} |a^{-1}\cdot f(a){>}\delta_{a,f(a)},$$

and by $\neg|a>$ we can assume that $\delta_{a,b} \neq \bot$ for all a, b. Fix $f : \Lambda_1 \to \Lambda_2$. Suppose there is some $a_1, a_2 \in \Lambda_1$ such that $a_1^{-1} \cdot f(a_1) \neq a_2^{-1} \cdot f(a_2)$. Then $\bigwedge_{a\in\Lambda_1} |a^{-1} \cdot f(a)>\delta_{a,f(a)} \equiv \underline{0}$ by s3 and 9.2. If on the other hand $a_1 \cdot f(a_1) = a_2^{-1} \cdot f(a_2) = a_f$, say, for all $a_1, a_2 \in \Lambda$ then

$$\bigwedge_{a\in\Lambda_1}|a^{-1} \cdot f(a)>\delta_{a,f(a)} \equiv |a_f>\bigwedge_{a\in\Lambda_1}\delta_{a.f(a)}.$$

We now apply the induction hypothesis and find some δ_{a_f} such that $\bigwedge_{a\in\Lambda_1} \delta_{a.f(a)} \equiv \delta_{a_f}$. It is now a simple matter to complete the rewriting using the tools already introduced.

This completes the case for $\phi = \phi_1 \to \phi_2$. Assume next that $\phi = \phi_1 \wedge \phi_2$, $\phi_1, \phi_2 \in$ NF. The only interesting case here is when $\phi_1 = \bigvee_{a\in\Lambda_1} |a>\phi_a$ and $\phi_2 = \bigvee_{b\in\Lambda_2} |b>\psi_b$. The other case are either already covered or in the case of t easily reducible to the present case. To rewrite into normal form first use distribution to obtain the form $\phi \equiv \bigvee_{a,b} |a>\phi_a \wedge |b>\psi_b$, and then 9.2 and s3 to obtain either $\phi \equiv \underline{0}$ if $\Lambda_1 \cap \Lambda_2$ is empty, or if not, $\phi \equiv \bigvee_{a\in\Lambda_1\cap\Lambda_2} |a>(\phi_a \wedge \psi_a)$. The induction hypothesis is then used in a routine way to rewrite $\bigvee_{a\in\Lambda_1\cap\Lambda_2} |a>(\phi_a \wedge \psi_a)$ into normal form. The remaining cases are straightforward and left to the reader, as is the check that the size of ϕ does not increase under normalisation.

10.2 The liveness case

Let again $\phi = \phi_1 \to \phi_2$, $\phi_1, \phi_2 \in$ NF. The case for one of ϕ_1 or ϕ_2 equal to $\underline{0}$ need not be considered. The case for one of ϕ_1 or ϕ_2 equal to $\bot$ is trivial.

a. $\phi_1 = \top$. Procees by case on ϕ_2.

   i. $\phi_2 = \top$. Then $\phi \equiv \top$.

   ii. $\phi_2 = \mathrm{t}$. Then $\phi \equiv \bot$ by l4, as $\mathrm{t} \equiv |e>\mathrm{t}$.

   iii. $\phi_2 = \bigvee_{b\in\Lambda_2} |b>\psi_b$. Here $\phi \equiv \bot$ by l4.

b. $\phi_1 = \mathrm{t}$. Here $\phi \equiv \phi_2$.

c. $\phi_1 = \bigvee_{a\in\Lambda_1} |a>\phi_a$. Proceed by cases on ϕ_2.

   i. $\phi_2 = \top$. Then $\phi \equiv \top$ by 8.1.4.

   ii. $\phi_2 = \mathrm{t}$. We first obtain $\phi \equiv \bigwedge_{a\in\Lambda_1}(|a>\phi_a \to |e>\mathrm{t})$. By 9.2, $|a>\phi_a$ is 2-satisfiable, so by l3 and 6.1.3 $\phi \equiv \bigwedge_{a\in\Lambda_1} |a^{-1}>(\phi_a \to \mathrm{t})$. By 8.2.9 we obtain $\phi \equiv \bot$ whenever Λ_1 contains more than one element. Otherwise let $\Lambda_1 = \{a\}$, and it is now a simple matter to apply the induction hypothesis to $\phi_a \to \mathrm{t}$ and obtain the desired normal form.

   iii. $\phi_2 = \bigvee_{b\in\Lambda_2} |b>\psi_b$. As in the proof for the safety case we obtain $\phi \equiv \bigvee_{f:\Lambda_1\to\Lambda_2} \bigwedge_{a\in\Lambda_1} |a^{-1} \cdot f(a)>\delta_{a,f(a)}$ where each $\delta_{a,f(a)}$ is 2-satisfiable and in normal form. Fix $f : \Lambda_1 \to \Lambda_2$. If there is some $a_1, a_2 \in \Lambda_1$ such that

$a_1^{-1} \cdot f(a_1) \neq a_2^{-1} \cdot f(a_2)$ then $\bigwedge_{a \in \Lambda_1} |a^{-1} \cdot f(a)> \delta_{a,f(a)} \equiv \bot$. Otherwise let $\Lambda_1 = \{a_f\}$ and then $\bigwedge_{a \in \Lambda_1} |a^{-1} \cdot f(a)> \delta_{a,f(a)} \equiv |a_f> \bigwedge_{a \in \Lambda_1} \delta_{a,f(a)}$, and the rewriting into normal form can then be completed by the induction hypothesis and standard reasoning.

Assume next that $\phi = \phi_1 \wedge \phi_2$, $\phi_1, \phi_2 \in \mathrm{NF}$. The case for one ϕ_1 and ϕ_2 equal to $\top$ is trivial. For the rest:

a. $\phi_1 = \mathrm{t}$. If $\phi_2 = \mathrm{t}$ then $\phi \equiv \mathrm{t}$ so assume instead that $\phi_2 = \bigvee_{b \in \Lambda_2} |b> \psi_b$. Then $\phi \equiv \bigvee_{b \in \Lambda_2} (|e> \mathrm{t} \wedge |b> \psi_b$, so if $e \notin \Lambda_2$ then $\phi \equiv \bot$ by 8.2.9. Otherwise $\phi \equiv |e>(\mathrm{t} \wedge \psi_b)$ which is easily rewritten into normal form using the induction hypothesis.

b. $\phi_1 = \bigvee_{a \in \Lambda_1} |a> \phi_a$. The proof proceeds as in case (a).

The remaining cases and the check that normalisation does not increase size is left to the reader.

11 Conclusion, and Future Work

Our aim has been to investigate the use of linear and relevant logics as logical handles on the static structure of processes, and in this framework explore the use of modal operators to account for dynamic behaviour. We have given three examples, one of which was studied in detail, and a number of completeness and characterisation results have been obtained concerning axiomatisations and the relationship to linear logic proper as well as to the computational interpretations of (term) models. Computationally the main example is rather weak in that it lacks a suitable notion of controllable choice. It is an important issue for future work to extend our approach in this direction. In Dam [8] one such extension is pursued, sacrificing, however, the algebraic interpretation of formulas and the tight relationship to linear logic. Another important issue is to consider asynchronous parallel composition as in CCS. One option is to try and reduce asynchrony to synchrony by introducing special idling actions as in e.g. [30].

It is important to note the strength of the completeness and decidability results obtained in the last part of the paper. Clearly they solve the problems of satisfiability and model checking posed in the introduction; the latter indeed in a compositional manner. Moreover a large range of entirely new correctness properties can now be decided which express structural properties of processes such as the following:

> Given process q, and specifications ϕ and ψ, does there exist a process p such that $p \models \phi$ and $p \times q \models \psi$?

Process p can be mechanically derived from the normal form of the formula $\neg\neg(\phi \wedge (\mathrm{cf}(q) \rightarrow \psi))$ where cf is the representation of processes as formulas of section 7.

There are clear potential applications of such procedures for instance in the area of program derivation. Note the relationship to the work on equation solving of e.g. Parrow [23]. Of course, part of the strength derives from the relative expressive weakness of the process and specification languages considered, and it is not clear how far the results of the present paper generalises, for instance to temporal properties. Indeed it may be that the completeness results achieved here are too strong, and that instead completeness should be sought only in much weaker forms, for instance for ground implications (formulas of the form $\phi \rightarrow \psi$ where neither ϕ nor ψ contain occurences of $\rightarrow$).

Basic to our approach is a concept of processes as a semilattice-ordered structure with the semilattice operation a choice operator required to be preserved by parallel composition. We have explored the close relations to algebraic models (such as quantales) of relevant and linear logics (c.f. Dunn [10], Abramsky and Vickers [3]). Moreover there are intimate relations to the models for BCK-logics of Ono and Komori [22], and for $\wedge/\vee$ distributive logics to the ternary relation model for relevant logics of Routley and Meyer [26] (see Dam [8] for a detailed exposition). While preservation of choice by parallel composition is natural in the synchronous case, if asynchronous parallel composition is to be modelled directly this is likely to be too strong, and only monotonicity with respect to the induced semilattice ordering should be expected.

It may be of interest to consider process-based interpretations of connectives other than the ones we have considered above, notably the De Morgan negation $\sim$, the intensional sum-operator $+$, and the linear modalities ! and ?. Given just the monoid structure of models, by distinguishing a constant formula $\perp$ the double negation construction of Girard [13] (the phase semantics) applies, and full propositional linear logic can be interpreted. Relating to the intended interpretation of the monoid operation as parallel composition this interpretation is however of dubious practical value. Our general models can be extended to cover De Morgan negation by an approach similar to that of Ono and Komori: A subset of prime elements and an involution $(\cdot)^*$ is presupposed and then $\sim$ is interpreted by $x \models \sim\phi$ iff for all prime $y \geq x$, $y^* \not\models \phi$. For the linear ? a possibility is to add a binary relation R which is reflexive and for which

1. if $1Rx$ then $1 \leq x$,
2. if $(x \sqcap y)Rz$ then there are x_1, y_1 such that xRx_1, yRy_1 and $x_1 \sqcap y_1 \leq z$,
3. if $x \times yRz$ then there are x_1, y_1 such that xRx_1, yRy_1 and $x_1 \times y_1 \leq z$,

and then satisfaction is extended by $x \models ?\phi$ iff there is some y such that $1 \leq y \leq x$, y idempotent (i.e. $y \times y = y$) and for all z, if yRz then $z \models \phi$.

Acknowledgements

Many thanks to Colin Stirling for innumerable useful discussions on the topics discussed in the present paper. Thanks are due also to the referees for their useful

advice and for pointing out various errors.

References

[1] M. Abadi and G. Plotkin. A logical view of composition and refinement. In *Proc. Eighteenth Ann. ACM Symp. Principles of Programming Languages*, pages 323–332, 1991.

[2] S. Abramsky and R. Jagadeesan. New foundations for the geometry of interaction. Manuscript, 1991.

[3] S. Abramsky and S. Vickers. Quantales, observational logic, and process semantics. Technical Report DOC 90/1, Imperial College, 1990.

[4] G. Allwein and J. M. Dunn. Kripke models for linear logic. Manuscript, Indiana University, 1992.

[5] A. Avron. The semantics and proof theory of linear logic. *Theoretical Computer Science*, 57:161–184, (North-Holland, 1988).

[6] S. Brookes, C. Hoare, and A. Roscoe. A theory of communicating sequential processes. *Journal ACM*, 31:560–599, 1984.

[7] C. Brown. Relating Petri nets to formulae of linear logic. Technical Report ECS-LFCS-89-87, University of Edinburgh, 1989.

[8] M. Dam. *Relevance Logic and Concurrent Composition.* PhD thesis, Dept. of Computer Science, University of Edinburgh, 1990. CST–66–90. Also published as ECS–LFCS–90–119.

[9] E. Dijkstra. *A Discipline of Programming.* Prentice-Hall, 1976.

[10] J. M. Dunn. Relevance logic and entailment. In D. Gabbay, F. Guenthner (eds.), *Handbook of Philosophical Logic, vol. III,* D. Reidel, pages 117–224, 1986.

[11] U. Engberg and G. Winskel. Petri nets as models of linear logic. In *Proc. 15th Coll. Trees in Algebra and Programming (CAAP),* Lecture Notes in Computer Science, 431, 1990.

[12] V. Gehlot and C. Gunter. A proof-theoretic operational semantics for true concurrency. In *Proc. International Conference on Applications and Theory of Petri Nets*, 1989.

[13] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–101, 1987.

[14] M. Hennessy. Synchroous and asynchronous experiments on processes. *Information and Control*, 59:36–83, 1983.

[15] M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, **32**:137–162, 1985.

[16] M. Hennessy and G. Plotkin. Finite conjunctive nondeterminism. In K. Voss, H. J. Genrich, and G. Rozenberg (eds.), *Concurrency and Nets*, pages 233–244, 1987.

[17] K. G. Larsen. A context dependent equivalence between processes. *Theoretical Computer Science*, 49:185–216, 1987.

[18] N. Marti-Oliet and J. Meseguer. From petri nets to linear logic. In *Proc. Category Theory and Computer Science*, 1989.

[19] R. Milner. Calculi for synchrony and asynchrony. *Theoretical Computer Science*, 25:267–310, 1983.

[20] R. Milner. *Communication and Concurrency.* Prentice Hall International, 1989.

[21] R. De Nicola and M. Hennessy. Testing equivalences for processes. *Theoretical Computer Science*, 34:83–133, 1984.

[22] H. Ono and Y. Komori. Logics without the contraction rule. *Journal of Symbolic Logic*, 50:169–201, 1985.

[23] J. Parrow. Submodule construction as equation solving in CCS. In Proc. Foundations of Software Technology and Theoretical Computer Science, *Lecture Notes in Computer Science*, **287**:103–123, 1987.

[24] G. D. Plotkin. A structural approach to operational semantics. Aarhus University report DAIMI FN-19, 1981.

[25] A. Pnueli. In transition from global to modular temporal reasoning about programs. In *Logics and Models for Concurrent Systems* K. R. Apt (ed.), NATO ASI series, pages 123–144, 1984.

[26] R. Routley and R. K. Meyer. The semantics of entailment, i. In H. Leblanc (ed.) *Truth, Syntax and Modality,* North-Holland, pages 199–243, 1973.

[27] C. Stirling. Modal logics for communicating systems. *Theoretical Computer Science*, 49:311–347, 1987.

[28] A. Urquhart. Semantics for relevant logics. *Journal of Symbolic Logic*, 37:159–169, 1972.

[29] G. Winskel. A complete proof system for SCCS with modal assertions. *Lecture Notes in Computer Science*, 206:392–410, 1985.

[30] G. Winskel. A category of labelled Petri nets and compositional proof system. In *Proc. 3rd Annual Symposium on Logic in Computer Science*, pages 142–154, 1988.