# Small Space to Small Width in Resolution

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## Space to Width: Old and New Proof

Resolution is the most well-studied proof system in proof complexity, and much research has gone into understanding not only proof size but also other measures such as width and space. In 2003, Atserias and Dalmau [AD03] resolved a major open question by proving that width is always a lower bound on space. We give a completely elementary alternative proof of the same result.

### Old Proof

- Finite model theory
- Ehrenfeucht-Fraïssé games
- No small width proof ⇒ Duplicator's strategy
- Duplicator's strategy ⇒ No small space proof

### New Proof

- Syntactic manipulation
- Negating whiteboards
- Small space whiteboard ⇒ Small width negative whiteboard
- Run proof in **reverse**

## Resolution

**CNF formula** = set of clauses (disjunctions)

Prove on **whiteboard** that CNF formula is unsatisfiable by deriving empty clause $\bot$.
Allowed to:

- Write down clauses of formula (axioms)
- Infer new clauses by resolution $\frac{C \vee x \quad D \vee \overline{x}}{C \vee D}$ or weakening $\frac{C}{C \vee D}$
- Erase clauses (to save space)

## Space and Width

**Space:** # clauses on whiteboard
*Example:* space $= 3$

**Width:** size of largest clause
*Example:* width $= 2$

$u$
$\overline{x} \vee z$
$y \vee u$

## Small Space

$\overline{z}$ | $\overline{z}$, $\overline{x} \vee \overline{y} \vee z$ | $x \vee y$ | $\overline{x} \vee \overline{y}$, $\overline{u} \vee x$ | $\overline{u} \vee \overline{y}$ | $\overline{u} \vee \overline{y}$, $\overline{u} \vee y$ | $\overline{u}$ | $\overline{u}$, $u$ | $\bot$

### Axiom Clauses

$u \quad \wedge$
$(\overline{u} \vee x) \quad \wedge$
$(\overline{u} \vee y) \quad \wedge$
$(\overline{x} \vee \overline{y} \vee z) \quad \wedge$
$\overline{z}$

### Negative Whiteboard

$\neg((\overline{u} \vee \overline{y}) \wedge (\overline{u} \vee y))$ → $(u \wedge y) \vee (u \wedge \overline{y})$ → $u \wedge (u \vee \overline{y}) \wedge (y \vee u) \wedge (y \vee \overline{y})$ → $u$

1. Negate whiteboard
2. Apply De Morgan's laws
3. Expand to CNF
4. Remove trivial and subsumed clauses

## Small Width

$\bot$ | $z$ | $z \vee x$, $z \vee y$ | $x$, $y$ | $x \vee u$, $y \vee u$, $y \vee \overline{x}$ | $u$, $y$ | $u$ | $u$ | | |

## Axiom Writing in Reverse

Derive clause $z$ from:

- Whiteboard $(z \vee x) \wedge (z \vee y)$
- Axiom clause $\overline{x} \vee \overline{y} \vee z$.

$$\dfrac{\dfrac{\overline{x} \vee \overline{y} \vee z \qquad z \vee x}{\overline{y} \vee z} \qquad z \vee y}{z}$$

Can be done for all axiom writing steps.

## Width of Reversed Proof

For a CNF formula $F$:

- Width of negative whiteboard $\leq$ Space of whiteboard
- Axiom writing in reverse ⇒ Extra width bounded by width of $F$

**Width $\leq$ Space + Formula width**

**Open Problem:** Relation between degree and space in Polynomial Calculus?

## References

[AD03] A. Atserias and V. Dalmau. A combinatorial characterization of resolution width. In *Proc. 18th Conference on Computational Complexity (CCC '03)*, 2003.

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