A Generalized Method for Proving Polynomial Calculus Degree Lower Bounds

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30th Annual Computational Complexity Conference

Portland, Oregon, USA
19 June 2015

Joint work with Jakob Nordström

Introduction

Topic: Proof complexity

Focus: Polynomial calculus (Gröbner basis calculations)

Goal: Degree lower bounds ($\Rightarrow$ Size lower bounds)

Lower Bounds via Expansion

Standard approach: Lower bounds from expansion.

Simplest example: Clause-variable incidence graph (CVIG).

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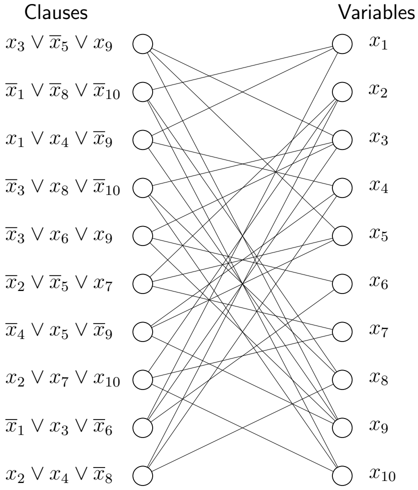

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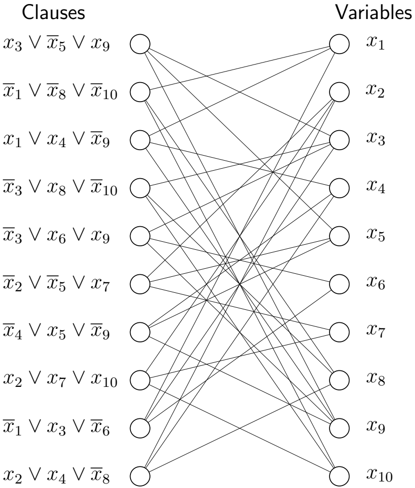
Subsets of left vertices have many unique neighbors on right.



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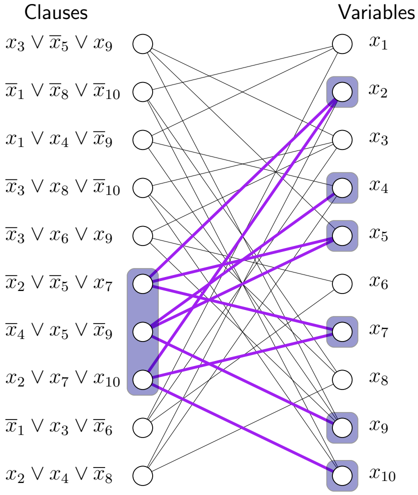

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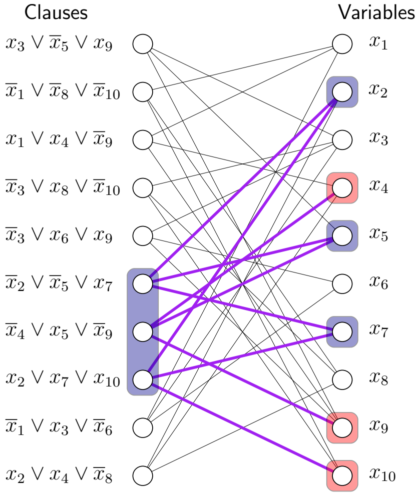

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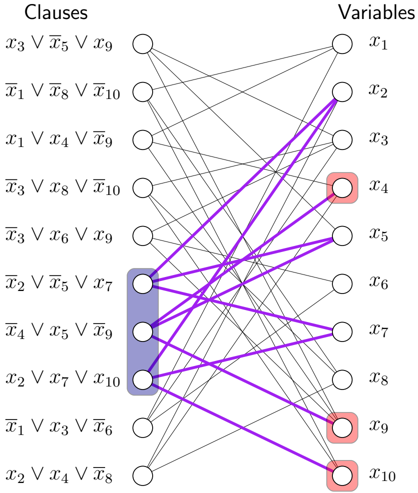

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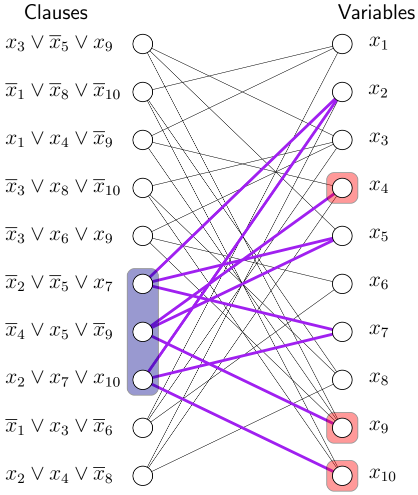

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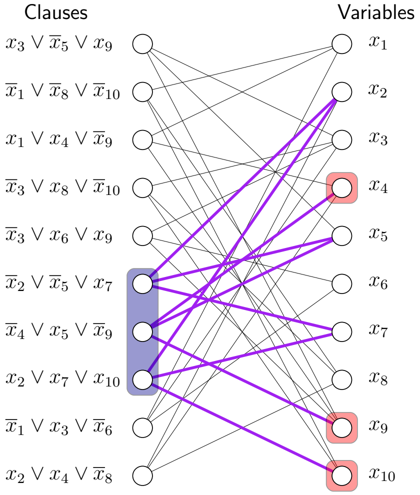
Need graph capturing underlying principle!



Our Results

Main Theorem (Informal)

Graph structure on formula such that expansion implies hardness in polynomial calculus.

Extends an approach from [Alekhnovich, Razborov '01].

Unifies (almost) all previous lower bounds.

Corollary

Functional pigeonhole principle is hard for polynomial calculus.

Resolves question in [Razborov '02].

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Warm-up: Use resolution to present main ideas and challenges.

Resolution

- **Input:** CNF formula $\mathcal{F}$

$$(x \vee \overline{y} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (x \vee y) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{x} \vee z)$$

- **Resolution rule:**

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

- **Goal:** Proof of unsatisfiability (refutation) = Derive empty clause $\perp$

Refer to clauses of formula as axioms.

Complexity Measures for Resolution

Size: number of steps in proof

Width: size of the largest clause

1.	$x \lor \overline{y} \lor z$	Axiom
2.	$\overline{y} \lor \overline{z}$	Axiom
3.	$x \lor \overline{y}$	Res$(1, 2)$
4.	$x \lor y$	Axiom
5.	x	Res$(3, 4)$
6.	$\overline{x} \lor \overline{z}$	Axiom
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Theorem (Ben-Sasson, Wigderson '99)

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Width lower bounds via expansion argument.

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Example: Tseitin Formulas

Given set of equations over $\mathbb{F}_2$.

$$
\begin{aligned}
x + w &= 0 \\
x + y &= 0 \\
y + w + z &= 1 \\
z &= 0
\end{aligned}
$$

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Encode as clauses.

Clauses

$x \vee \overline{w}$

$\overline{x} \vee w$

$x \vee \overline{y}$

$\overline{x} \vee y$

$y \vee w \vee z$

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$y \vee \overline{w} \vee \overline{z}$

$\overline{z}$

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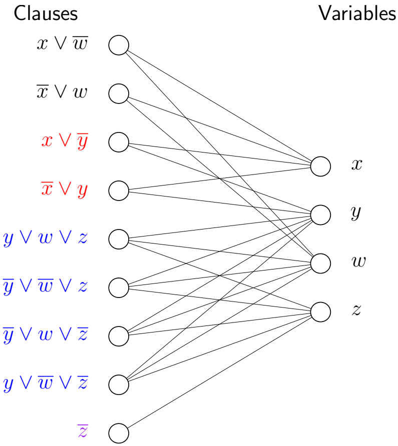

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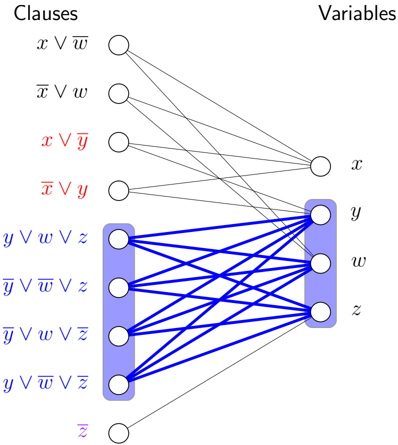

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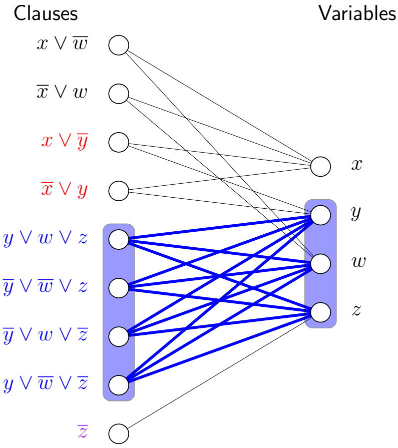
Encode as clauses.

Does CVIG expand? No!

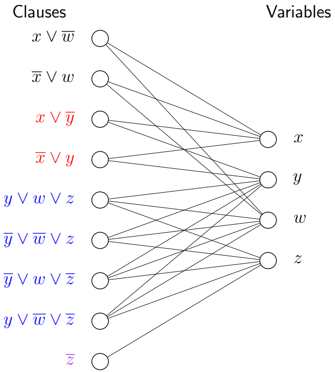
Graph should encode equations not clauses!

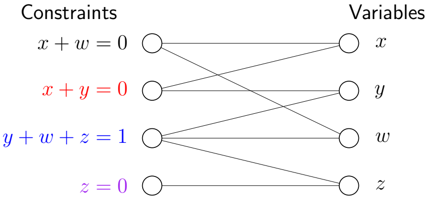


Constraint-Variable Incidence Graph

Have single vertex for each constraint on the left.

Put edge if variable appears in constraint.





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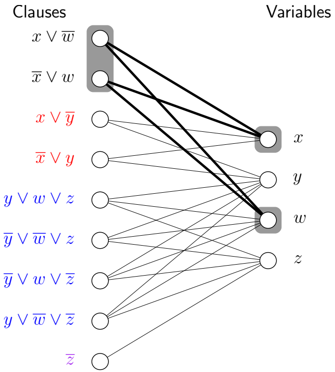

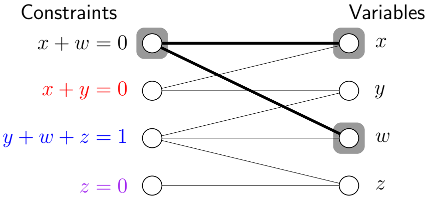

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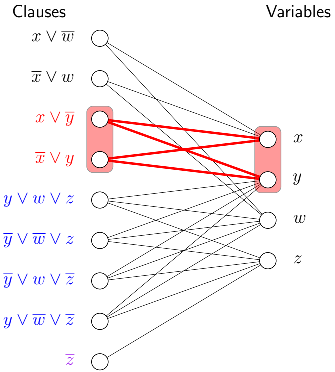

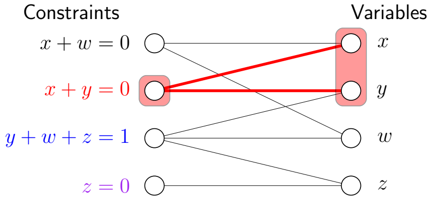

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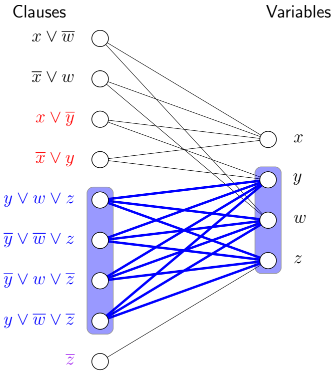

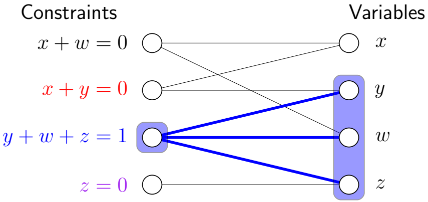

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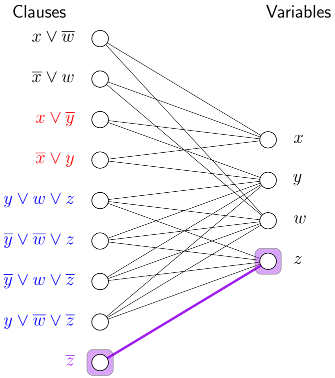

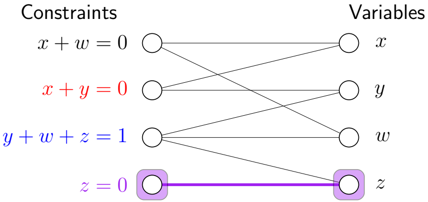

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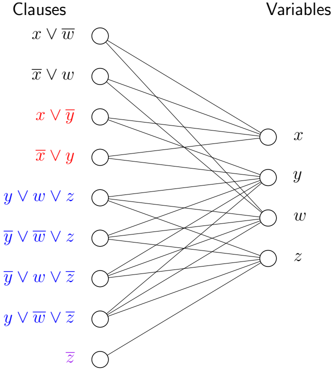

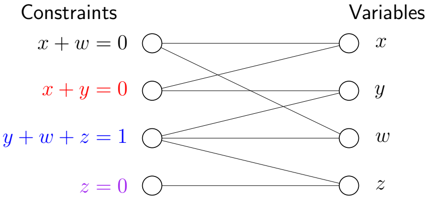

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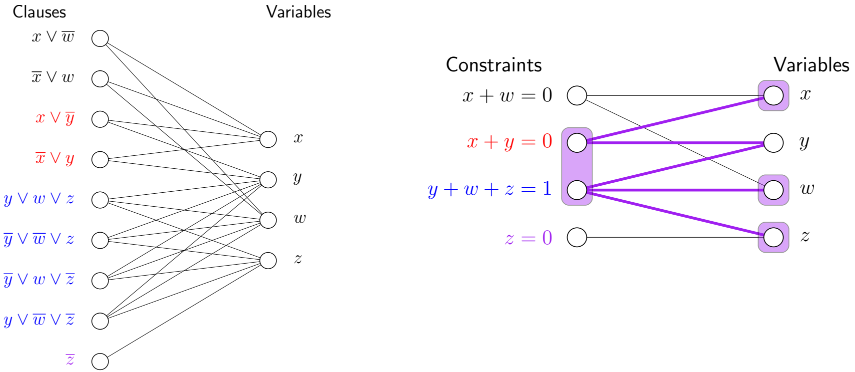
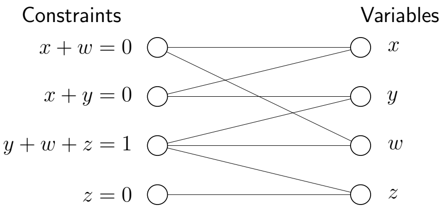


The constraint-variable incidence graph expands!

Proof Sketch of Tseitin Lower Bound



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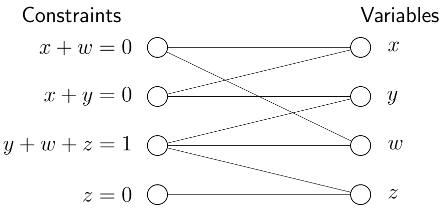

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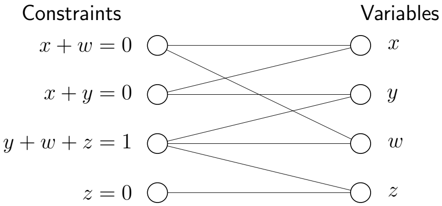

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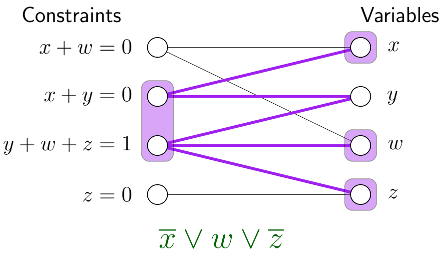

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Contradiction $\perp$:	All constraints
Halfway through:	Clause C depending on medium-sized set S

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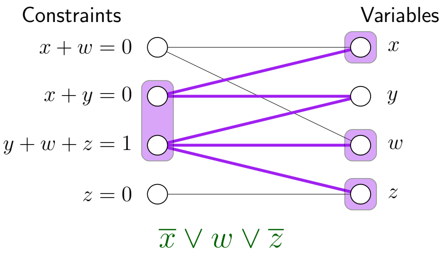

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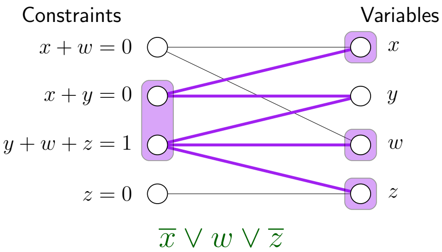

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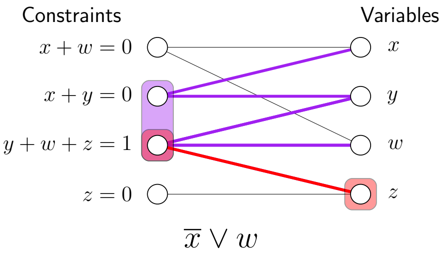

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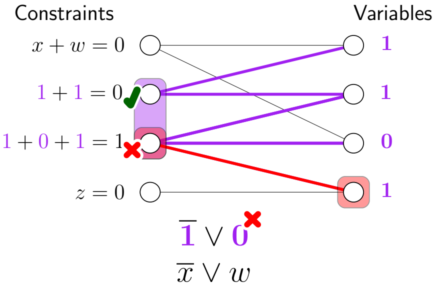

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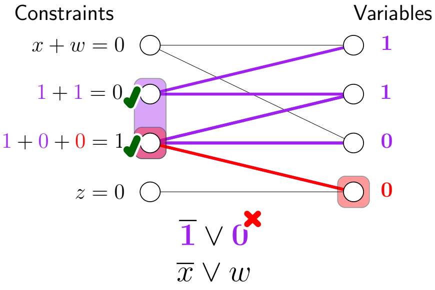

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Resolution Lower Bounds

Resolution edge game on (P, x)

1. Adversary provides assignment ρ to all variables.
2. Can flip x to some b so that P is satisfied.

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Theorem (Ben-Sasson, Wigderson '99)

If from formula $\mathcal{F} = \bigwedge_{P \in \mathcal{F}} P$, we can form graph $\mathcal{G}(\mathcal{F})$ such that

- *$\mathcal{G}(\mathcal{F})$ is expanding, and*
- *for all edges (P, x), P is satisfied by flipping x,*

then refuting $\mathcal{F}$ requires large width.

Polynomial Calculus [CEI '96, ABRW '00]

Lines are polynomial equations over some field $\mathbb{F}$.

- **Input:** Polynomial equations encoding Boolean constraints

  Clause encoded as: $x \vee \overline{y} \vee z \;\rightarrow\; \overline{x} y \overline{z} = 0$

  Additional axioms: $x^2 - x = 0$ and $x + \overline{x} - 1 = 0$

- **Linear combination:**

$$\frac{p = 0 \qquad q = 0}{\alpha p + \beta q = 0}$$

- **Variable multiplication:**

$$\frac{p = 0}{xp = 0}$$

- **Goal:** Derive $1 = 0$ showing that constraints are unsatisfiable

Complexity Measures for Polynomial Calculus

Size: number of monomials in proof

Degree: max degree of monomial

1.	$x\overline{y} = 0$	Axiom
2.	$y = 0$	Axiom
3.	$y + \overline{y} - 1 = 0$	Axiom
4.	$\overline{y} - 1 = 0$	Lin$(2, 3)$
5.	$x\overline{y} - x = 0$	Mul$(4, x)$
6.	$x = 0$	Lin$(1, 5)$
7.	$x + \overline{z} + 1 = 0$	Axiom
8.	$\overline{z} + 1 = 0$	Lin$(6, 7)$
9.	$\overline{z} = 0$	Axiom
10.	$1 = 0$	Lin$(8, 9)$

Complexity Measures for Polynomial Calculus

Size: number of monomials in proof 17

Degree: max degree of monomial

1.	$x\overline{y}^{\mathbf{1}} = 0$	Axiom
2.	$y^{\mathbf{2}} = 0$	Axiom
3.	$y^{\mathbf{3}} + \overline{y}^{\mathbf{4}} - 1^{\mathbf{5}} = 0$	Axiom
4.	$\overline{y}^{\mathbf{6}} - 1^{\mathbf{7}} = 0$	Lin$(2, 3)$
5.	$x\overline{y}^{\mathbf{8}} - x^{\mathbf{9}} = 0$	Mul$(4, x)$
6.	$x^{\mathbf{10}} = 0$	Lin$(1, 5)$
7.	$x^{\mathbf{11}} + \overline{z}^{\mathbf{12}} + 1^{\mathbf{13}} = 0$	Axiom
8.	$\overline{z}^{\mathbf{14}} + 1^{\mathbf{15}} = 0$	Lin$(6, 7)$
9.	$\overline{z}^{\mathbf{16}} = 0$	Axiom
10.	$1^{\mathbf{17}} = 0$	Lin$(8, 9)$

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Complexity Measures for Polynomial Calculus

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Degree: max degree of monomial 2

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2.	$y = 0$	Axiom
3.	$y + \overline{y} - 1 = 0$	Axiom
4.	$\overline{y} - 1 = 0$	Lin$(2, 3)$
5.	$x\overline{y} - x = 0$	Mul$(4, x)$
6.	$x = 0$	Lin$(1, 5)$
7.	$x + \overline{z} + 1 = 0$	Axiom
8.	$\overline{z} + 1 = 0$	Lin$(6, 7)$
9.	$\overline{z} = 0$	Axiom
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Complexity Measures for Polynomial Calculus

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$$\mathbf{Size} \gtrsim \exp(\mathbf{Degree})$$

Used in:

- Buss, Grigoriev, Impagliazzo, Pitassi '99
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Polynomial calculus exponentially stronger than resolution.

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1. Commit to assignment $x = b$ ahead of time.
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Can't win this game for Tseitin.

Main Theorem (Tentative Version)

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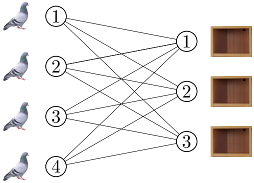
Not enough to prove functional pigeonhole principle hard!

Pigeonhole Principle (PHP)

Statement: $n+1$ pigeons can fit into n holes

Variable $x_{1,3}$ is true if pigeon 1 sits in hole 3



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Pigeons

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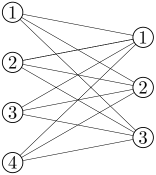

Holes

$\overline{x}_{1,1} \vee \overline{x}_{2,1}$ $\quad$ $\overline{x}_{2,1} \vee \overline{x}_{3,1}$
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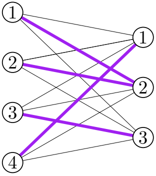

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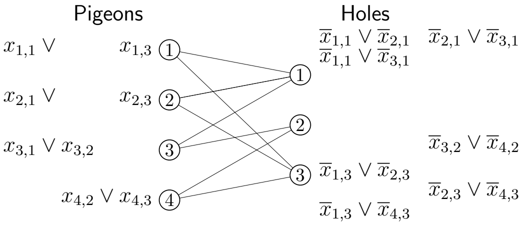

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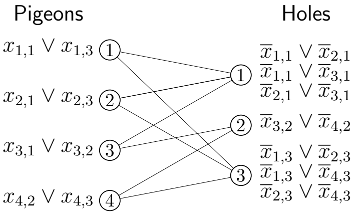

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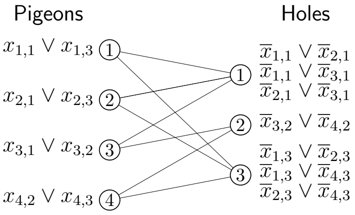

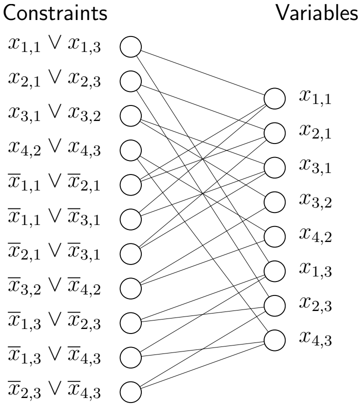

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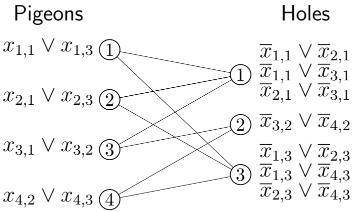

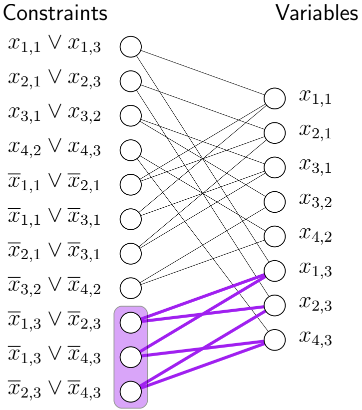

Again CVIG not expanding!

Proving PHP Lower Bound

Isolate hole axioms from graph and group hole variables together!

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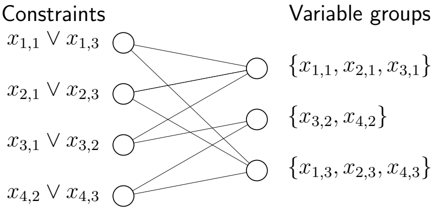

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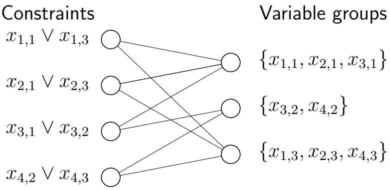

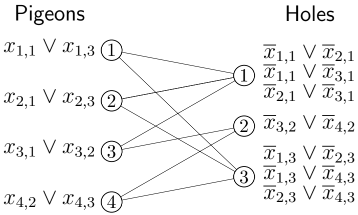

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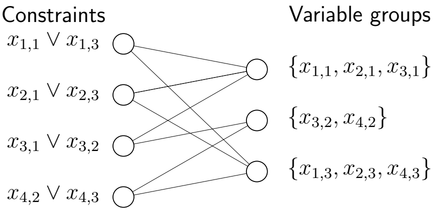

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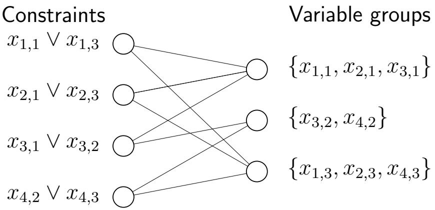

Change the game: Assign group so that hole axioms (E) aren't violated!

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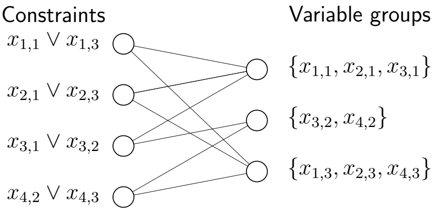

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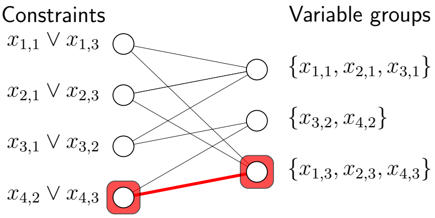

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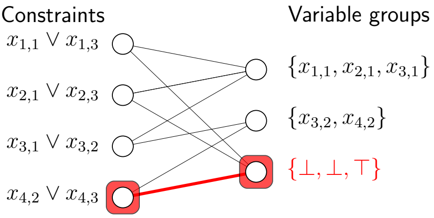

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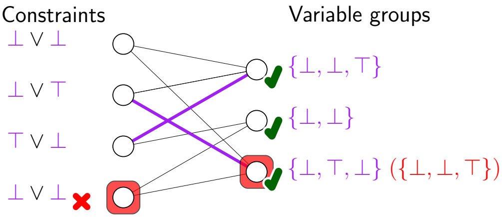

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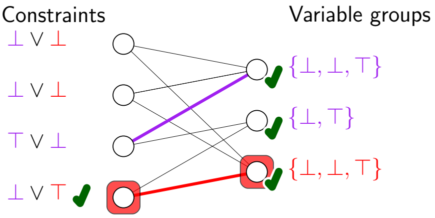

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Gives common framework for previous lower bounds.

- Expanding CNF [Alekhnovich, Razborov '01]
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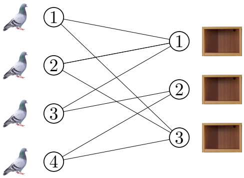
Proves functional PHP hard.

PHP Variants



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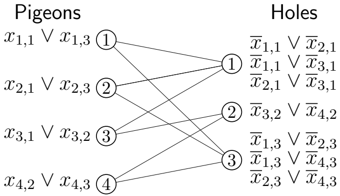

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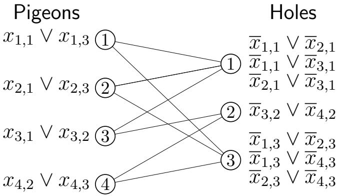

- Can have fat pigeons which are assigned to multiple holes.

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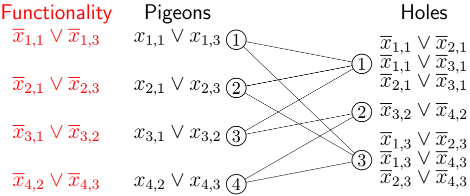

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  ⇒ Add functionality axioms (makes mapping 1-to-1).

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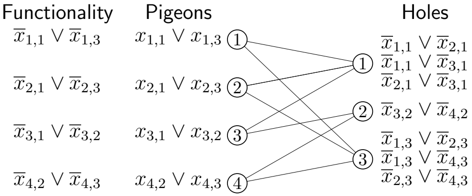

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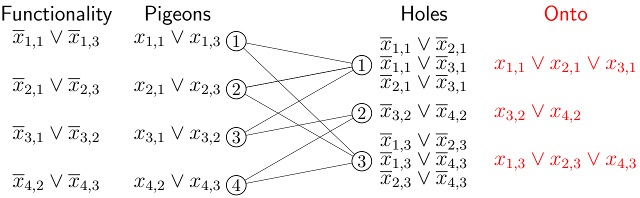
  $\Rightarrow$ Add functionality axioms (makes mapping 1-to-1).
- Can have hole with no pigeons.

PHP Variants



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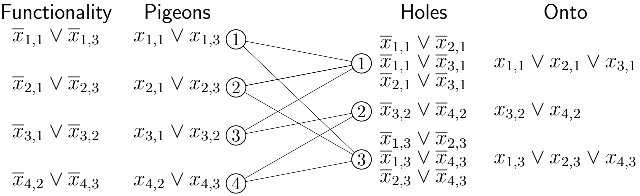

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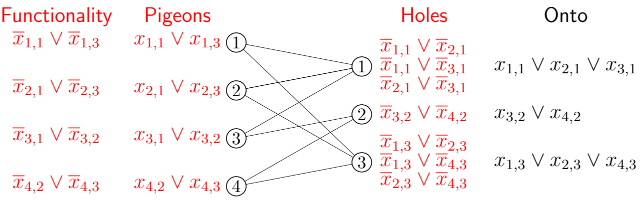

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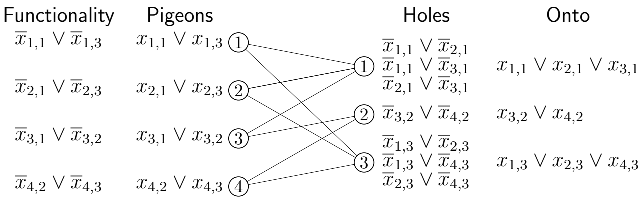
Functional PHP = PHP + Functionality

PHP Variants



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PHP Variants

Functionality	Pigeons	Holes	Onto
$\overline{x}_{1,1} \lor \overline{x}_{1,3}$	$x_{1,1} \lor x_{1,3}$ (1)	(1) $\overline{x}_{1,1} \lor \overline{x}_{2,1}$, $\overline{x}_{1,1} \lor \overline{x}_{3,1}$, $\overline{x}_{2,1} \lor \overline{x}_{3,1}$	$x_{1,1} \lor x_{2,1} \lor x_{3,1}$
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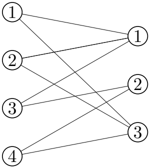
$\overline{x}_{3,1} \lor \overline{x}_{3,2}$	$x_{3,1} \lor x_{3,2}$ (3)	(3) $\overline{x}_{1,3} \lor \overline{x}_{2,3}$, $\overline{x}_{1,3} \lor \overline{x}_{4,3}$, $\overline{x}_{2,3} \lor \overline{x}_{4,3}$	$x_{1,3} \lor x_{2,3} \lor x_{4,3}$
$\overline{x}_{4,2} \lor \overline{x}_{4,3}$	$x_{4,2} \lor x_{4,3}$ (4)		



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Onto-PHP = PHP + Onto

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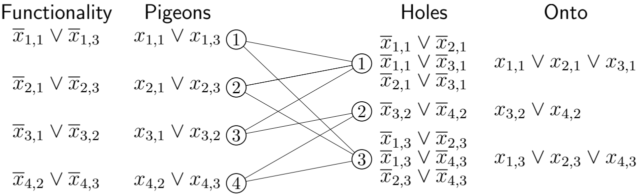

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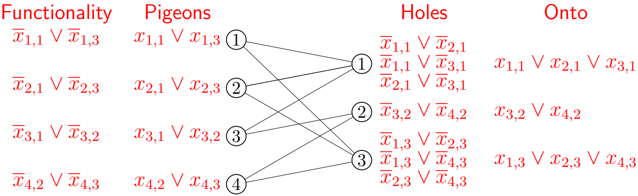
PHP Variants



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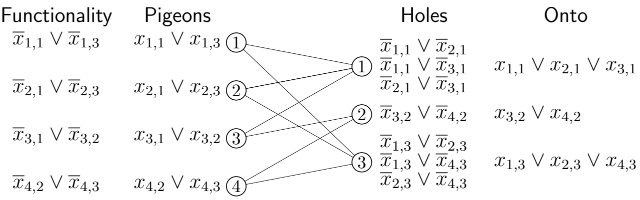

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This work

Observe that [AR '01] proves hardness of Onto-PHP.

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Hardness of PHP Variants

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FPHP	hard [Haken '85]	hard [MN '15]
Onto-PHP	hard [Haken '85]	hard [AR '01]
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This work

Observe that [AR '01] proves hardness of Onto-PHP.

Prove that FPHP is hard in polynomial calculus.

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- Prove polynomial calculus lower bounds for other formulas!

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- Find truly general method capturing all PC degree lower bounds!

  We generalize part of [AR '01] that doesn't capture [BGIP '99].

Summary

Generalized method for degree lower bounds

- Unified framework for previous lower bounds.
- Hardness of Functional PHP.

Further directions

- Extend techniques to other formulas.
- Devise non-degree-based size lower bound techniques.

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Thank you for your attention!