Long Proofs of (Seemingly) Simple Formulas

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Joint work with Jakob Nordström

SAT Solvers: Triumphs and Limits

State-of-the-art CDCL SAT solvers

Very successful in practice, can solve instances with millions of variables

Exist small instances with just a few hundred variables that are hard

Natural questions

1. When does CDCL work and why?
2. What is the smallest formula that is infeasible in practice?

This work: Focus on second question

(Some) Theoretical Hardness Results

Pigeonhole principle [Haken '85]

Claims existence of matching between $n + 1$ pigeons and n holes

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Random 3-CNF formulas [Chvátal, Szemerédi '88]

Randomly sampled 3-CNF formula

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In practice become infeasible at around 200-300 variables

Small Infeasible Instances

Spence '10 proposed formulas based on cardinality constraints

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Showed experimentally that these formulas harder than other benchmarks

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Our results

Theory: Prove exponential lower bounds

Experiments: Compare theory and practice

Resolution

Resolution — Basis for CDCL solvers

- **Input:** CNF formula F

  $(x \vee \overline{y} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (x \vee y) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{x} \vee z)$

- **Resolution rule:**

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

- **Goal:** Proof of unsatisfiability (refutation)
  = Derive empty clause $\perp$

This talk: All formulas unsatisfiable

Example refutation

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Refutation length: $\#$ clauses in refutation

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Subset Cardinality Formulas

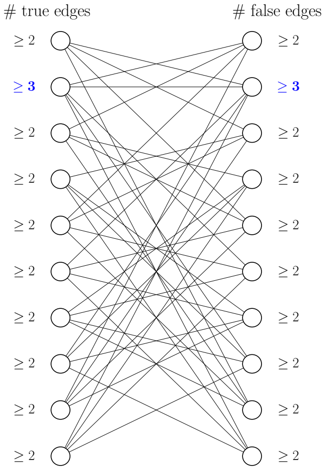
Cardinality constraints on 4-regular bipartite graph with one added edge

Variables: edges in graph

Clauses: vertex cardinality constraints

Left: majority of edges true

Right: majority of edges false



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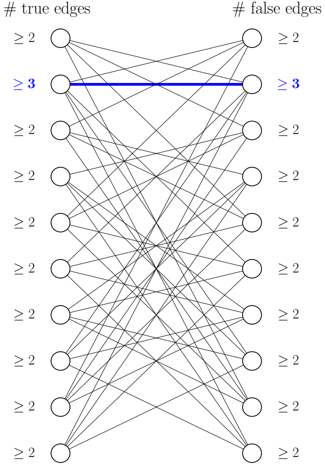

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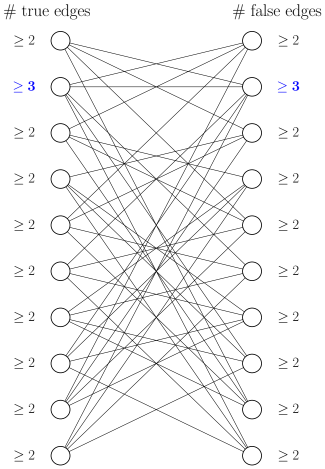

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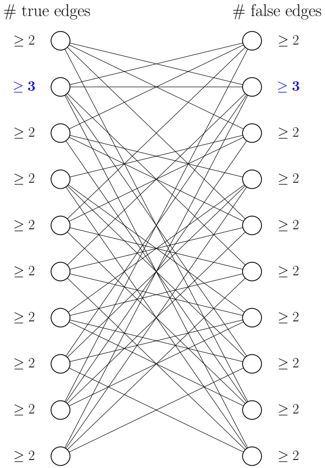
$\Rightarrow$ Loops of length 4 are forbidden



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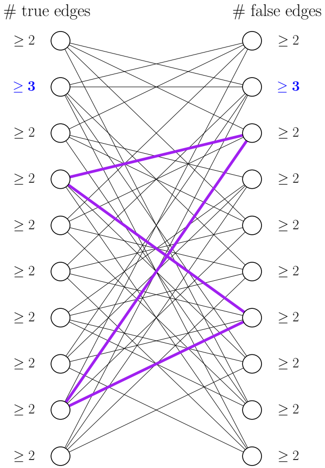


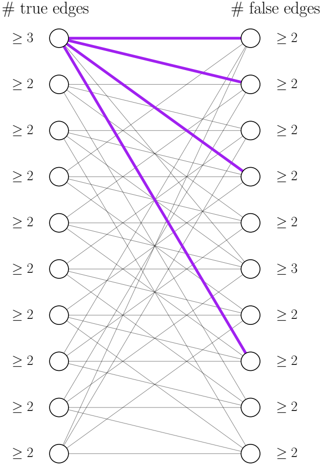
Subset cardinality formula

Fixed Bandwidth Formulas

Fixed bandwidth

Variant of subset cardinality formulas

Neighbors defined by cyclic shifts of original pattern



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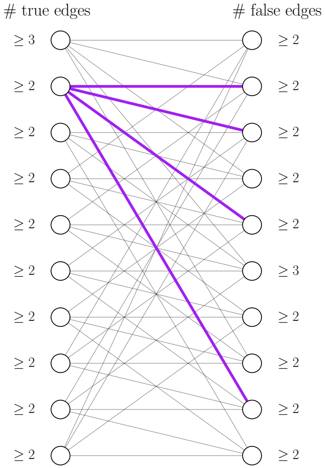

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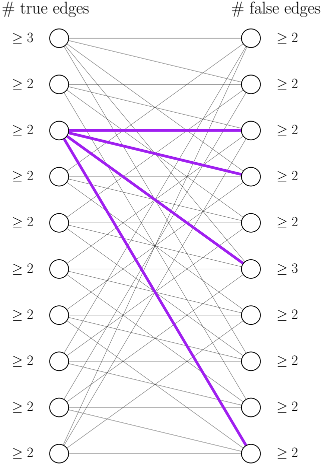

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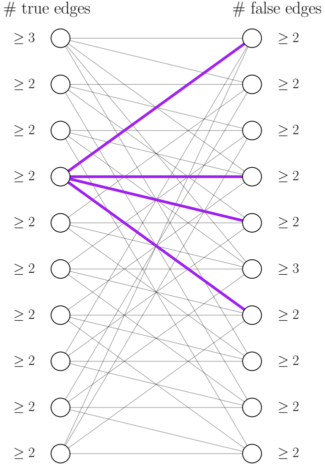

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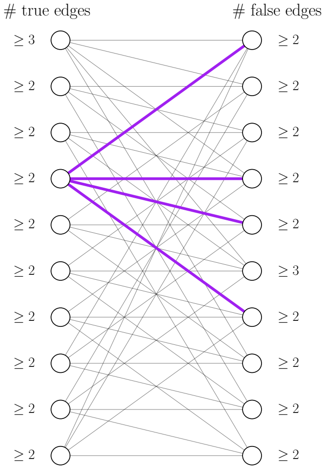
- Quadrangle-free
- Polynomial length refutations [Van Gelder and Spence '10]



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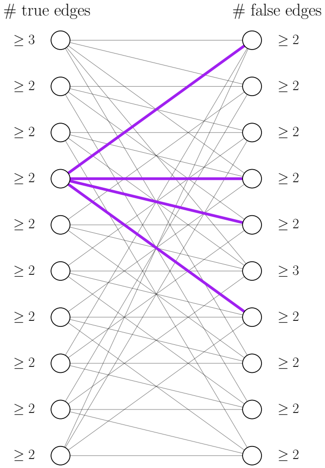
⇒ Quadrangle-freeness not sufficient for lower bounds

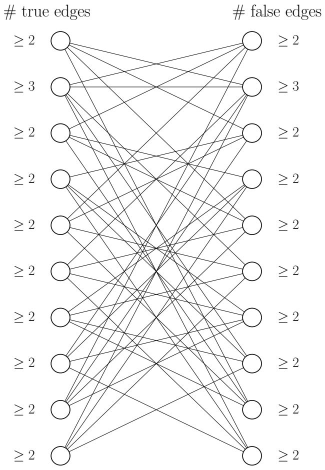


Fixed bandwidth

Expanders Graphs and Subset Cardinality Formulas

Expander graphs are ubiquitous in theoretical computer science

Bipartite expanders:
Subsets of left vertices have many neighbors on right



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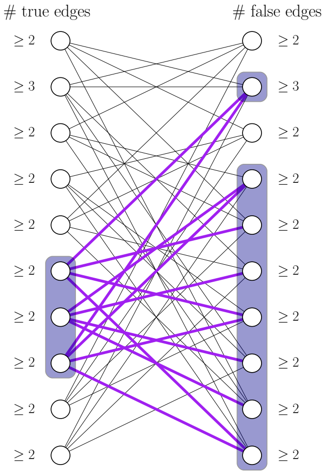

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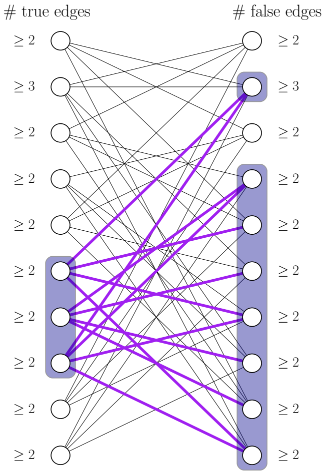
  $\Rightarrow$ Exponential lower bound



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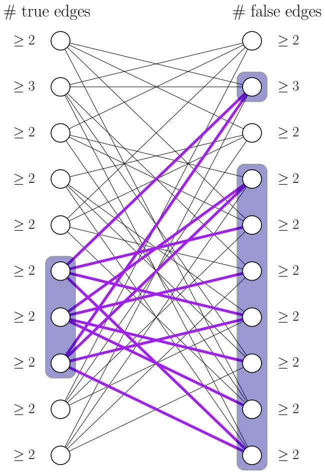
Fixed bandwidth graphs:
Decent neighbor spread, but deteriorates as size grows



Key Connection

Pigeonhole principle (PHP) formulas

Matching in complete bipartite graph with $n+1$ left and n right vertices

Graph PHP: hard even for non-complete graph if it's expanding

Subset cardinality formulas

Similar flavor to graph pigeonhole principle formulas

Key insight: reduce subset cardinality formulas to graph PHP formulas

Standard Tool: Restrictions

$$(x \vee \overline{y} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (x \vee y) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{x} \vee z)$$

Restriction:

Assign truth value to some variables and simplify formula

True literal: remove clause

False literal: shrink clause

Example refutation

1.	$x \vee \overline{y} \vee z$	Axiom
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Standard fact:

Restrictions preserve refutations

Lower bound on restricted formula
$\Rightarrow$ Lower bound on original formula

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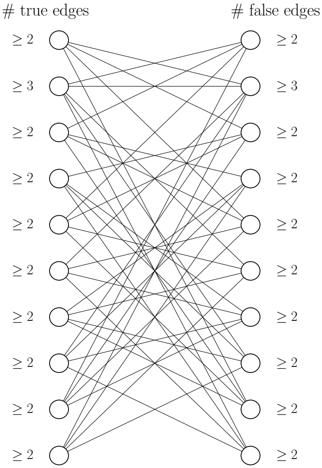
From Subset Cardinality to Pigeonhole Principle Formulas

Restriction:

- Take added edge



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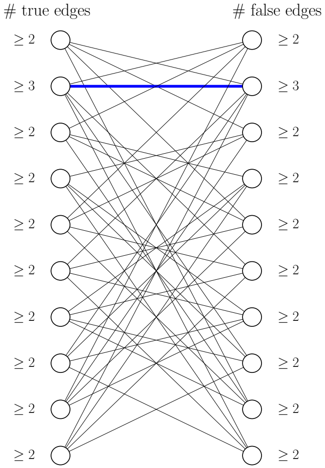
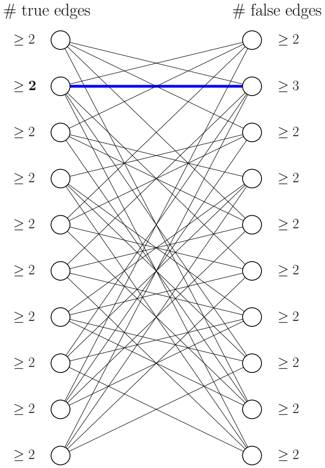


From Subset Cardinality to Pigeonhole Principle Formulas

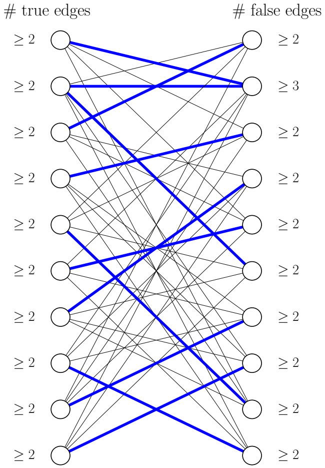
Restriction:

- Take added edge $\rightarrow$ set to true



Restriction:

- Take added edge → set to true
- Find matching



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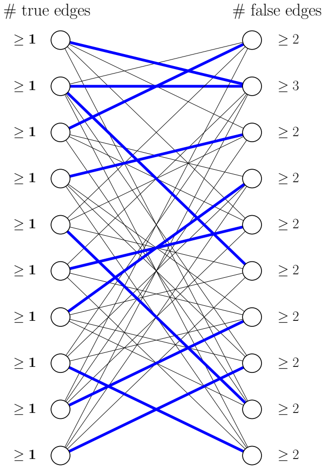

From Subset Cardinality to Pigeonhole Principle Formulas

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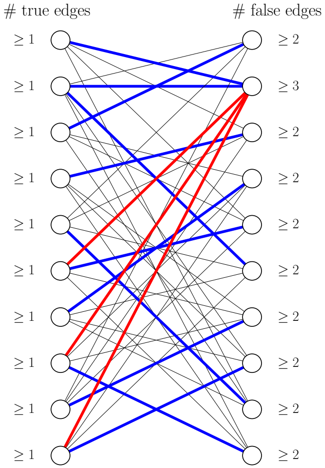

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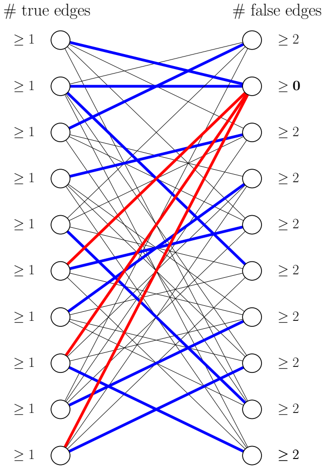

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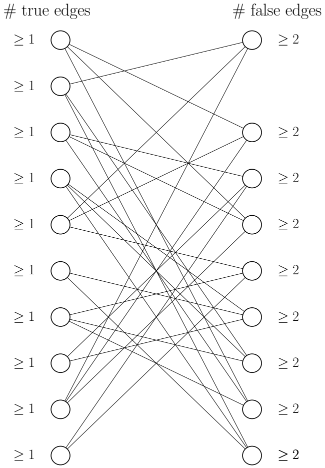
Restricted formula is graph pigeonhole principle



Graph Pigeonhole Principle Lower Bound

Prove lower bound for resolution and polynomial calculus (extending resolution with algebraic reasoning)

Use techniques in:

[Ben-Sasson and Wigderson '99] for resolution

[Alekhnovich and Razborov '01] for polynomial calculus

Cannot use quite as black box, but need minor technical tweaks

Putting the Pieces Together

- GPHP graph is expander $\Rightarrow$ Exponential lower bound for GPHP
- Restriction $\Rightarrow$ Exponential lower bound for subset cardinality formulas
- Random graphs expand almost surely (as size of graph increases)

Theorem

Random instances of **subset cardinality formulas** *are almost surely* **exponentially hard** *to refute in resolution and polynomial calculus.*

Experiments

Formulas:

- Subset cardinality formulas on random 4-regular graphs
- Fixed bandwidth formulas

Compared with:

- Random 3-CNF formulas
- Tseitin formulas on random 3-regular graphs

Equipment:

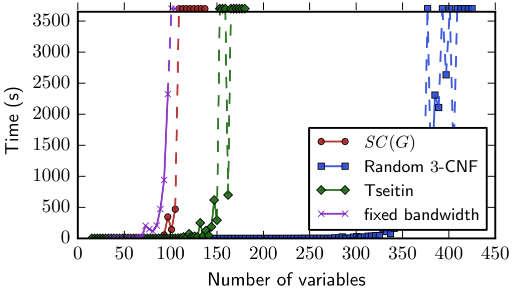
Computer with 2 quad-core AMD Opteron 2.2 Ghz CPUs (2374 HE) and 16 GB of memory

Only one solver running on computer at any given time

Timeout: 1 hour

Solvers used: Glucose 2.2, March-rw, Lingeling-ala

Subset Cardinality Formulas Vs. Other Benchmarks



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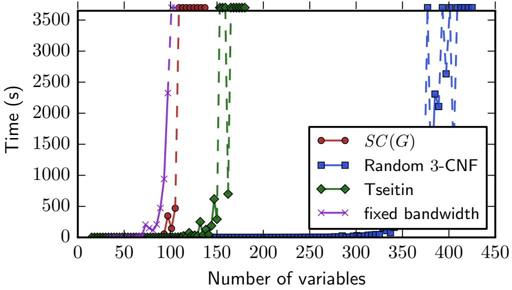

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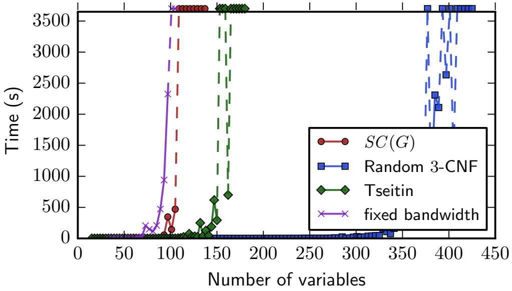

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Intriguingly, fixed bandwidth formulas hardest, although easy in theory!

Fixed Bandwidth Formulas Harder than Random Instances

Question: Are hidden constants in polynomial upper bound too large?

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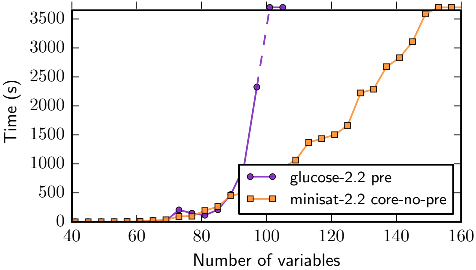

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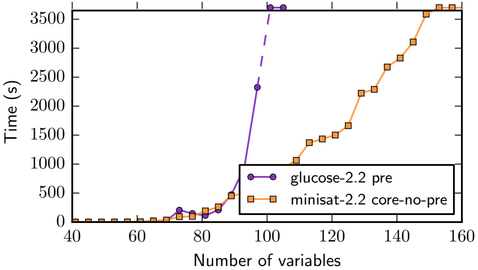

Doesn't seem like it. Manually fixed variable ordering: Solver runs faster

Maybe expansion is still the cause?
⇒ Higher for fixed bandwidth than random graphs in tested range

Open Problems

Why are fixed bandwidth instances hardest in practice?

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2. CDCL with current heuristics doesn't fully explore power of resolution

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Challenge formula from [VS '10] solved in 0.5 seconds by Sat4j

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Hard formulas for CDCL+Gaussian elimination+Cardinality constraints?

Investigate formulas in [Markström '06]

Prove resolution (or even polynomial calculus) lower bounds

Compare practical hardness with that of subset cardinality formulas

Summary

Our contributions

- Theoretical hardness of subset cardinality formulas
- Experimental comparison with standard benchmarks

Conclusions

- Subset cardinality formulas remain hardest for CDCL solvers
- Tested range: fixed bandwidth formulas harder than random instances

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- Find small, hard instances even for solvers using algebraic and cardinality reasoning
- Understand power of CDCL with currently used heuristics compared to resolution

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Thank you for your attention!