From Small Space to Small Width in Resolution

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Joint work with Yuval Filmus, Massimo Lauria, Jakob Nordström, and Marc Vinyals

Resolution

Resolution

- **Input:** CNF formula F

$$(x \vee \overline{y} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (x \vee y) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{x} \vee z)$$

- **Resolution rule:**

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

- **Goal:** Proof of unsatisfiability (refutation) = Derive empty clause $\perp$

Refer to clauses of formula as axioms

Resolution Size, Space, and Width

Can represent refutation as

- annotated list or
- DAG

1.	$x \vee \overline{y} \vee z$	Axiom
2.	$\overline{y} \vee \overline{z}$	Axiom
3.	$x \vee \overline{y}$	Res$(1, 2)$
4.	$x \vee y$	Axiom
5.	x	Res$(3, 4)$
6.	$\overline{x} \vee \overline{z}$	Axiom
7.	$\overline{x} \vee z$	Axiom
8.	$\overline{x}$	Res$(6, 7)$
9.	$\bot$	Res$(5, 8)$

Resolution Size, Space, and Width

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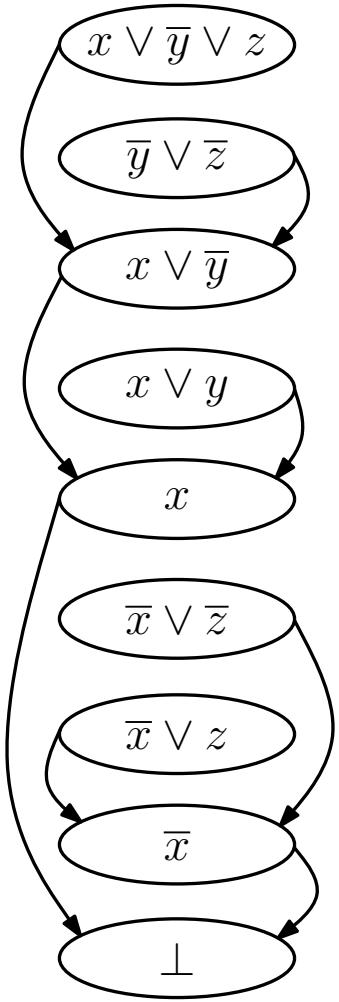

Resolution Size, Space, and Width

Can represent refutation as

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Size: number of steps in refutation

Space: memory usage (at step t: # clauses before t used after t)

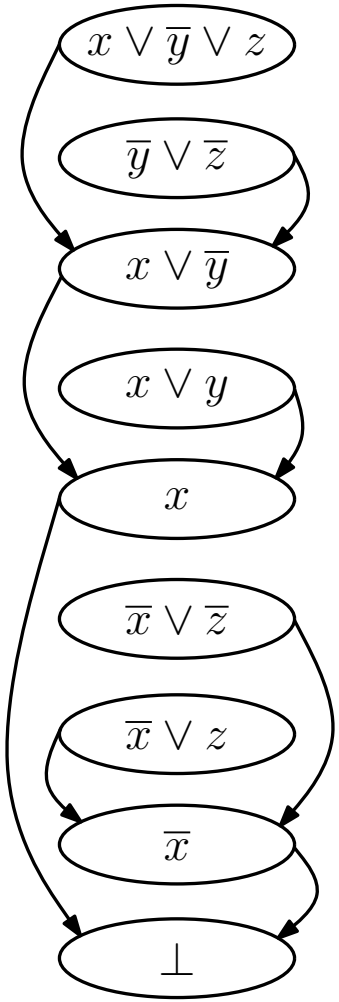
Width: size of the largest clause

Example:

Size

Space

Width



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Example:

Size 9
Space
Width

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2.	$\overline{y} \vee \overline{z}$	Axiom
3.	$x \vee \overline{y}$	Res$(1, 2)$
4.	$x \vee y$	Axiom
5.	x	Res$(3, 4)$
6.	$\overline{x} \vee \overline{z}$	Axiom
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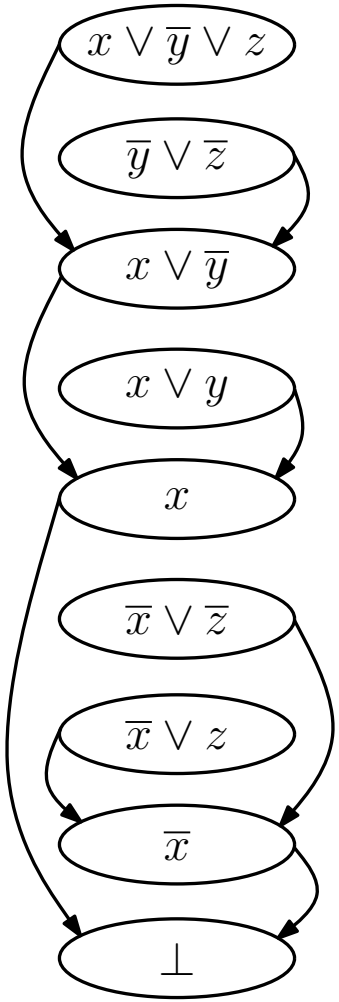

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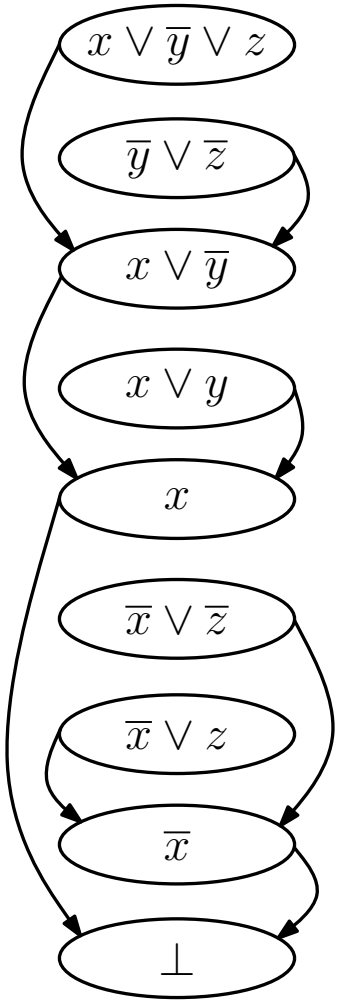
Example:

Size 9
Space 0
Width



Space at current step 0

Resolution Size, Space, and Width

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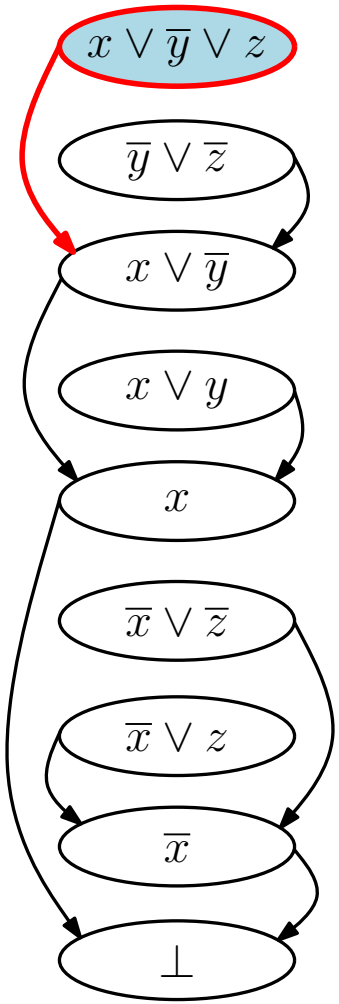
Example:

Size 9
Space 1
Width



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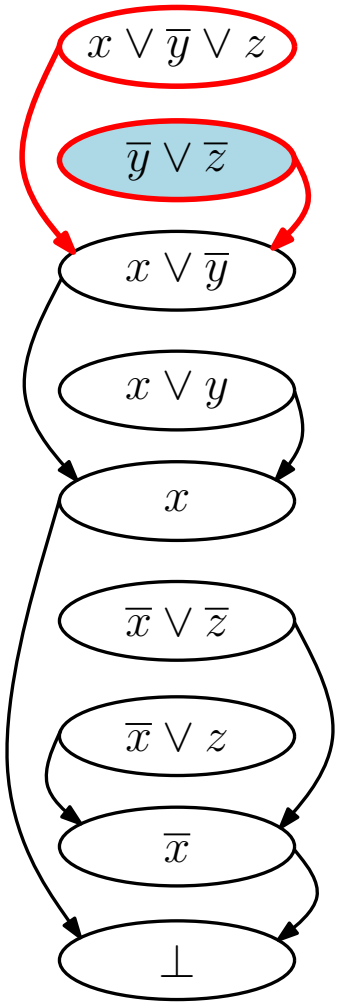
Example:

Size	9
Space	2
Width	



Space at current step 2

Resolution Size, Space, and Width

Can represent refutation as

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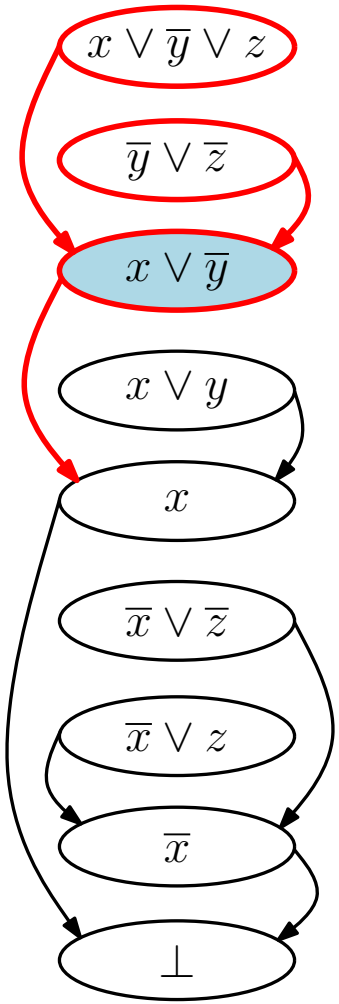
Example:

Size	9
Space	3
Width	



Space at current step 3

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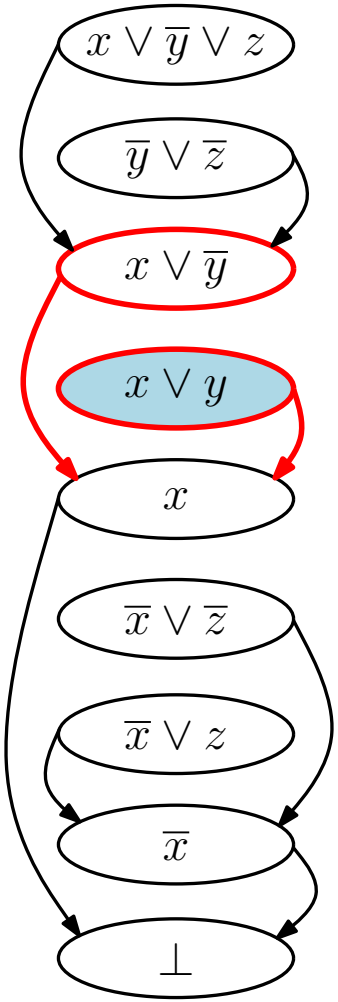

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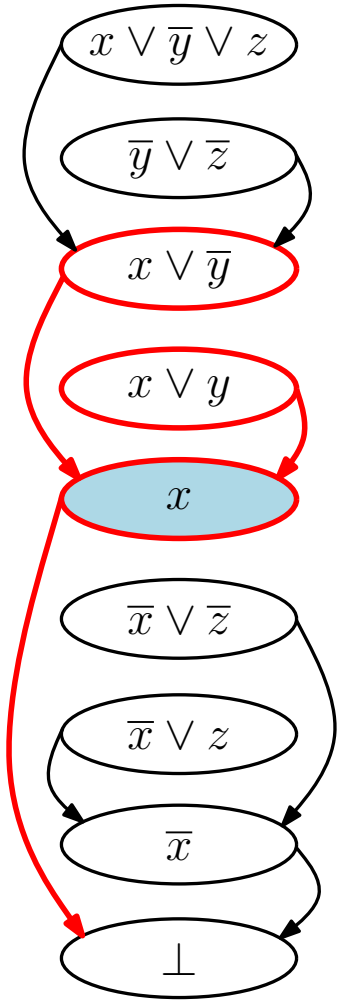

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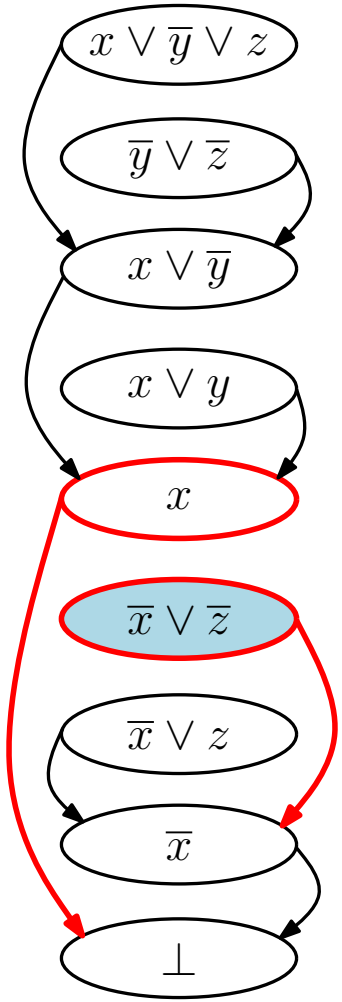

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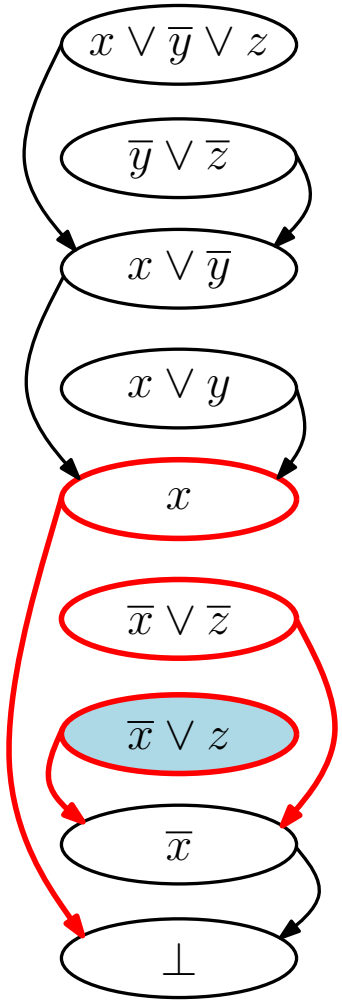

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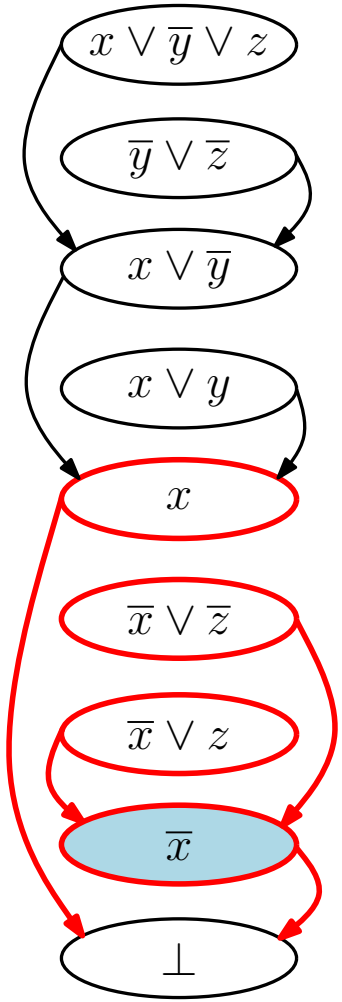
Example:

Size	9
Space	4
Width	



Space at current step 4

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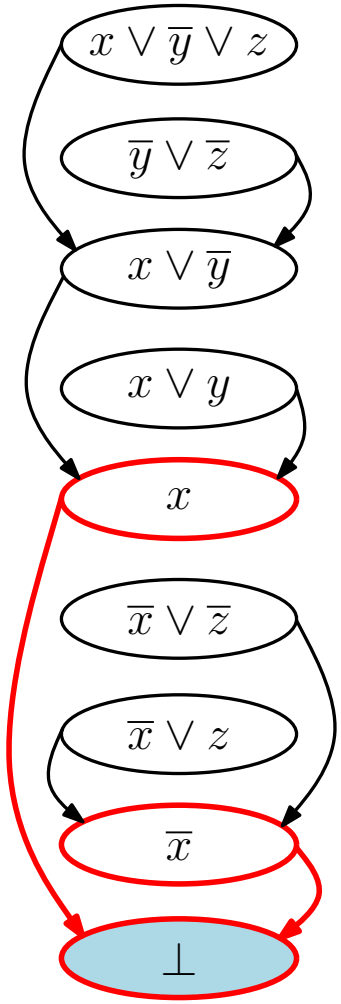

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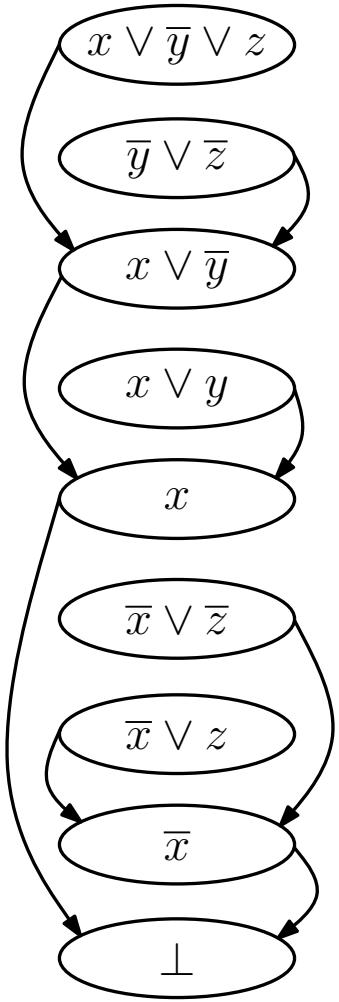

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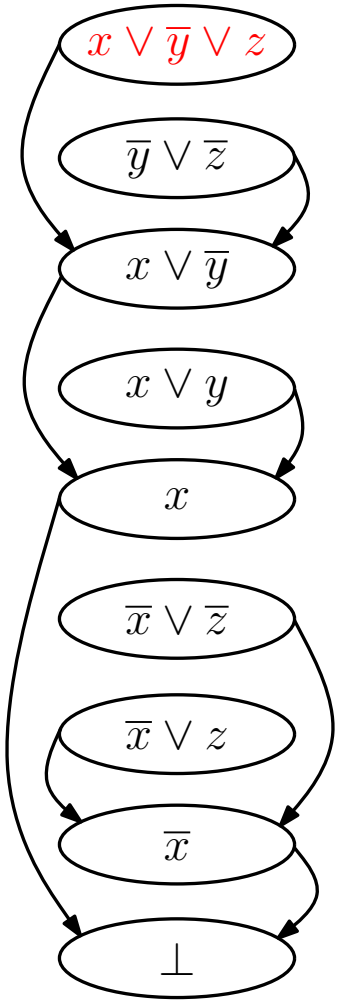

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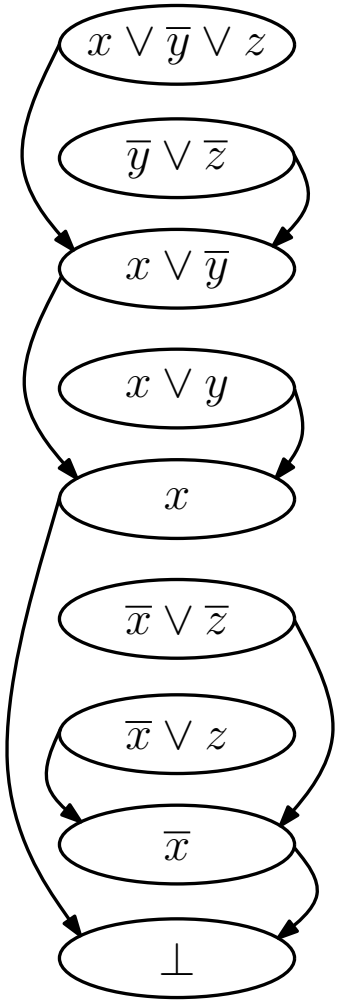

Space at current step 0

Relation Between Width and Size/Space

Width helps us understand size and space

Makes most sense for small width formulas — focus on k-CNFs

Size: Ben-Sasson and Wigderson '99

$$\log(\textbf{Size}) \gtrsim \textbf{Width}$$

Proof by syntactically manipulating short refutation into narrow refutation

Space: Atserias and Dalmau '03

$$\textbf{Space} \geq \textbf{Width}$$

More involved proof in terms of strategies for Ehrenfeucht-Fraïssé games

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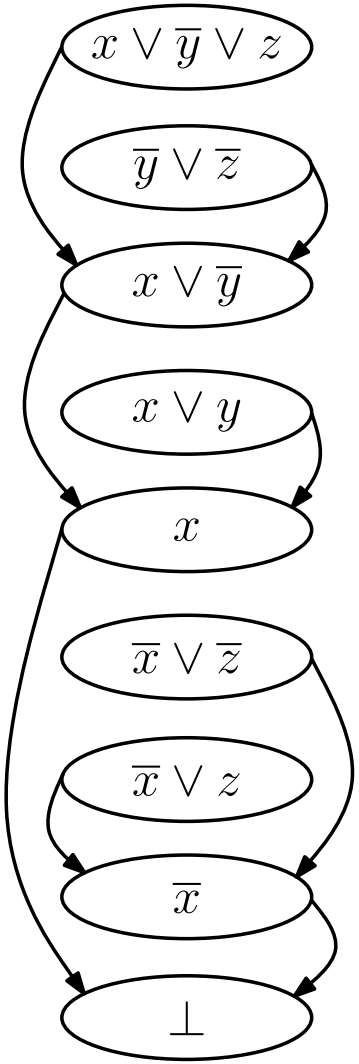
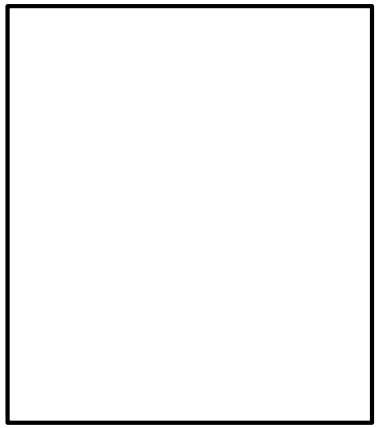
More involved proof in terms of strategies for Ehrenfeucht-Fraïssé games

Our result: Simple purely syntactic proof

Razborov independently obtained a similar proof

Whiteboard Interpretation of Space





Refutation presented on whiteboard

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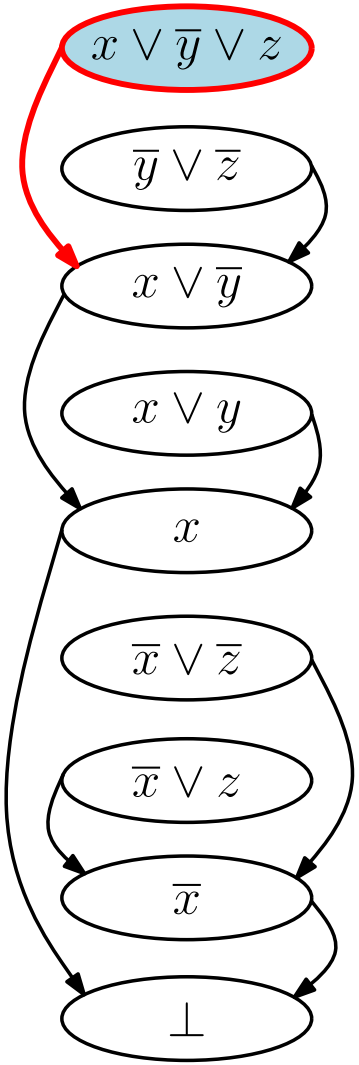

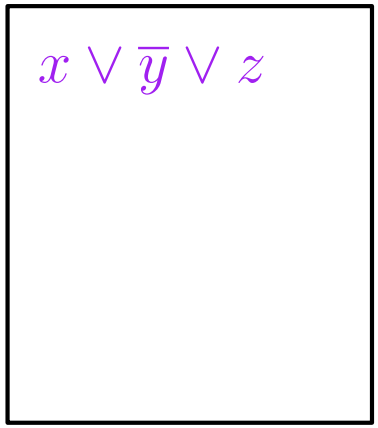

Refutation presented on whiteboard

- Write down axioms

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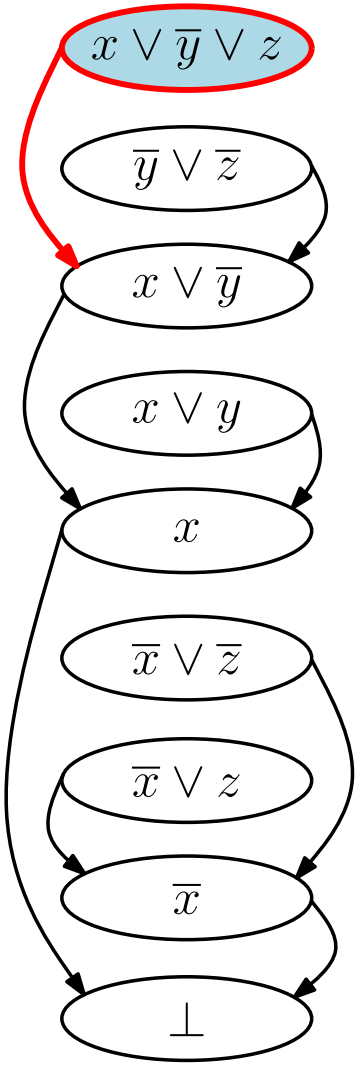

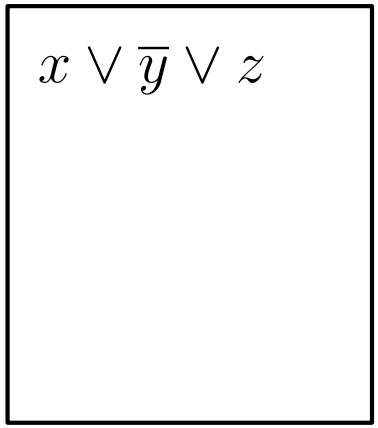

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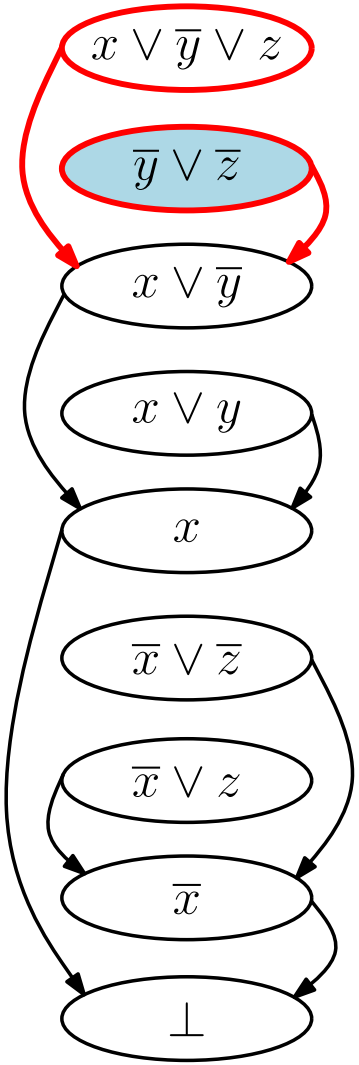

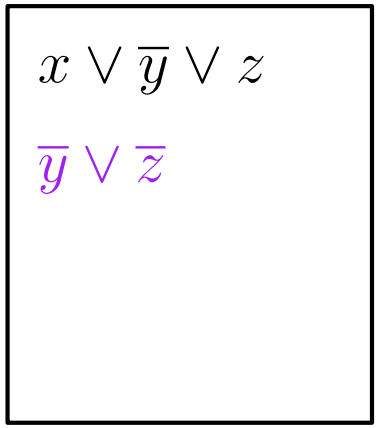

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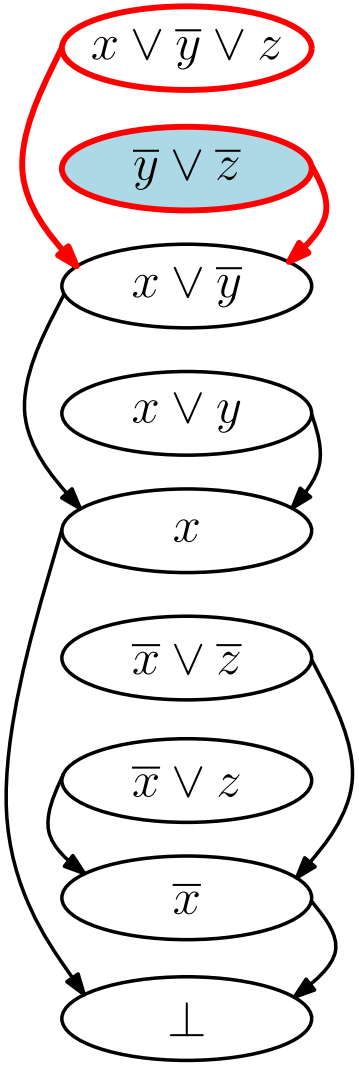

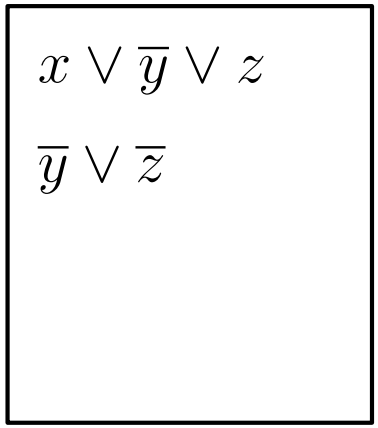

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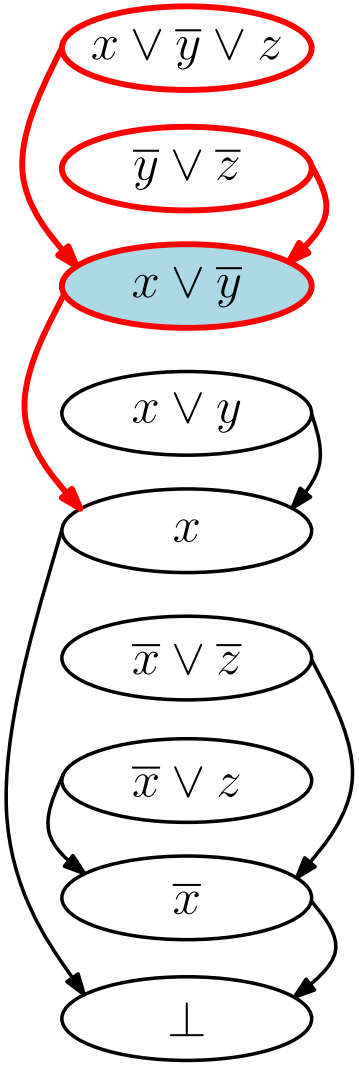


$x \vee \overline{y} \vee z$

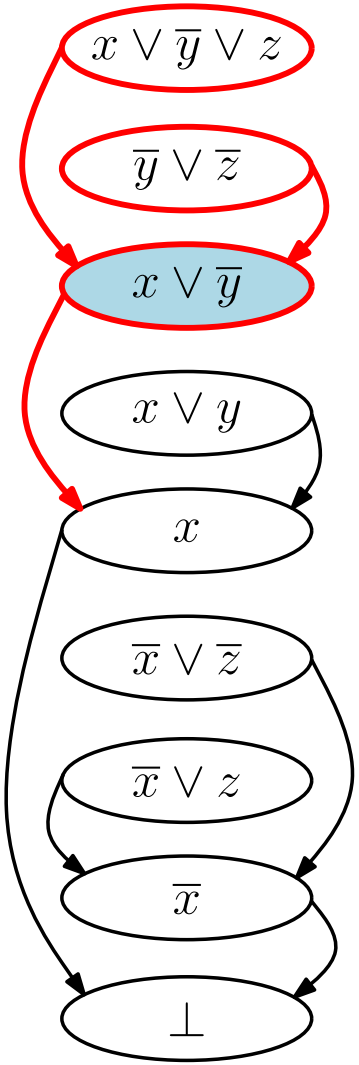
$\overline{y} \vee \overline{z}$

$x \vee \overline{y}$

Refutation presented on whiteboard

- Write down axioms
- Use resolution rule

Whiteboard Interpretation of Space



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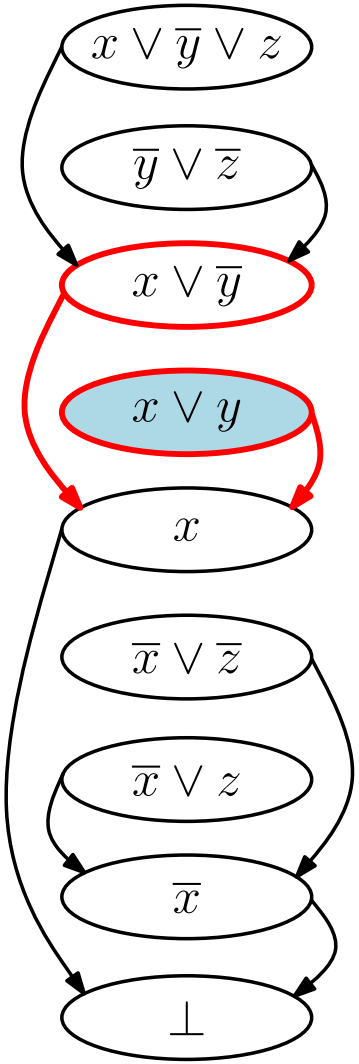

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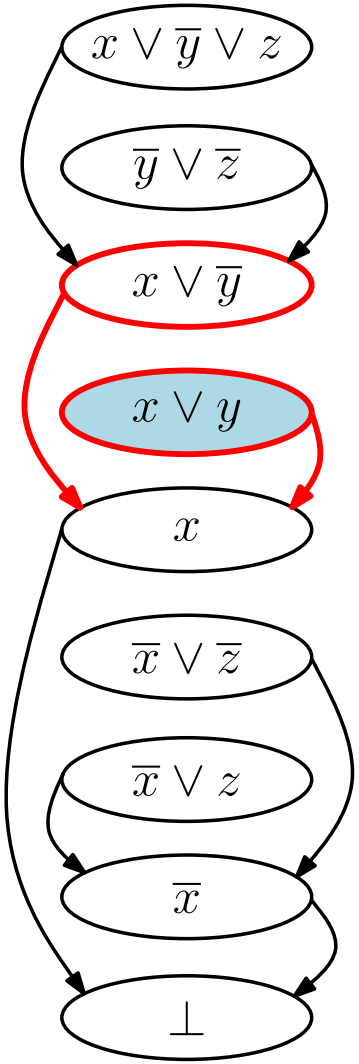
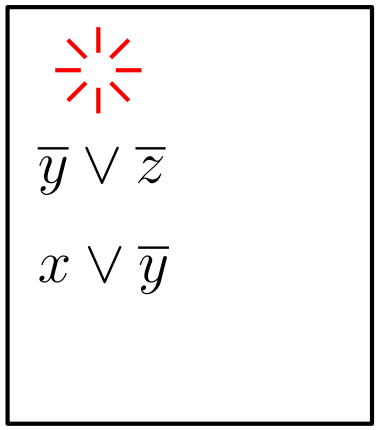
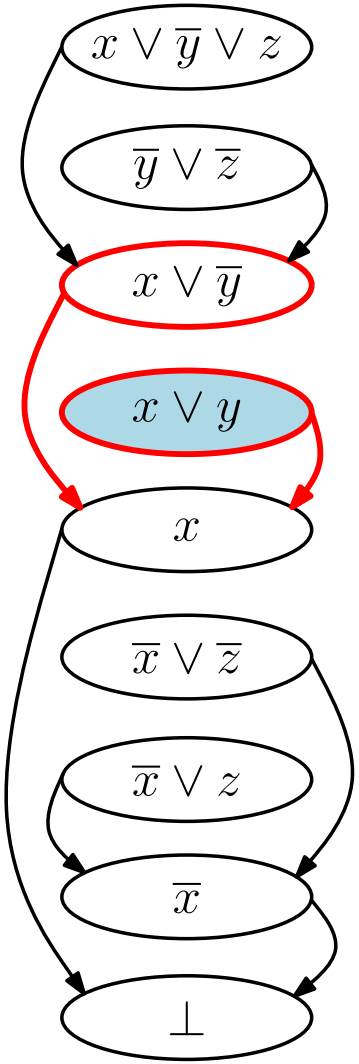


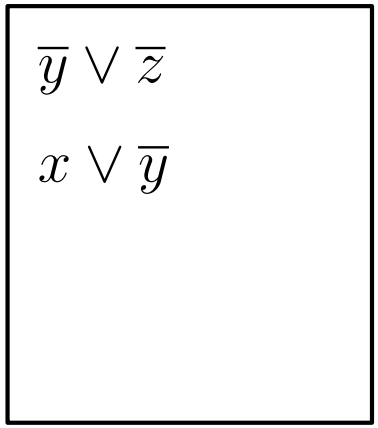


Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
- Erase clause

Whiteboard Interpretation of Space





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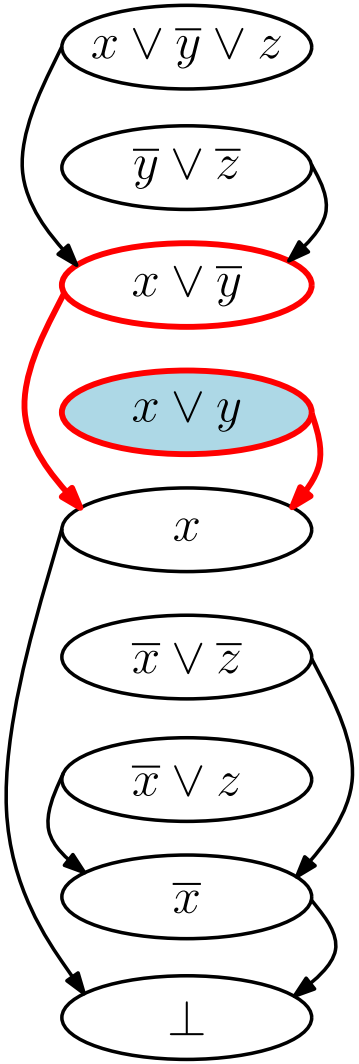

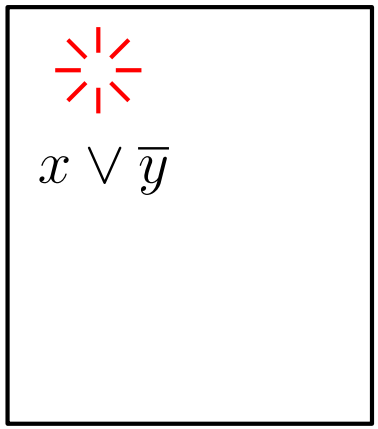

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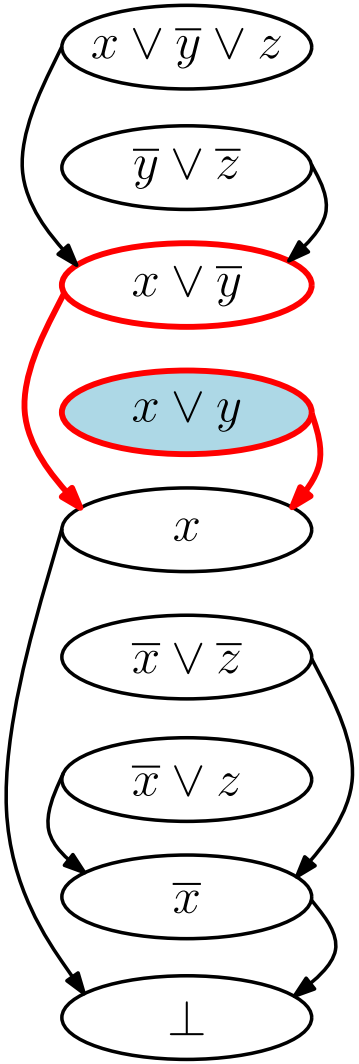

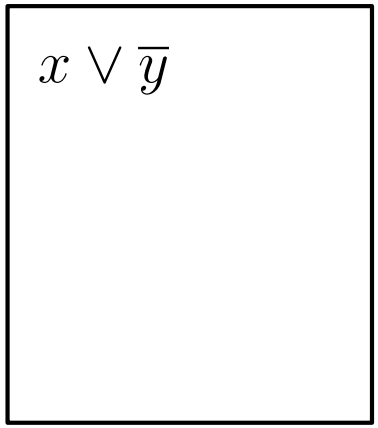

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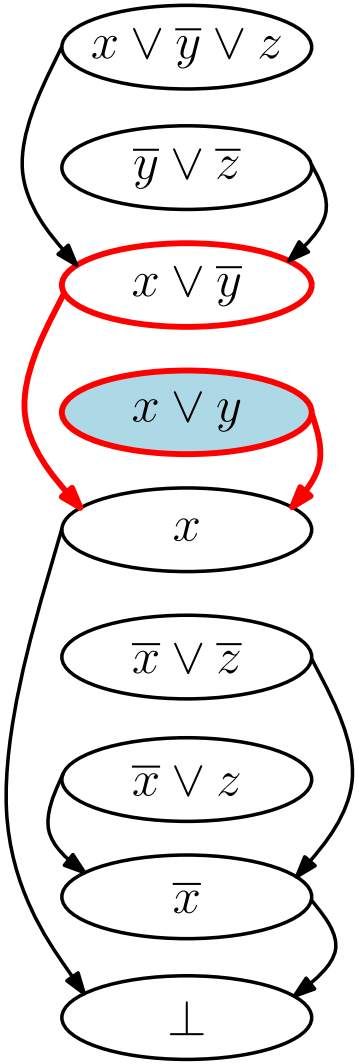

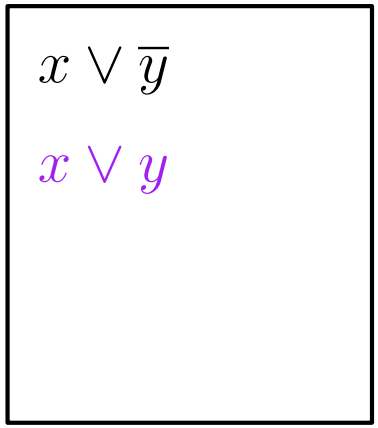

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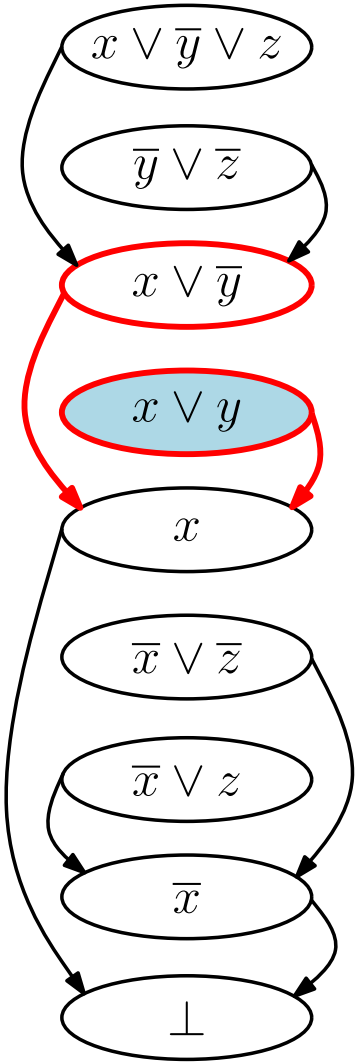

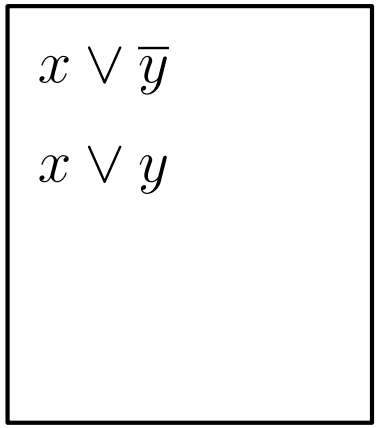

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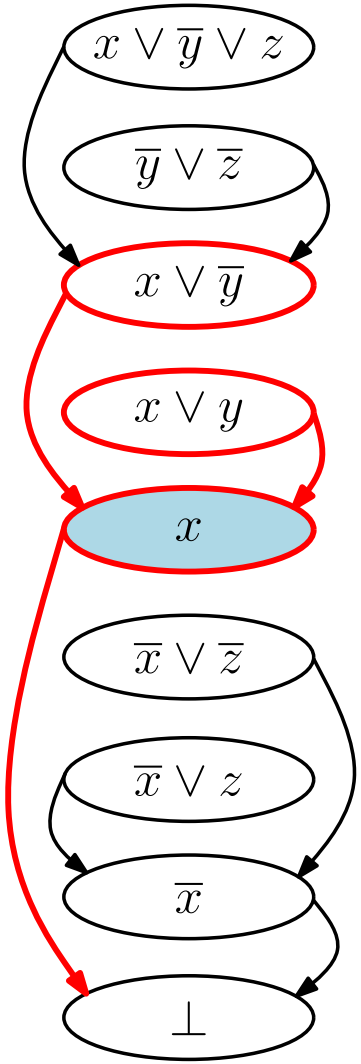
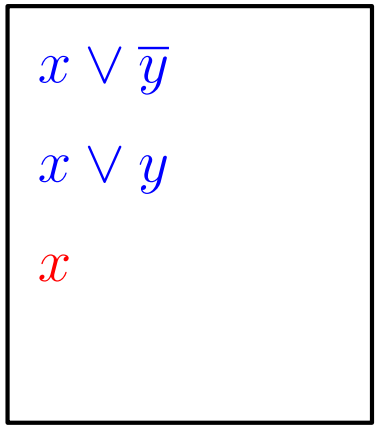
- Erase clause

Whiteboard Interpretation of Space

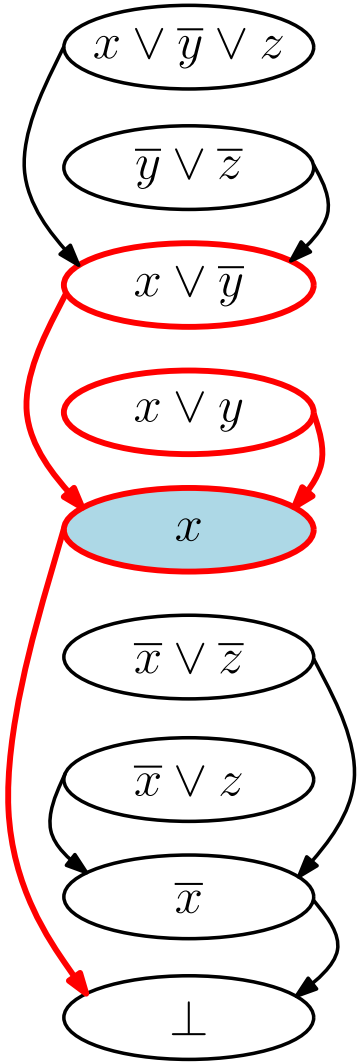


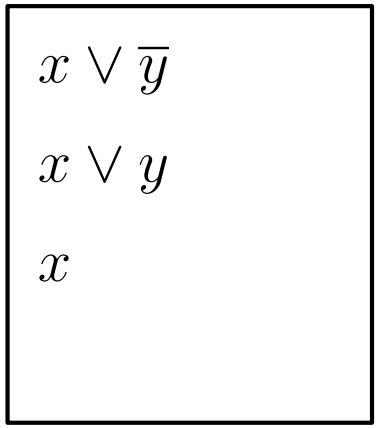


Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
- Erase clause

Whiteboard Interpretation of Space





Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
- Erase clause

Space: max # clauses on board

Proof Idea in One Slide

No finite model theory and no Ehrenfeucht-Fraïssé games

Want to turn small-height whiteboard into small-width one

$$x \vee \overline{y} \vee \overline{z} \vee v \vee w$$

$$\overline{y} \vee \overline{z} \vee \overline{w} \vee x$$

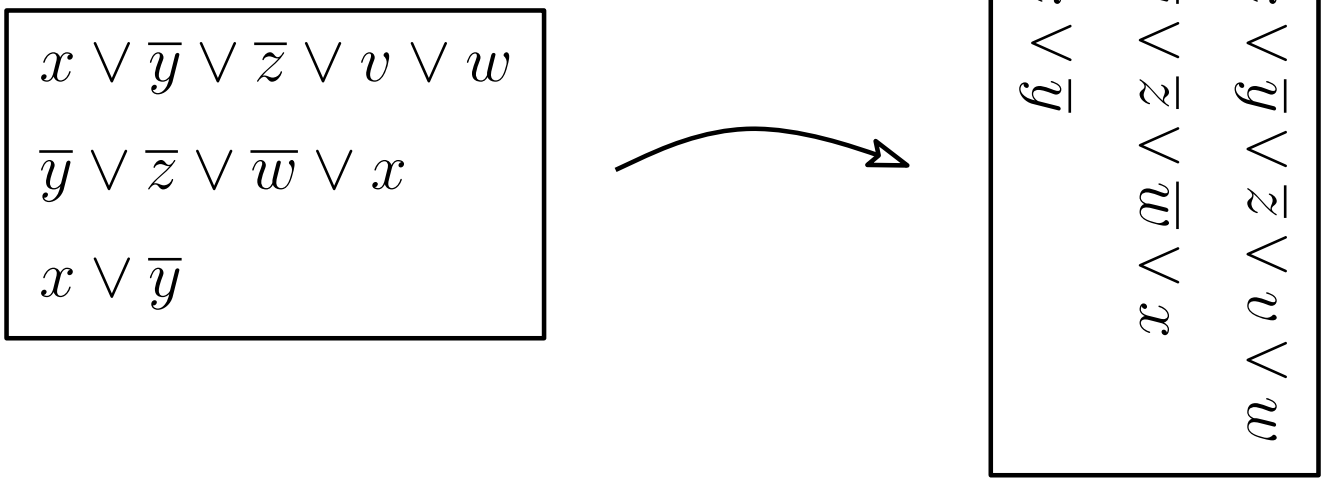
$$x \vee \overline{y}$$

Proof Idea in One Slide

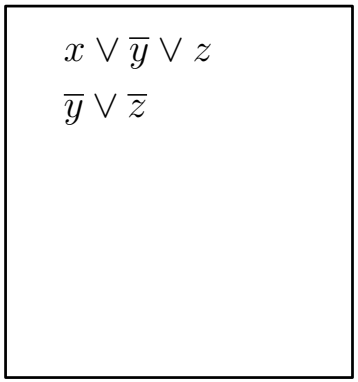
No finite model theory and no Ehrenfeucht-Fraïssé games

Want to turn small-height whiteboard into small-width one

Rotate whiteboard and get narrow whiteboard



In Somewhat More Detail...

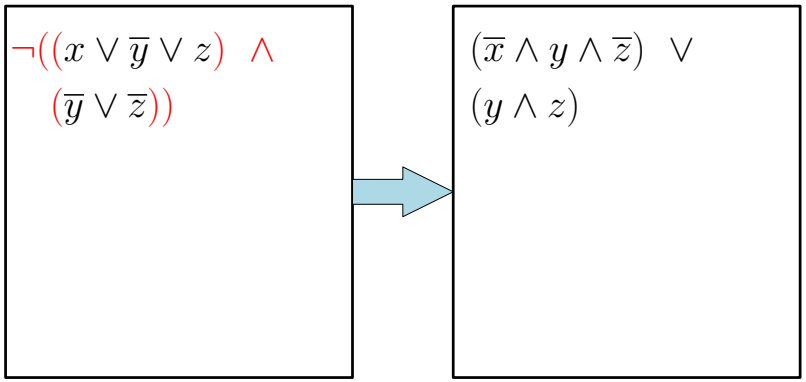


In Somewhat More Detail...

$\neg((x \vee \overline{y} \vee z) \wedge (\overline{y} \vee \overline{z}))$

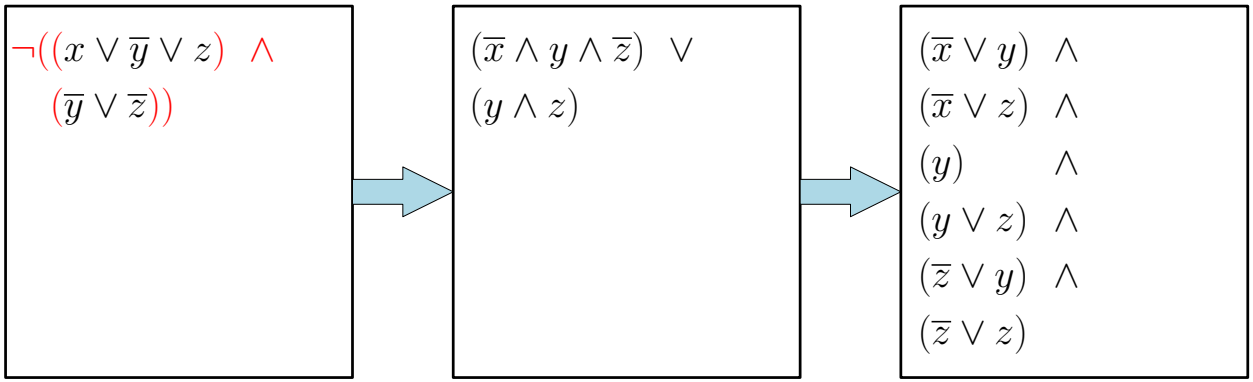
- View clauses on whiteboard as CNF and negate

In Somewhat More Detail...



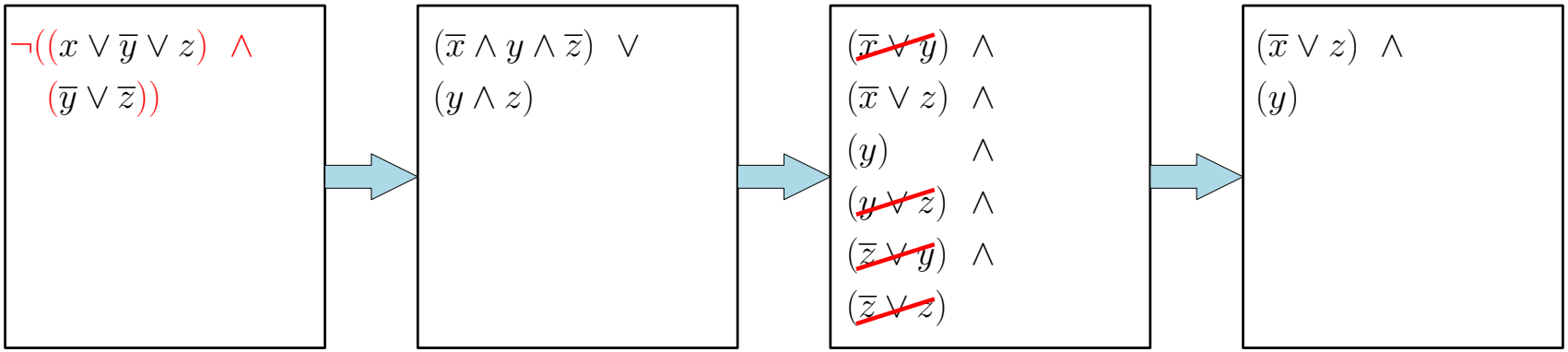
- View clauses on whiteboard as CNF and negate
- Apply DeMorgan's rules

In Somewhat More Detail...



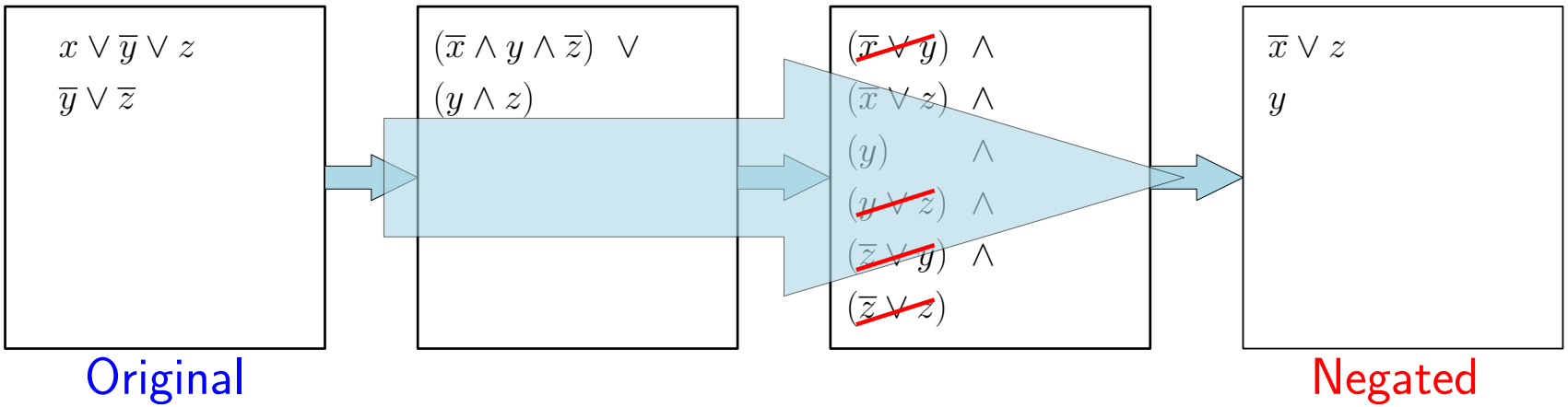
- View clauses on whiteboard as CNF and negate
- Apply DeMorgan's rules
- Expand the formula into CNF by distributing OR over ANDs

In Somewhat More Detail...



- View clauses on whiteboard as CNF and negate
- Apply DeMorgan's rules
- Expand the formula into CNF by distributing OR over ANDs
- Remove trivial and redundant clauses

In Somewhat More Detail...

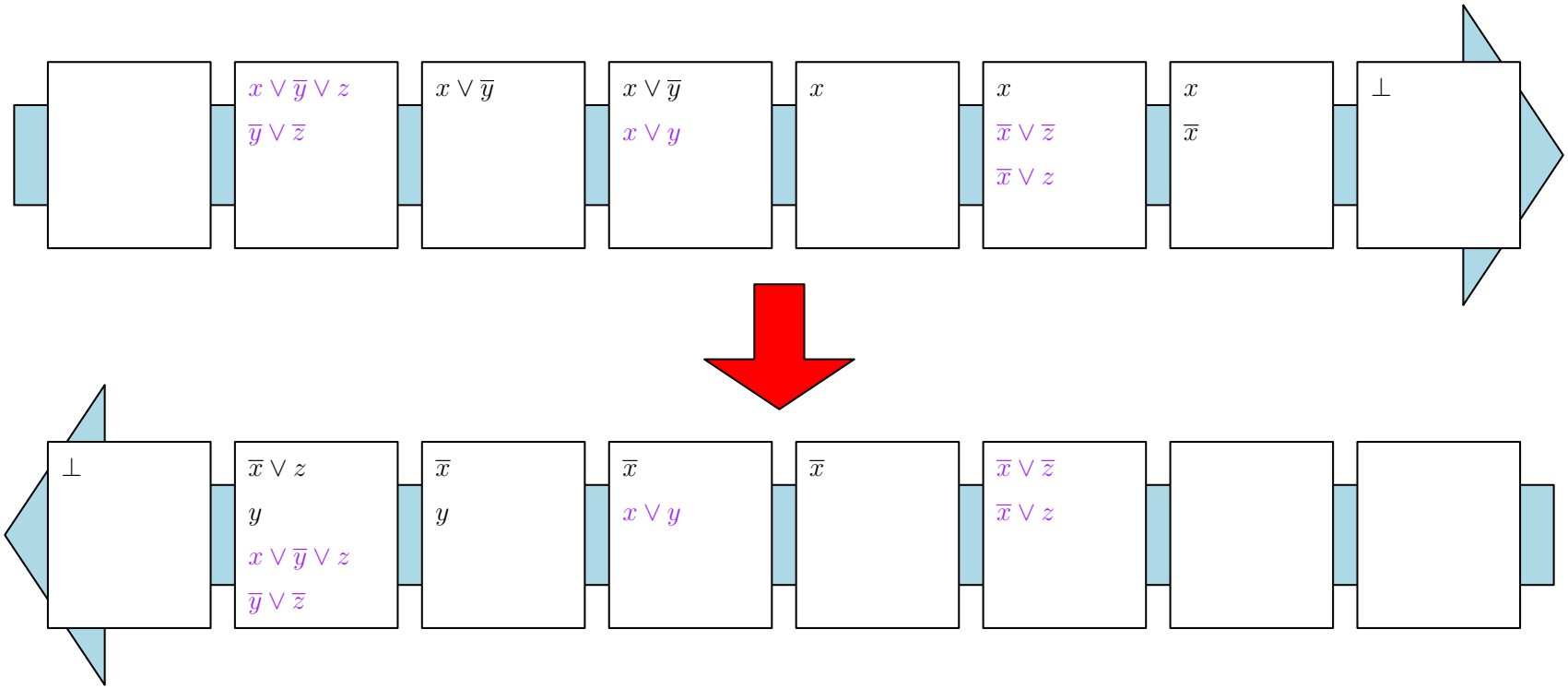


- View clauses on whiteboard as CNF and negate
- Apply DeMorgan's rules
- Expand the formula into CNF by distributing OR over ANDs
- Remove trivial and redundant clauses
- Write CNF on whiteboard

Space (# clauses) of Original $\geq$ Width of Negated

Consequences of Negation

Negate every whiteboard and run refutation in reverse



Note: Empty whiteboard turns into contradiction and vice versa

Small space refutation is transformed into narrow one

Missing Details

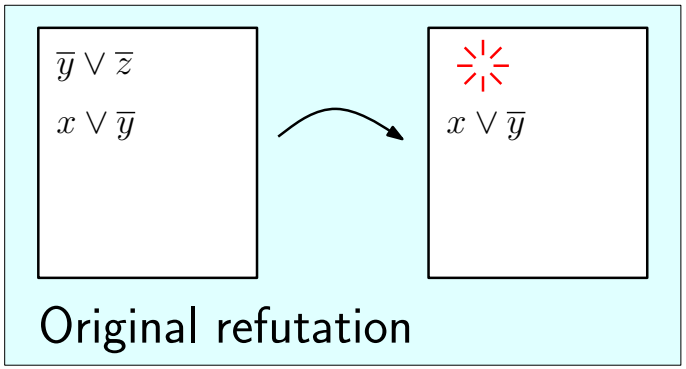
Need two things

1. Prove we have backbone of resolution refutation
2. Fill in missing details (without blowing up width)

Proof by case analysis over derivation steps:

- Axiom download
- Resolution rule application
- Clause erasure

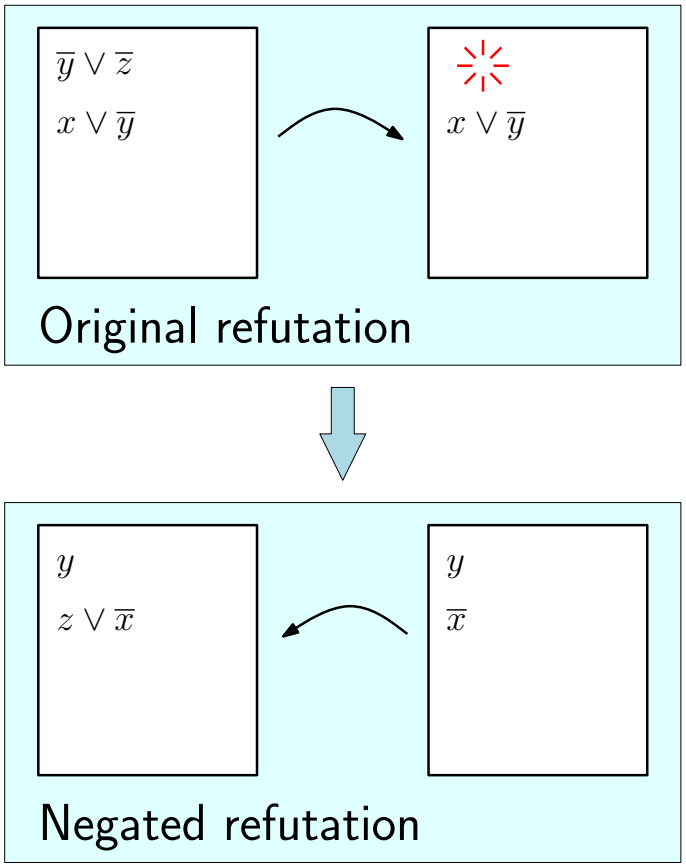
Clause Erasure



Original: Erasure weakens whiteboard

Right board weaker than left board

Clause Erasure



Original: Erasure weakens whiteboard

Right board weaker than left board

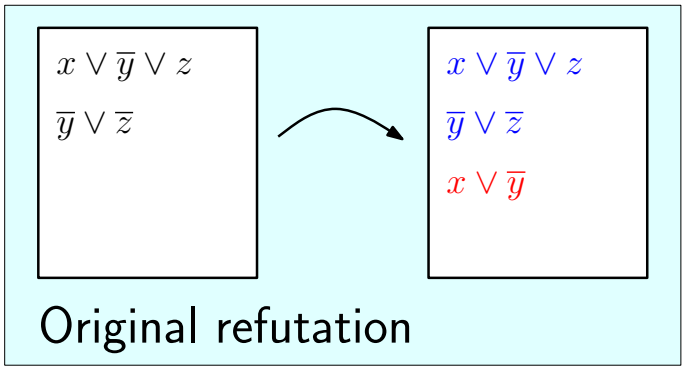
Negated: Negation inverts relation

Left board weaker than right board

Negated refutation run in reverse!

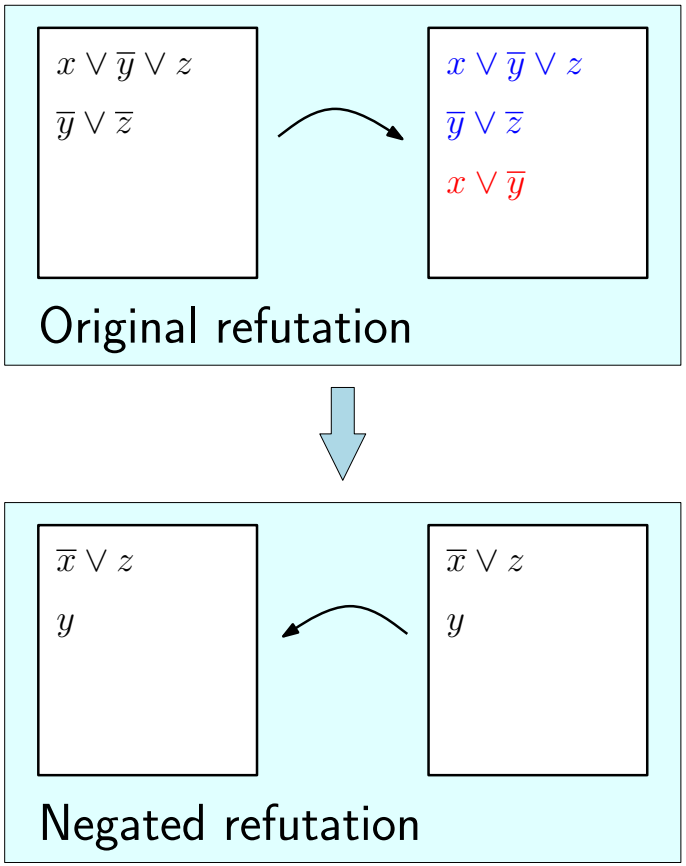
Can skip weaker whiteboards

Resolution Rule Application



Original: No change in semantic content

Resolution Rule Application

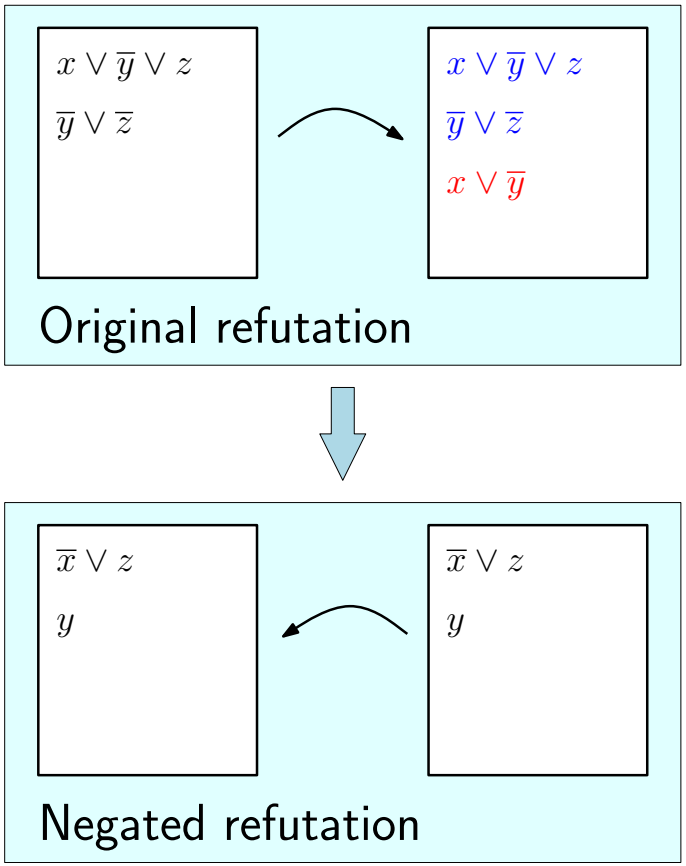


Original: No change in semantic content

Negated: No change in syntactic content (after prunning redundant clauses)

Whiteboard stays the same!

Resolution Rule Application



Original: No change in semantic content

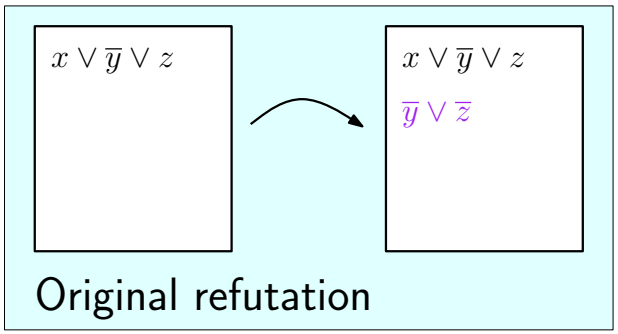
Negated: No change in syntactic content (after prunning redundant clauses)

Whiteboard stays the same!

Note: No work done thus far!

Axiom Download

Original: Add axiom A to whiteboard



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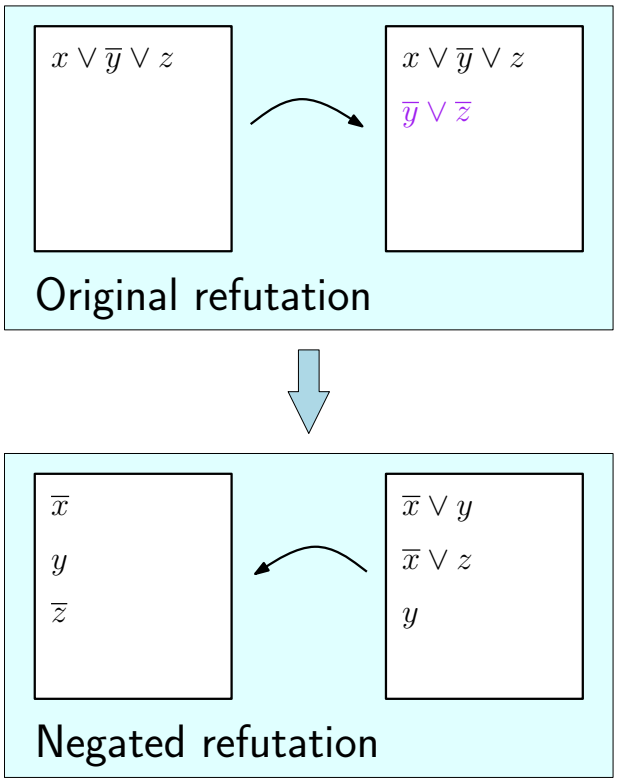

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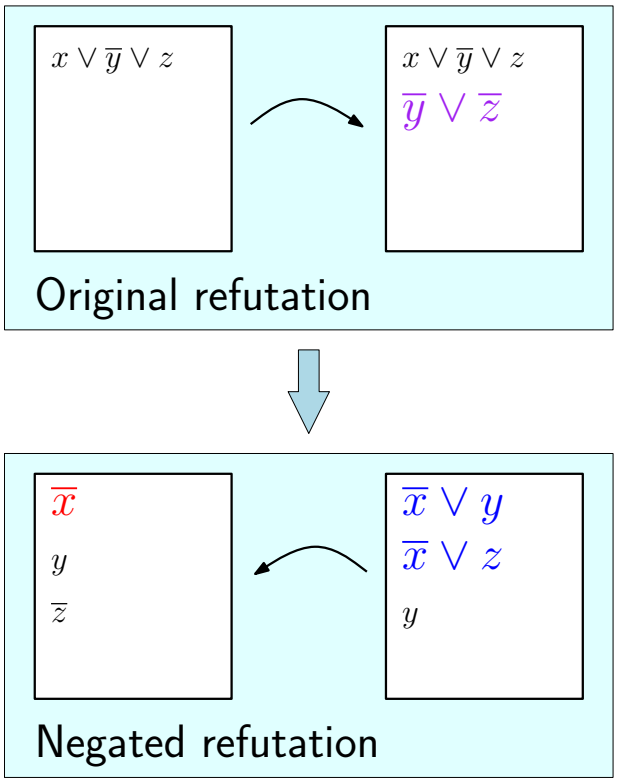
Negated: For every literal $a \in A$ add $\overline{a}$ to all clauses of whiteboard

Use clauses $C \vee \overline{a}$ and A to derive C

Axiom Download



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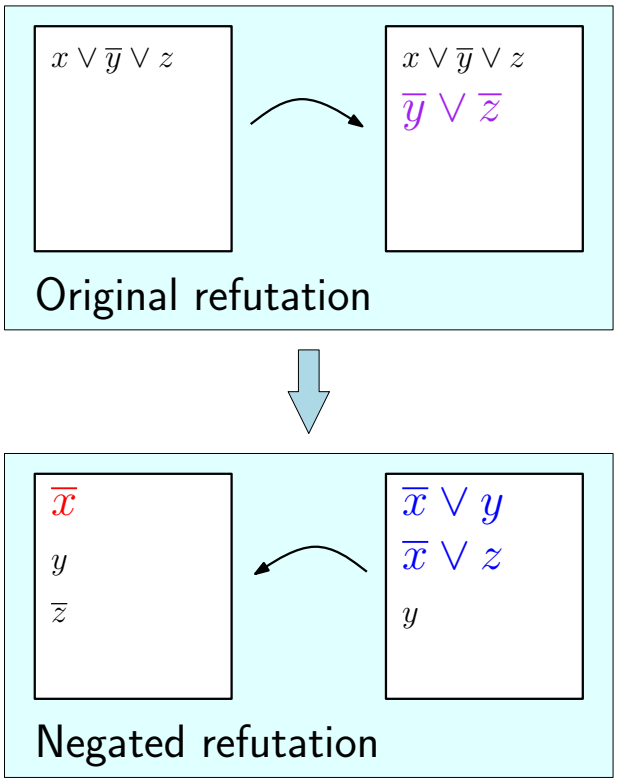
$$\dfrac{\dfrac{\overline{y} \vee \overline{z} \qquad \overline{x} \vee y}{\overline{x} \vee \overline{z}} \qquad \overline{x} \vee z}{\overline{x}}$$

Adds constant width to derivation

Axiom Download



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Adds constant width to derivation

Theorem

Space $\geq$ Width

Open Problem: Similar Problem for Polynomial Calculus

Polynomial calculus

Stronger proof system based on algebraic reasoning

Lines are polynomial equations instead of clauses

Degree of refutation analogous to width in resolution

Size: Impagliazzo, Pudlák, and Sgall '99

$$\log(\text{Size}) \gtrsim \text{Degree}$$

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$$\log(\text{Size}) \gtrsim \text{Degree}$$

Open Problem

Is Space $\geq$ Degree in polynomial calculus?

Original motivation for our work

We show our approach is unlikely to work (see paper for details)

Concluding Remarks

- Space upper bounds width in resolution [Atserias and Dalmau '03]
- This work: New simple proof of this theorem
- **Open problem:** Space-degree relation in polynomial calculus?

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Thank you for your attention!