Ordered coloring grids and related graphs

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Abstract

We investigate a coloring problem, called ordered coloring, in grids and some other families of grid-like graphs. Ordered coloring (also known as vertex ranking) is related to conflict-free coloring and other traditional coloring problems. Such coloring problems can model (among others) efficient frequency assignments in cellular networks. Our main technical results improve upper and lower bounds for the ordered chromatic number of grids and related graphs. To the best of our knowledge, this is the first attempt to calculate exactly the ordered chromatic number of these graph families.

Keywords: ordered coloring, vertex ranking, conflict-free coloring

1 Introduction

In this paper we focus on the problem of computing efficient *ordered colorings* (also known as *vertex rankings*) for grids and related graphs. Ordered colorings are defined as follows:

Definition 1.1 *An* ordered coloring *of $G = (V, E)$ with k colors is a function $C\colon V \to \{1, \dots, k\}$ such that for each simple path p in G the maximum color assigned to vertices of p occurs in* exactly one *vertex of p.*

The problem of computing ordered colorings is a well-known and widely studied problem (see e.g. [9]) with many applications including VLSI design [10] and parallel Cholesky factorization of matrices [11]. The problem is also interesting for the Operations Research community, because it has applications in planning efficient assembly of products in manufacturing systems [8]. In general, it seems the vertex ranking problem can model situations where interrelated tasks have to be accomplished fast in parallel (assembly from parts, parallel query optimization in databases, etc.)

Another motivation for the study of ordered colorings comes from more recent research into an area of coloring problems inspired by wireless mobile networks, called conflict-free (CF) colorings. The study of conflict-free colorings originated in the work of Even et al. [6] and Smorodinsky [14]. Conflict-free coloring models frequency assignment for cellular networks. A cellular network consists of two kinds of nodes: *base stations* and *mobile agents*. Base stations have fixed positions and provide the backbone of the network; they are modeled by vertices in V. Mobile agents are the clients of the network and they are served by base stations. This is done as follows: Every base station has a fixed frequency; this is modeled by the coloring C, i.e., colors represent frequencies. If an agent wants to establish a link with a base station it has to tune itself to this base station's frequency. Since agents are mobile, they can be in the range of many different base stations. To avoid interference, the system must assign frequencies to base stations in the following way: For any range, there must be a base station in the range with a frequency that is not reused by some other base station in the range. One can solve the problem by assigning n different frequencies to the n base stations. However, using many frequencies is expensive, and therefore, a scheme that reuses frequencies, where possible, is preferable. CF-coloring problems have been the subject of many recent papers due to their practical and theoretical interest (see e.g. [13, 7, 3, 5, 1]).

In the case where the ranges of the mobile agents are modeled by paths on the graph, the CF-coloring problem is very closely connected to the vertex ranking problem as defined above, since every path contains a uniquely colored vertex (i.e., a base station with a unique and maximum frequency). In fact, many approaches in the CF literature use ordered colorings because the latter are easier to argue about. In addition, the topologies we study in this paper are of special interest in this setting because they can model frequency assignment in a Manhattan-like environment,where base stations are approximately placed on a regular grid and this gives us additional motivation to calculate the exact ordered chromatic number of the grid.

In general graphs, finding the exact ordered chromatic number of a graph is NP-complete [12] and there is a $O(\log^2 n)$ polynomial time approximation algorithm [2], where n is the number of vertices. Since the problem is generally hard, it makes sense to study specific graph topologies and the focus of this paper is the calculation of the ordered coloring number of several grid-like families of graphs. Our main focus are grid graphs, which can be formally defined as follows: An $m_1 \times m_2$ *grid* is a graph with vertex set $\{0, \dots, m_1 - 1\} \times \{0, \dots, m_2 - 1\}$ and edge set $\{\{(x_1, y_1), (x_2, y_2)\} \mid |x_1 - x_2| + |y_1 - y_2| \leq 1\}$. In a standard drawing of the grid graph, vertex (x, y) is drawn at point (x, y) in the plane. The grid can also be defined as the *cartesian product* of two paths $P_{m_1} \times P_{m_2}$.

It is known [9] that for general planar graphs the ordered chromatic number is $O(\sqrt{n})$. Grid graphs are planar and therefore the $O(\sqrt{n})$ bound applies. One might expect that, since the graph families we study have a relatively simple and regular structure, it should be easy to calculate their ordered chromatic numbers. This is why it is rather striking that, even though it is not hard to show upper and lower bounds that are only a small constant multiplicative factor apart, the *exact* value of these ordered chromatic numbers is not known. The main contribution of this paper is to further improve on these upper and lower bounds and to the best of our knowledge this is the first such attempt.

Paper organization. In the rest of this section we provide the necessary definitions and some preliminary known results that will prove useful in the remainder. In Section 2 we present our results improving the known upper bounds on the ordered chromatic number of grids, tori and related graphs, while Section 3 deals with the lower bounds. Conclusions and open problems are presented in Section 4.

1.1 Preliminaries

First, let us remark that Definition 1.1 is not the typical definition found in the literature. Instead the more standard definition is:

Definition 1.2 *An* ordered *k-coloring of a graph G is a function $C\colon V(G) \to \{1,\ldots,k\}$ such that for every pair of distinct vertices v, v', and every path p from v to v', if $C(v) = C(v')$, there is an internal vertex v'' of p such that $C(v) < C(v'')$. The* ordered chromatic number *of a graph G, denoted by $\chi_{\mathrm{o}}(G)$, is the minimum k for which G has an ordered k-coloring.*

It is not hard to show that the two definitions are equivalent [9]. We prefer to use Definition 1.1 because it is closer to the definition of CF-colorings. CF-coloring can be seen as a relaxation of ordered coloring: In every path there must be a uniquely colored vertex, but its color does not necessarily need to be the maximum occurring in the path.

A concept that will prove useful in the remainder (especially for proving lower bounds) is that of a graph minor.

Definition 1.3 *A graph X is a* minor *of Y, denoted as $X \preccurlyeq Y$, if there is a subgraph G of Y, and a sequence $G_0,\ldots,G_k$, with $G_0 = G$ and $G_k = X$, such that $G_i = G_{i-1}/e_{i-1}$, where $e_{i-1} \in E(G_{i-1})$ (i.e., edge e_{i-1} is* contracted *in G_{i-1}), for $i \in \{1,\ldots,k\}$. Edge contraction is the process of merging both endpoints of an edge into a new vertex, which is connected to all neighbors of the two endpoints.*

It is not difficult to prove, with the help of a recoloring argument, that the ordered chromatic number is monotone with respect to minors.

Proposition 1.4 *If $X \preccurlyeq Y$, then $\chi_{\mathrm{o}}(X) \le \chi_{\mathrm{o}}(Y)$.*

In the rest of this section we provide ordered colorings for some graphs with (relatively) few edges.

Chain. Ordered coloring of a chain is equivalent to CF-coloring a chain and is better known as conflict-free coloring with respect to intervals [3]. Exactly, $1+\lfloor \lg n \rfloor$ colors are needed: For $n = 2^k - 1$, the coloring is defined recursively as follows: The middle vertex receives the maximum color k so the left and right sides (with $2^{k-1} - 1$ vertices each) can freely use the same colors and are colored recursively.

Ring. To color a *ring*, we use the above coloring of a chain. We pick an arbitrary vertex v and color it with a unique and maximum color. The remaining vertices form a chain that we color with the method described above. This method colors a ring of n vertices with $2 + \lfloor \lg(n-1) \rfloor$ colors.

Grid. To color the $m \times m$ grid, denoted by G_m, we can use the previous idea of the recursive coloring. We simply divide the grid in 4 equal grids of half size and recursively color them using exactly the same colors for each. To make this possible we should use unique colors in the middle row and column, as we did for the middle vertex of the chain. So, we use m unique maximum colors for the middle row and then about $\frac{m}{2}$ unique colors for the middle column (the same above and under the middle row). This method requires about $3m$ colors. However, this coloring remains proper even if we add two edges in every internal face of the standard drawing of G_m. This indicates that $3m$ is not optimal and in fact, in section 2, we improve the above upper bound.

There is also a lower bound of $\chi_o(G_m) \geq m$ from [9]. Another proof [2] is immediate from the fact that the *treewidth* and *pathwidth* of a graph G are at most the *minimum elimination tree height* [11] of G. We provide yet another proof, based on minors:

Proposition 1.5 *If G_m is the $m \times m$ grid, $\chi_o(G_m) \geq m$.*

Proof. By induction. Base: For $m = 1$, it is true, as $\chi_o(K_1) = 1$. For the inductive step, consider a Hamilton path p of G_m, with $m > 1$. If G_m is ordered colored, then there is a vertex v with a unique color in p (and thus in G). So, for some v, $\chi_o(G_m) = 1 + \chi_o(G_m - v)$. However, for every v, $G_{m-1} \preccurlyeq G_m - v$. Therefore, from proposition 1.4, $\chi_o(G) \geq 1 + \chi_o(G_{m-1})$ and from the inductive hypothesis, $\chi_o(G) \geq 1 + m - 1 = m$. □

In section 3, we improve the above lower bound.

2 Upper bounds

In this and the next section we show how to color several grid-like families of graphs. We are mainly interested in the $m \times m$ (square) grid. In order to color the grid efficiently we rely on separators whose removal leaves some subgraphs of the grid to be colored. The subgraphs we will rely on are the rhombus R_x, the wide-side triangle T_x, and the right triangle O_x. These are depicted in Figures 1, 2, and 3 and similar formal definitions are not hard to infer. Another graph topology we will investigate is the *torus*, which is a variation of the grid with wraparound edges added, connecting the last vertex of every row (and column) with the first. The torus graph $\widehat{G}_m$ can also be defined as the cartesian product of two cycles $C_m \times C_m$. A summary of our upper bound results can be seen on Table 1. It is interesting that the golden ratio $\phi \approx 1.618$ appears in some of these bounds.

graph	upper bound	based on
G_m	$2.519\,m$	R_m, O_m
R_m	$1.500\,m$	-
T_m	$1.118\,m$	R_m
O_m	$1.618\,m$	T_m
$\widehat{G}_m$	$3.500\,m$	R_m

Table 1: Summary of upper bounds. The last column indicates on which upper bounds each result is based.

As was evident in the examples of the previous section, one strategy for constructing an ordered coloring of a graph is to attempt to find a separator, that is, a set of vertices whose removal disconnects the graph. The vertices of this set are all assigned distinct colors that will be the maximum colors used in the graph. This way, we can recursively construct a coloring for the components formed by the deletion of the separator, since paths connecting vertices from different components have a unique maximum vertex in the separator.

The problem is then, to find a separator that is small and divides the graph into components of as low chromatic number as possible.

In the proofs we give below, we partition the graphs with the help of separators. All results are in the order of m, so without further mention we do not include terms logarithmic on m. These terms might be introduced by constant additive terms in a recursive bound. We are also omitting, in most cases floors and ceilings, because we are interested in asymptotic behavior. In that sense, a result like, for example, $\chi_o(G_m) \leq 2.67m$ should be read as an asymptotic upper bound of $2.67m \pm o(m)$.

In order to find improved upper bounds we need to find more intricate separators than those of the last example of the previous section. The idea is to use separators along diagonals in the grid. We will also need to find efficient colorings of some subgraphs that are left after we remove diagonal-like separators. That is the reason why we first present efficient colorings for the rhombus and the triangles.

In the figures of the following sections thicker lines indicate the selection of separator vertices which will receive unique and maximum colors. Thinner lines that lie on different sides of a thick line may reuse the same color range.

2.1 Rhombi and Triangles

The rhombus. The rhombus R_x is the first subgraph of the grid shown in Figure 1. It has height x. We have the following upper bound:

Proposition 2.1 $\chi_o(R_x) \leq 3x/2$.

Proof. Use a diagonal separator to cut the rhombus in half ($x/2$ unique colors are used), then cut also the remaining parts in half with a diagonal separator ($x/4$ unique colors, used in both parts). This is shown in figure 1. Therefore, we have the recursive formula $\chi_o(R_x) \leq x/2 + x/4 + \chi_o(R_{\lfloor x/2 \rfloor})$, which implies $\chi_o(R_x) \leq 3x/2$. □

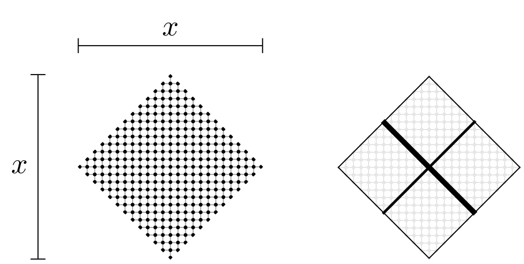

Figure 1: The rhombus subgraph R_x and its separation

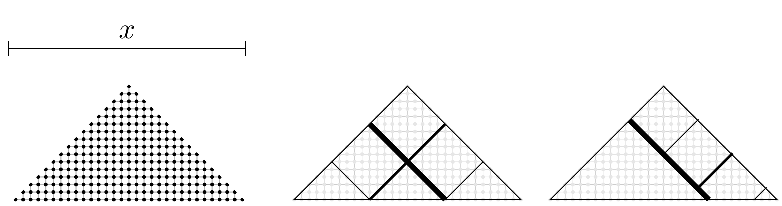

Figure 2: The wide side triangle T_x and its separations

The wide side triangle. The triangle T_x is the subgraph of the grid shown in figure 2. Its long side has length x. First, we give a simple upper bound:

Proposition 2.2 $\chi_o(T_x) \le 7x/6 \approx 1.167x$.

Proof. See the first separation of the wide-side triangle in Figure 2. Use a separator diagonally, parallel to one of the diagonal sides of the triangle T_x, with $2x/6$ unique colors. In the two remaining parts, separate diagonally by using separators parallel to the other diagonal side of the triangle T_x; each of those separators uses $x/6$ unique colors. With one more use of $x/6$ unique colors, we end up with connected components that are subgraphs of the rhombus $R_{2x/6}$. Therefore, $\chi_o(T_x) \le 2x/6 + x/6 + x/6 + \chi_o(R_{\lfloor 2x/6 \rfloor})$, and since by proposition 2.1, $\chi_o(R_x) \le 3x/2$, we have $\chi_o(T_x) \le 7x/6$. □

An improved upper bound can be obtained by the previous one, by making the observation that the graph on the left of the thickest separator in Figure 2 is also a wide side triangle. Thus, we may try to color it recursively in the same way. However, this would not improve the bound because the graph that remains on the right side uses $\frac{5x}{6}$ colors anyway. This indicates that the thickest separator would be better positioned if we moved it slightly to the right, since it seems that the remaining graph on the right side requires more colors.

Suppose that we move it slightly to the right, as in the last part of Figure 2 and that the ratio of its length over the length of the long side of the triangle is w (previously we had $w = 1/3$). We will optimize with respect to this w. Now, the rhombi on the right have length $x(1-2w)$, and the separators between them have length $x(1-2w)/2$. From the previously shown upper bound for the rhombus, and the fact that we need two sets of colors for the separators we conclude that the right part needs at most $\frac{5}{2}x(1-2w)$ colors. Assuming that the two parts are well balanced, the whole triangle needs at most $wx + \frac{5}{2}x(1-2w)$ colors. The triangle formed on the left of the separator has length $2wx$, thus from the above it needs $2w^2x + \frac{5}{2}(2wx)(1-2w)$ and in order for the balancing assumption to hold this must be equal to the number of colors used in the right part. Thus, we have $2w^2 + 5w(1-2w) = \frac{5}{2}(1-2w)$, which implies $w = \frac{5-\sqrt{5}}{8} \approx 0.345$. It is not hard to verify that using a separator of this length all the above arguments hold. Thus, we reach the following conclusion:

Proposition 2.3 $\chi_o(T_x) \le \sqrt{5}x/2 \approx 1.118x$.

The right triangle. The right triangle O_x is the subgraph of the grid shown in figure 3. It has height x. We have the following upper bound:

Proposition 2.4 $\chi_o(O_x) \le \phi x = \frac{\sqrt{5}+1}{2}x \approx 1.618x$.

Proof. See figure 3. Use a separator diagonally to form two wide side triangles whose long sides are of length x. We have the formula $\chi_o(O_x) \le x/2 + \chi_o(T_x)$ and since by proposition 2.2, $\chi_o(T_x) \le \sqrt{5}x/2$, we have $\chi_o(O_x) \le \frac{\sqrt{5}+1}{2}x = \phi x$, where we denote by ϕ the golden ratio. □

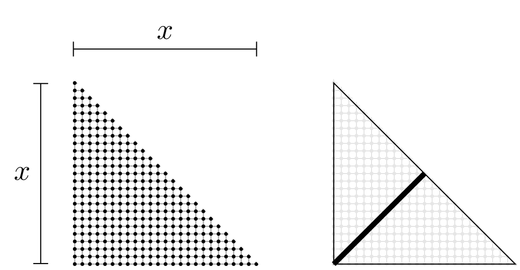

Figure 3: The right triangle O_x and its separation

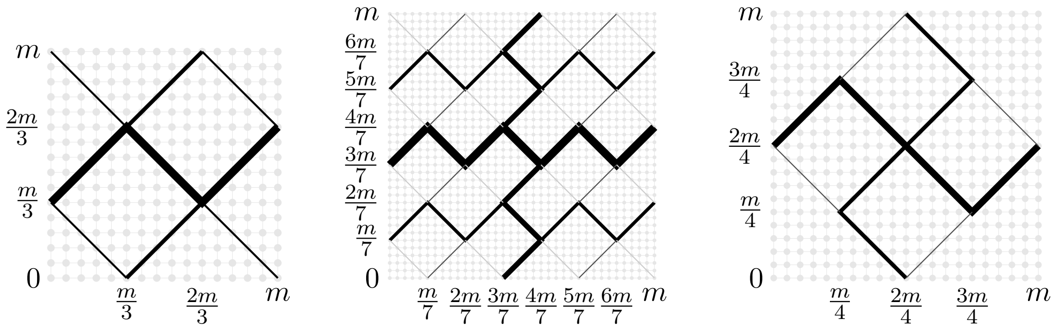

Figure 4: $8m/3$, $18m/7$ and $(7 + 2\phi)m/4$ upper bounds

2.2 Grids and tori

An $8m/3$ upper bound for square grids. In the first part of figure 4, we show how an $m \times m$ grid has to be partitioned with the help of separators to achieve an $8m/3$ upper bound.

The separators use m, $m/3$, and $m/3$ colors. After the removal of the separators, the remaining components are all subgraphs of a rhombus of height $2m/3$. By proposition 2.1, each remaining component can be colored with m colors. In total, $8m/3$ colors are required:

Proposition 2.5 $\chi_o(G_m) \leq 8m/3 \approx 2.6667m$.

An $18m/7$ upper bound for square grids. In the second part of figure 4, we show how an $m \times m$ grid has to be partitioned with the help of separators to achieve an $18m/7$ upper bound. The separators use m, $3m/7$, $3m/7$, $m/7$, and $m/7$ colors. Then, we have rhombi of height $2m/7$ that remain and, by proposition 2.1, each rhombus can be colored with $3m/7$ colors. In total, we have $18m/7$ colors:

Proposition 2.6 $\chi_o(G_m) \leq 18m/7 \approx 2.5714m$.

A $(7+2\phi)m/4$ upper bound for square grids. In the third part of figure 4 we show how an $m\times m$ grid can be partitioned to achieve a $(7 + 2\phi)m/4$ upper bound. We will show in the following section that shrinking this particular partition gives the best currently known result. The separators use $m + m/2 + m/4 = 7m/4$ unique colors. The remaining subgraphs of the grid to be colored are rhombi of height $m/2$ and right triangles of height $m/2$.By propositions 2.1 and 2.4 they can be colored with $3m/4$ and $\phi m/2$ colors respectively. Therefore the total use of colors is $7m/4 + \max(3m/4, \phi m/2) = (7 + 2\phi)m/4$.

Improving the upper bound by extending and shrinking colorings. The above upper bound may be slightly improved by extending or shrinking the underlying grid. The reason is that, even though for the most part the grid is partitioned into rhombi, different subgraphs are formed at its edges.

In the case of the $8m/3$ and $18m/7$ bounds, we can see that wide side triangles are formed. For each of these we can use the same set of colors as for the rhombi formed further inside the grid, but since the rhombi are of twice the size of the triangles extending the grid to the point where the triangles use the same number of colors as the rhombi will not increase the total number of colors used. For example, for the $8m/3$ coloring, if the coloring is extended by $m\left(\frac{1}{\sqrt{5}} - \frac{1}{3}\right)$ in every side (up, down, left, right), then one can color the new grid of side length $m' \approx 1.228m$ with $\frac{4(13+3\sqrt{5})}{31}m'$ colors. Thus:

Proposition 2.7 $\chi_o(G_m) \leq \frac{4(13+3\sqrt{5})}{31}m \approx 2.544m$.

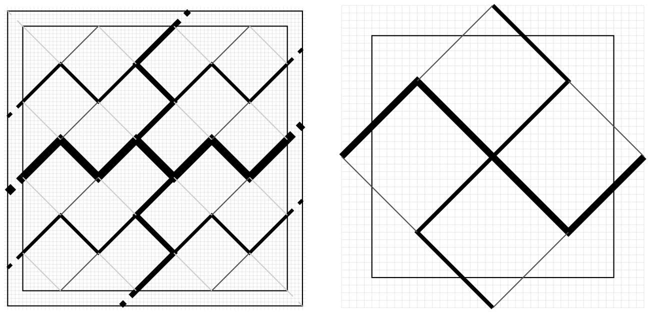

Figure 5: A $18m/7$ coloring extended and a $(7 + 2\phi)m/4$ coloring shrunk

In the case of the last coloring, we follow the opposite approach of shrinking the coloring. Four right triangles are formed, each using more colors than the rhombi. Therefore, slightly shrinking the grid so that the right triangles use the same number of colors as the rhombi improves the result. The optimal amount of shrinking is $x = \left(\frac{1}{4} - \frac{3}{8\phi}\right) m$ from each side (up, down, left, right). The remaining grid has side $m' \approx 0.9635m$ and can be colored with $6\frac{\phi+1}{2\phi+3}m'$ colors. Thus:

Proposition 2.8 $\chi_o(G_m) \leq 6\frac{\phi+1}{2\phi+3}m \approx 2.519m.$

Torus. An efficient coloring of the torus $\widehat{G}_m$ is as follows: Use the two diagonals as separators (at most $2m$ vertices). The remaining two connected components are subgraphs of the rhombus R_m which can be colored with at most $3m/2$ colors. Therefore, we have the following proposition.

Proposition 2.9 $\chi_o(\widehat{G}_m) \leq 2m + 3m/2 = 3.5m.$

Rectangular grids. Intuitively, a rectangular grid begins to resemble a chain when one of its dimensions is much smaller than the other, i.e., $m_2 \gg m_1$. We may attempt to exploit this observation in the following manner: given a grid with m_1 rows and m_2 columns, pick the m_1-th column, the $2m_1$-th column, ..., the $(\lfloor m_2/m_1 \rfloor \cdot m_1)$-th column. These $\left\lfloor \frac{m_2}{m_1} \right\rfloor$ will be used as separators, thus partitioning the graph into $m_1 \times m_1$ grids, which will use the same colors. However, they do not all need distinct colors, because we can color them in a way similar to the coloring of a chain: the middle column receives the highest colors, then we color recursively the columns to the left and those to the right. This results to an upper bound of $\chi_o(G_{m_1,m_2}) \leq m_1 \left\lceil (1 + \log(\left\lfloor \frac{m_2}{m_1} \right\rfloor)) \right\rceil + \chi_o(G_{m_1,m_1})$.

However, the above upper bound can be further improved slightly. Instead of using columns as separators we may use a zig-zag line starting from the top left corner and proceeding diagonally to the right until it hits the bottom, the to the right and up again, and so on. This requires the same number of colors for the separators, since we can still color them in a chain-like fashion, but now wide-side triangles are formed (instead of grids), each of length $2m_1$, thus we reach the following conclusion:

Proposition 2.10 $\chi_o(G_{m_1,m_2}) \leq m_1 \left\lceil (1 + \log(\left\lfloor \frac{m_2}{m_1} \right\rfloor)) \right\rceil + \sqrt{5}m$

The above result is close to being optimal when $m_2 \gg m_1$ as we will see in the next section. However, when m_2 is not much larger than m_1, more careful strategies need to be examined.

3 Lower bounds

In the first part of this section we prove lower bounds on the ordered chromatic number of square grids and tori. Then we move on to prove lower bounds for rectangular grids.

An important observation is the following: suppose we are given an optimal ordered coloring of a graph, and let $c_1, c_2, \ldots, c_k$ be the colors used, in decreasing order. If c_i is the first color in this order assigned to more than one vertex, then vertices with colors $c_1, \ldots, c_{i-1}$ must form a separator, otherwise the path connecting the two vertices of color c_i would not have a unique maximum vertex. Thus, we can reason about lower bound by reasoning about separators: examine cases on the size and shape of the separator formed by the highest colors of an optimal coloring and then, for each case, argue that the size of the separator plus the ordered chromatic number of one of the remaining components is higher than a desired lower bound. Moreover, it is enough to consider only *minimal separators*, as shown in [4] (a separator S is minimal if for every vertex v of S, $S \setminus \{v\}$ is not a separator).

In order to argue that the ordered chromatic number of a remaining component is high we will rely heavily on Proposition 1.4 and make use of induction.

We start with the torus lower bound, because the separators are simpler in this case.

Proposition 3.1 $\chi_o(\widehat{G}_m) \geq \frac{3m}{2}$ *(for $m \geq 2$).*

Proof. By induction: For $m = 2$ the proposition holds.

Suppose that we are given an optimal coloring of a torus $\widehat{G}_m$. Since the torus has no "sides" the separator must enclose an area of the torus. The smallest possible such separator is a set of the form $\{(x-1, y), (x, y+1), (x, y-1), (x+1, y)\}$, i.e., four vertices enclosing a single vertex (we call this kind of separator a cross). The length l of a separator will be $\max(|x_i - x_j| + 1)$ for $(x_i, y_i), (x_j, y_j)$ vertices of the separator. Similarly the height of a separator is $\max(|y_i - y_j| + 1)$. We distinguish between two cases:

Case 1: The separator formed by the highest colors encloses more than one vertex. Without loss of generality, suppose that the separator's length is at least as much as its height. Then the separator must consist of at least $2l - 2$ vertices. We also know that $l > 3 \Rightarrow l \geq 4$, otherwise the separator would enclose a single vertex only. Removing the separator will leave two components, one of which will have $\widehat{G}_{m-l}$ as a minor. Therefore, $\chi_o(\widehat{G}_m) \geq 2l - 2 + \chi_o(\widehat{G}_{m-l}) \geq \frac{3m}{2} + \frac{l}{2} - 2 \geq \frac{3m}{2}$.

Case 2: The separator formed by the highest colors is a cross. It is not hard to see intuitively that this cannot lead to an optimal coloring, because our goal when using separators should probably be to balance the chromatic number of the components that will be formed, since only the maximum one matters. To show that this is the case, consider the following argument: let c_i be the color of vertex $(x, y-1)$, that is, a vertex outside the cross, but adjacent to two of its vertices. If it is unique, then $\chi_o(\widehat{G}_m) \geq 4 + 1 + \chi_o(\widehat{G}_{m-3})$, because the removal of the cross and this unique color leaves a graph with $\widehat{G}_{m-3}$ as a minor. Thus, $\chi_o(\widehat{G}_m) \geq 5 + \frac{3m}{2} - \frac{9}{2} > \frac{3m}{2}$. Now, if it is not unique it must be separated from its other appearances in the graph by a separator. If the separator is not a cross, similar reasoning as in case 1 proves the lower bound. If it is, we have two crosses contained in a 5×5 area. Therefore, $\chi_o(\widehat{G}_m) \geq 8 + \chi_o(\widehat{G}_{m-5}) > \frac{3m}{2}$. □

We continue with a $4m/3$ lower bound for square grids, where the separators might also contain vertices on the sides of the grid (i.e., vertices with degree less than four).

Proposition 3.2 *For $m \geq 2$, $\chi_o(G_m) \geq \frac{4m}{3}$.*

Proof. Since we want to prove a $4m/3$ lower bound, we consider only separators of size $|S| \leq 4m/3$. The sides of the grid are the four paths of m vertices with $x = 0$, $x = m-1$, $y = 0$, and $y = m-1$, respectively. For the grid G_m we have the following cases of minimal separators.

Case I: The separator does not contain any vertex of the sides. This case is similar to the case of the torus. The size of the separator $|S| = s \geq 4$ and $G - S$ contains a $G_{m-(\lfloor s/2 \rfloor+1)}$ minor. Therefore, by induction, with such a separator, at least $s + (4/3)(m - (\lfloor s/2 \rfloor + 1)) \geq 4m/3$ colors are needed (because $s \geq 4$).

Case II: The separator touches at most two adjacent sides (i.e., sides that share a common vertex) of the grid. Then, $|S| = s \geq 2$ and $G - S$ contains a $G_{m-\lceil s/2 \rceil}$ subgraph. Therefore, by induction, with such a separator, at least $s + (4/3)(m - \lceil s/2 \rceil) \geq 4m/3$ colors are needed (because $s \geq 2$).

Case III: The separator touches two non-adjacent sides. In that case, the separator has size $|S| = s \geq m$. Consider the four square grid subgraphs $G_{\lceil m/2-s/6 \rceil}$ of the grid G_m that touch the four corners of G_m. It is not difficult to see that a separator of size s can not touch all four of the above subgraphs. Therefore, by induction, with such a separator, at least $s + \frac{4}{3}\lceil \frac{m}{2} - \frac{s}{6} \rceil \geq 4m/3$ colors are needed (because $s \geq m$). □

Finally, we proceed to prove lower bounds for rectangular grids.

Proposition 3.3 $\chi_o(G_{m_1,2m_1}) \geq 2m_1$

Proof. By induction. For $m_1 = 1$ the proposition holds.

Let S be the separator formed by the highest colors. If $|S| \geq 2m_1$ then the proposition trivially holds. If $|S| < 2m_1$ then the separator can not touch both of the far sides of the grid. Thus, its removal will give us a component having height m_1. If $|S| < m_1$ the separator cannot touch two sides that are opposite each other. Therefore, its removal will give a graph with $G_{m_1-(|S|/2),2m_1-|S|}$ as a minor and thus $\chi_o(G_{m_1,m_2}) \geq |S| + 2m_1 - |S| = 2m_1$.

Finally, suppose that $m_1 \leq |S| < 2m_1$. The separator can not span a length of more than $|S|$ vertices, therefore one of the components formed must have $G_{(2m_1-|S|)/2,m_1}$ as a minor. Thus, $\chi_o(G_{m_1,2m_1}) \geq |S| + 2m_1 - |S| = 2m_1$. □

Proposition 3.4 $\chi_o(G_{m_1,m_2}) \geq m_1 \left\lfloor \log\left(\frac{m_2}{m_1} + 1\right) \right\rfloor$

Proof. First, note that for $m_2 < 7m_1$ the proposition follows from previous propositions. Therefore, we will deal with $m_2 \geq 7m_1$.

Let S be the separator formed by the highest colors. If $S < m_1$ then the removal of S must leave a component with $G_{m_1,m_2-|S|}$ as a minor. $\chi_o(G_{m_1,m_2}) \geq |S| + \chi_o(G_{m_1,m_2-|S|}) \geq |S| + m_1 \left\lfloor \log(\frac{m_2-|S|}{m_1} + 1) \right\rfloor > m_1 \left\lfloor \log(\frac{m_2}{m_1} + 1) \right\rfloor$.

If $m_1 \leq |S| \leq m_2 - 2m_1$ then, as in the previous proof, at least one $G_{m_1,(m_2-|S|)/2}$ minor is formed. Thus, $\chi_o(G_{m_1,m_2}) \geq |S| + \chi_o(G_{m_1,(m_2-|S|)/2}) \geq |S| + m_1 \left\lfloor \log(\frac{m_2-|S|}{2m_1} + 1) \right\rfloor$. It is not hard to verify, using elementary calculus, that the latter is minimized when $|S| = m_1$ in which case $\chi_o(G_{m_1,m_2}) \geq m_1 + m_1 \left\lfloor \log(\frac{m_2+m_1}{2m_1}) \right\rfloor = m_1 \left\lfloor \log(\frac{m_2}{m_1} + 1) \right\rfloor$.

Finally, for $m_2 - 2m_1 < |S|$, let $m_2 = km_1$. Then $|S| > (k-2)m_1$, while $\log(\frac{m_2}{m_1} + 1) = \log(k+1)$. We have that, $\chi_o(G_{m_1,m_2}) \geq |S| > (k-2)m_1$, therefore if $k - 2 > \lfloor \log(k+1) \rfloor$ the proposition holds. But we know that $k \geq 7$, which satisfies the previous inequality. □

4 Open problems

The most important problem still left open is of course the exact value of $\chi_o(G_m)$. For small values of m the correct answer seems to be $2m - 1$. If this is true, it means that there is some room for improvement in both the upper and the lower bounds.

Another area for future research may be the online version of the problem, where vertices of the grid are "activated" one by one, and the coloring must remain proper throughout the process. Relevant results in the case of chains can be found in [1, 3].

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