Improved Inapproximability for TSP

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The Traveling Salesman Problem

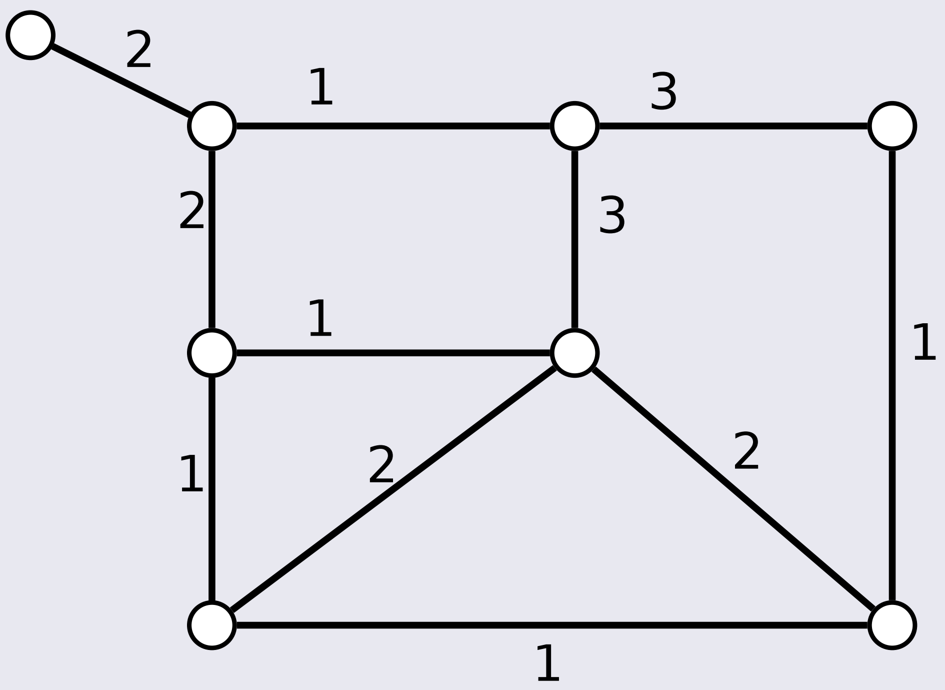
Input:

- An edge-weighted graph $G(V, E)$

Objective:

- Find an ordering of the vertices $v_1, v_2, \ldots, v_n$ such that $d(v_1, v_2) + d(v_2, v_3) + \ldots + d(v_n, v_1)$ is minimized.
- $d(v_i, v_j)$ is the shortest-path distance of v_i, v_j on G

The Traveling Salesman Problem



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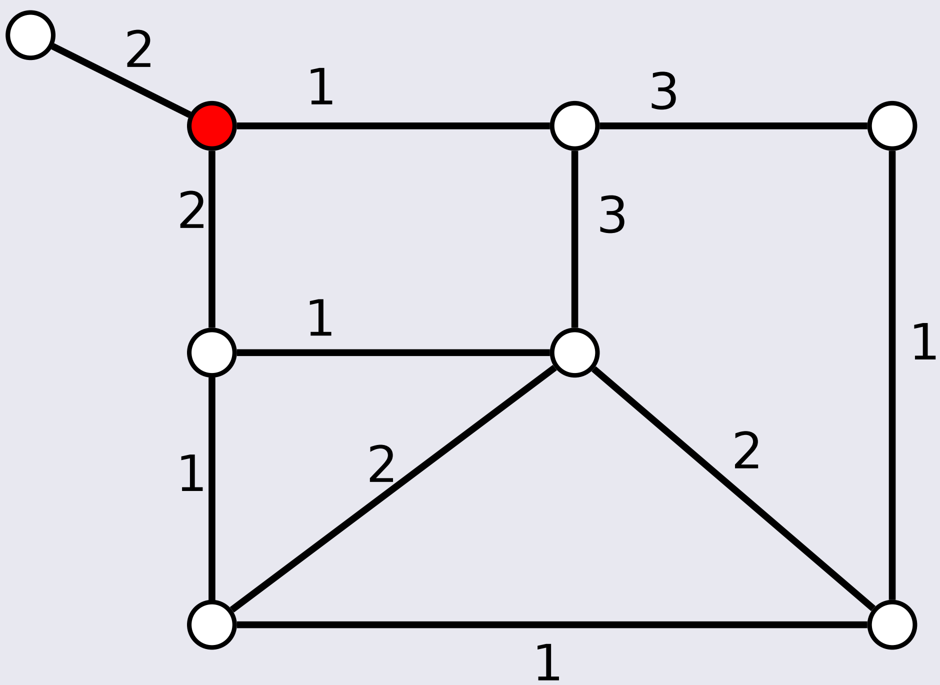

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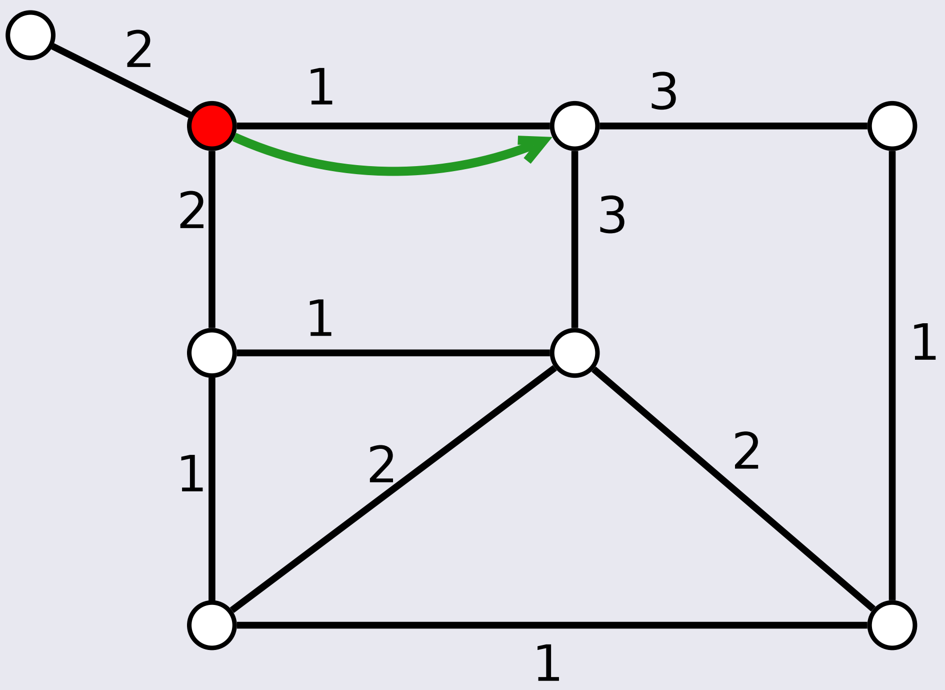

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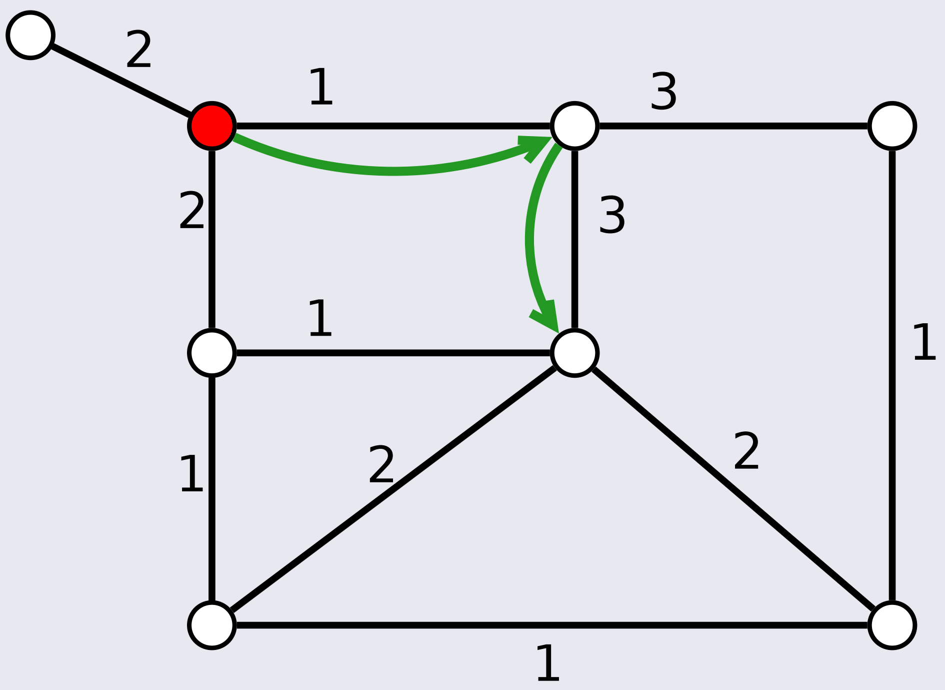
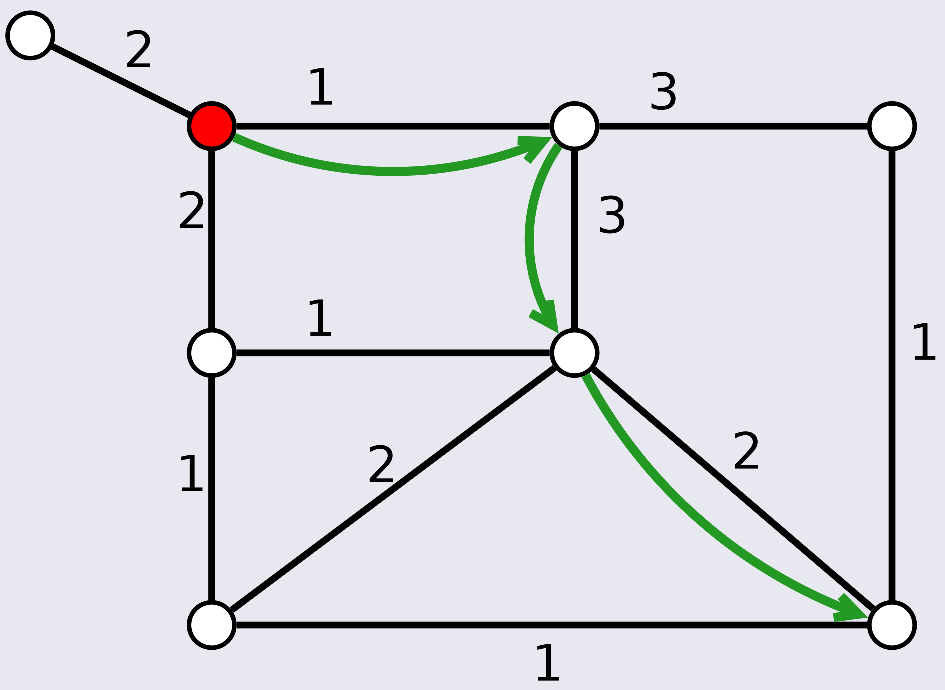


The Traveling Salesman Problem



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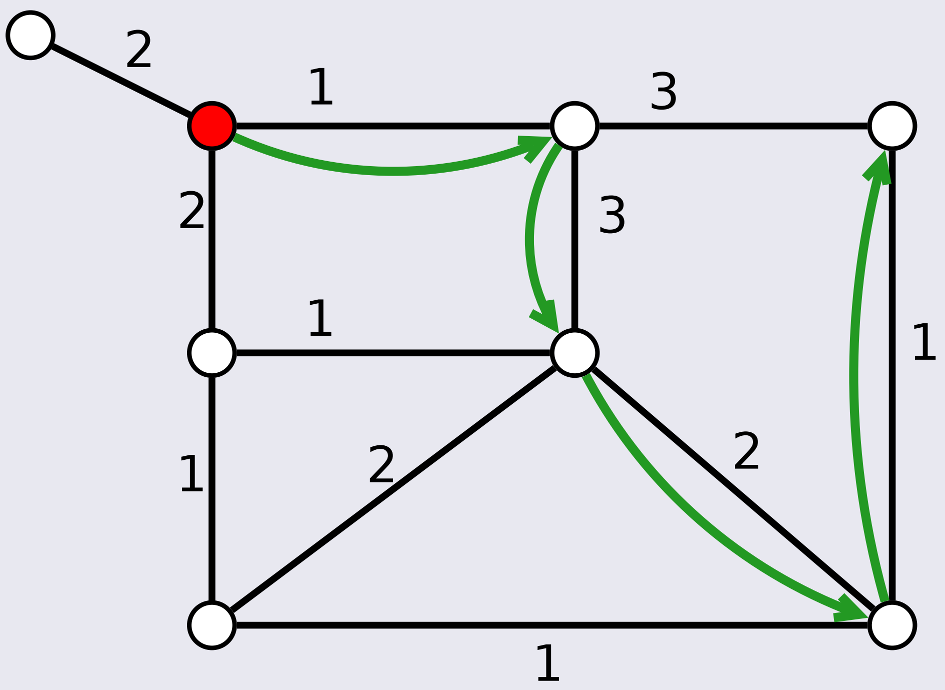
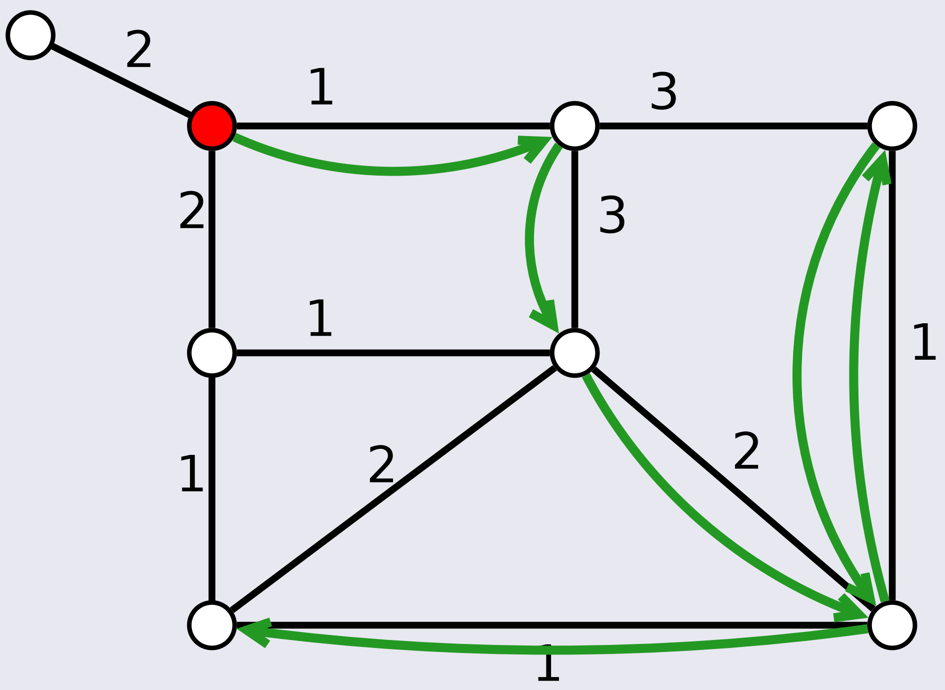
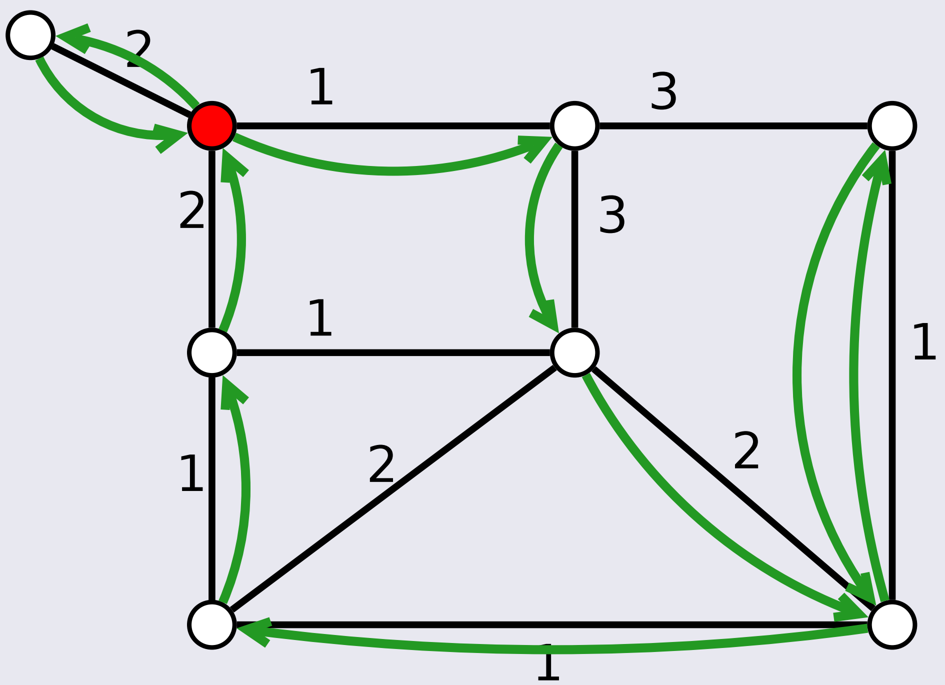


The Traveling Salesman Problem




2
1
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ACME
INSURANCE

- $\frac{3}{2}$ approximation (Christofides 1976)

For graphic (un-weighted) case

- $\frac{3}{2} - \epsilon$ approximation (Oveis Gharan et al. FOCS '11)
- 1.461 approximation (Mömke and Svensson FOCS '11)
- $\frac{13}{9}$ approximation (Mucha STACS '12)
- 1.4 approximation (Sebö and Vygen arXiv '12)

TSP Approximations – Lower bounds

- Problem is APX-hard (Papadimitriou and Yannakakis '93)
- $\frac{5381}{5380}$-inapproximable (Engebretsen STACS '99)
- $\frac{3813}{3812}$-inapproximable (Böckenhauer et al. STACS '00)
- $\frac{220}{219}$-inapproximable (Papadimitriou and Vempala STOC '00, Combinatorica '06)

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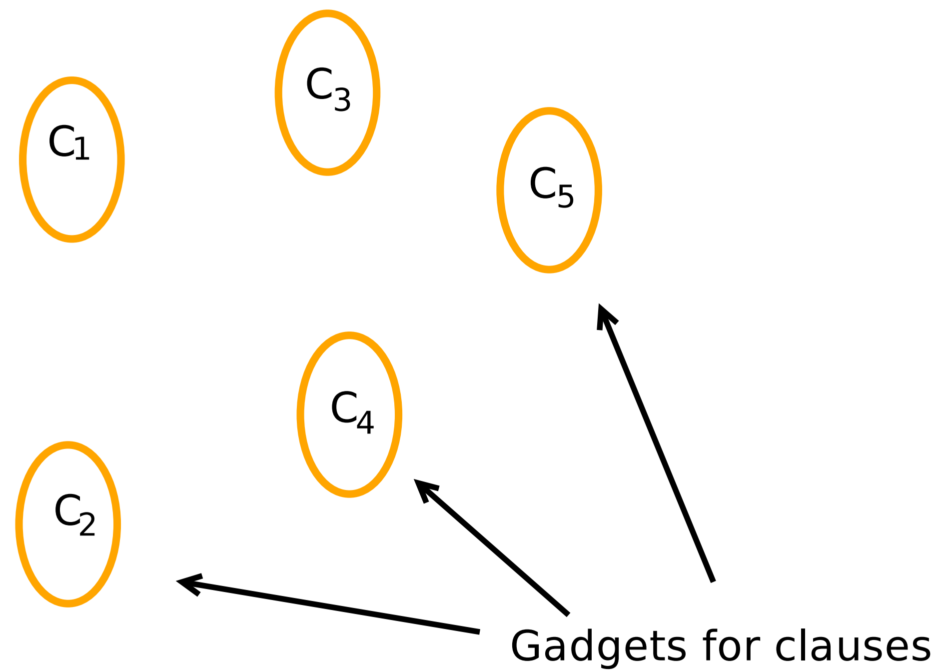
This talk:

Theorem
There is no $\frac{185}{184}$-approximation algorithm for TSP, unless P=NP.

Reduction Technique

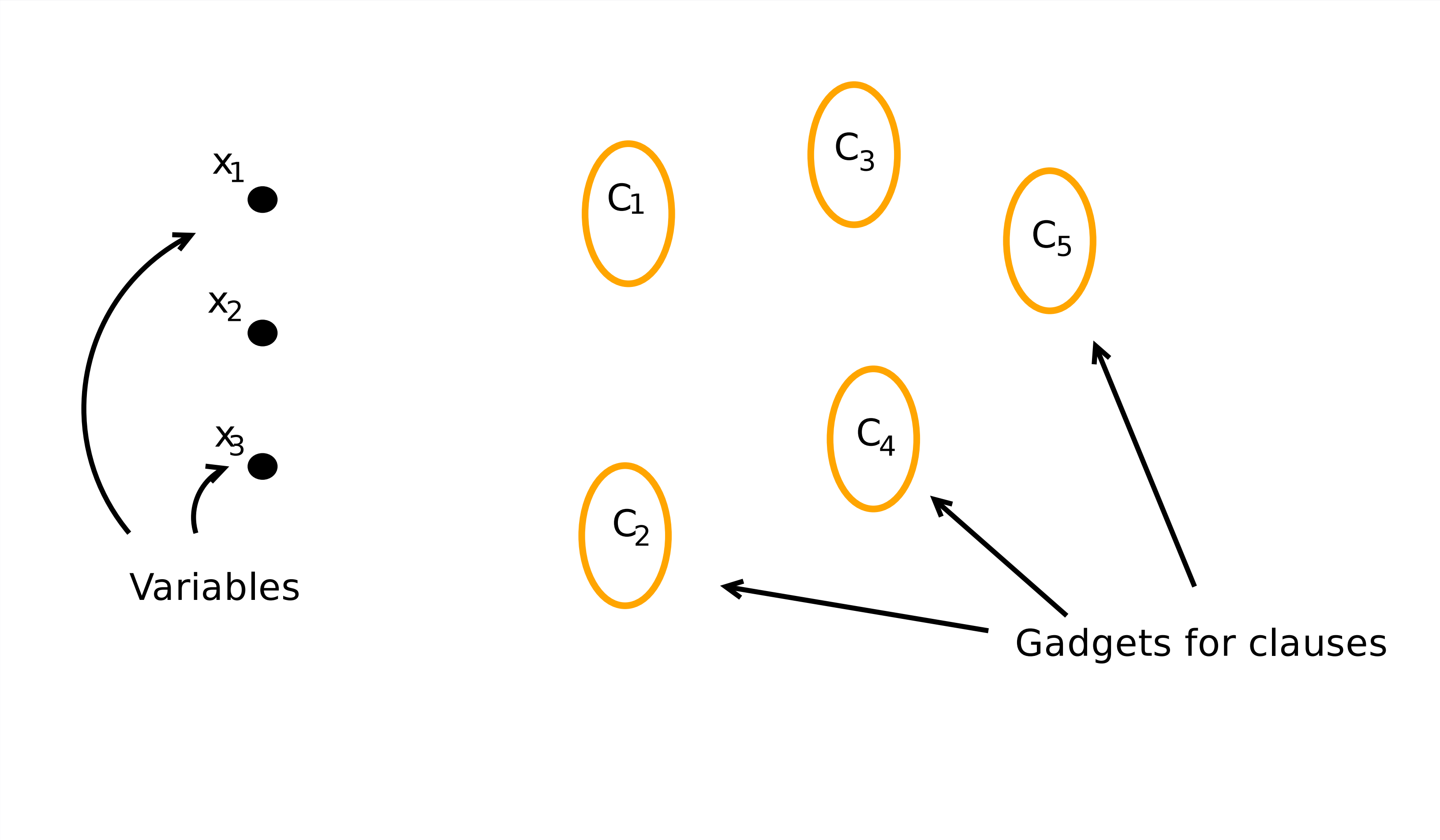
We reduce some inapproximable CSP (e.g. MAX-3SAT) to TSP.

Reduction Technique



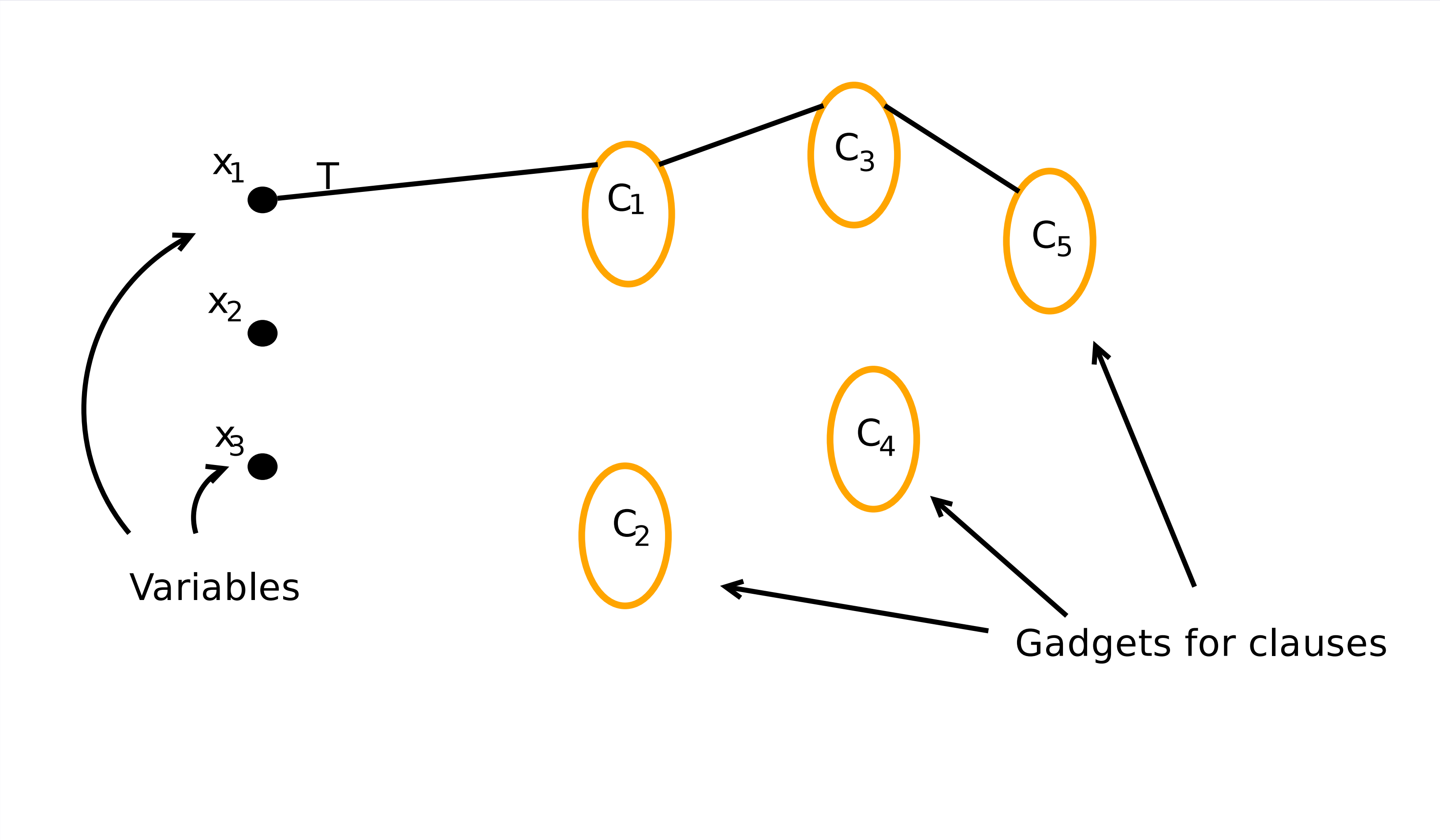
First, design some gadgets to represent the clauses

Reduction Technique



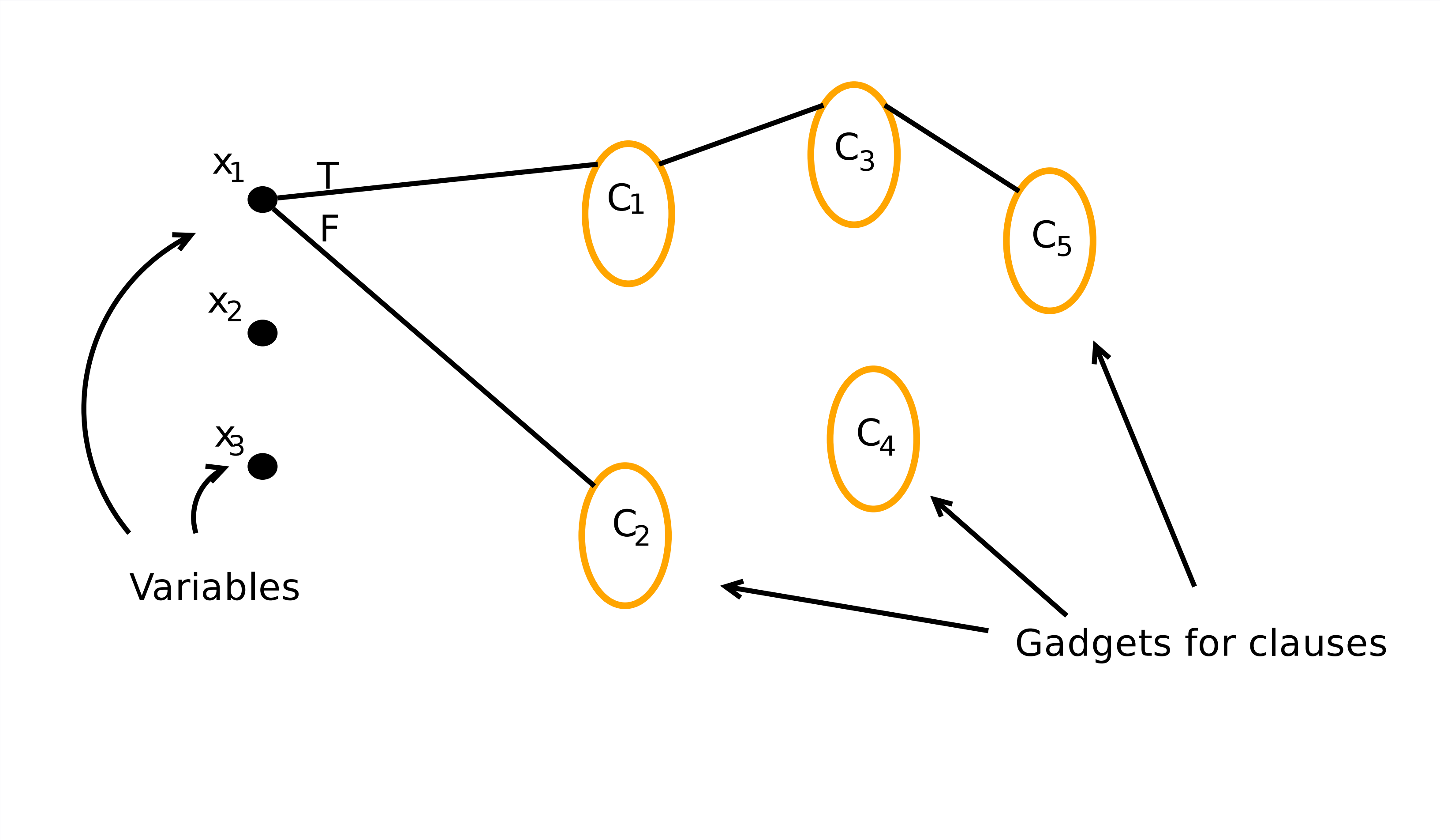
Then, add some choice vertices to represent truth assignments to variables

Reduction Technique



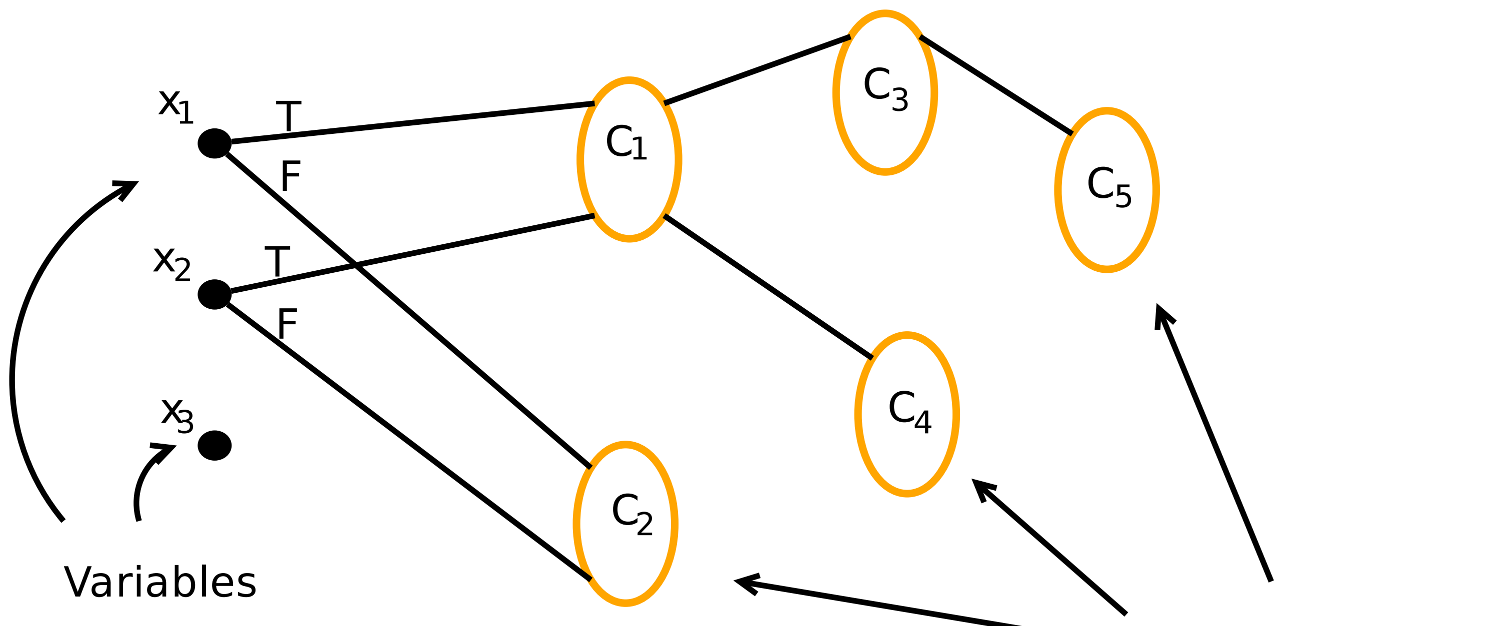
For each variable, create a path through clauses where it appears positive

Reduction Technique

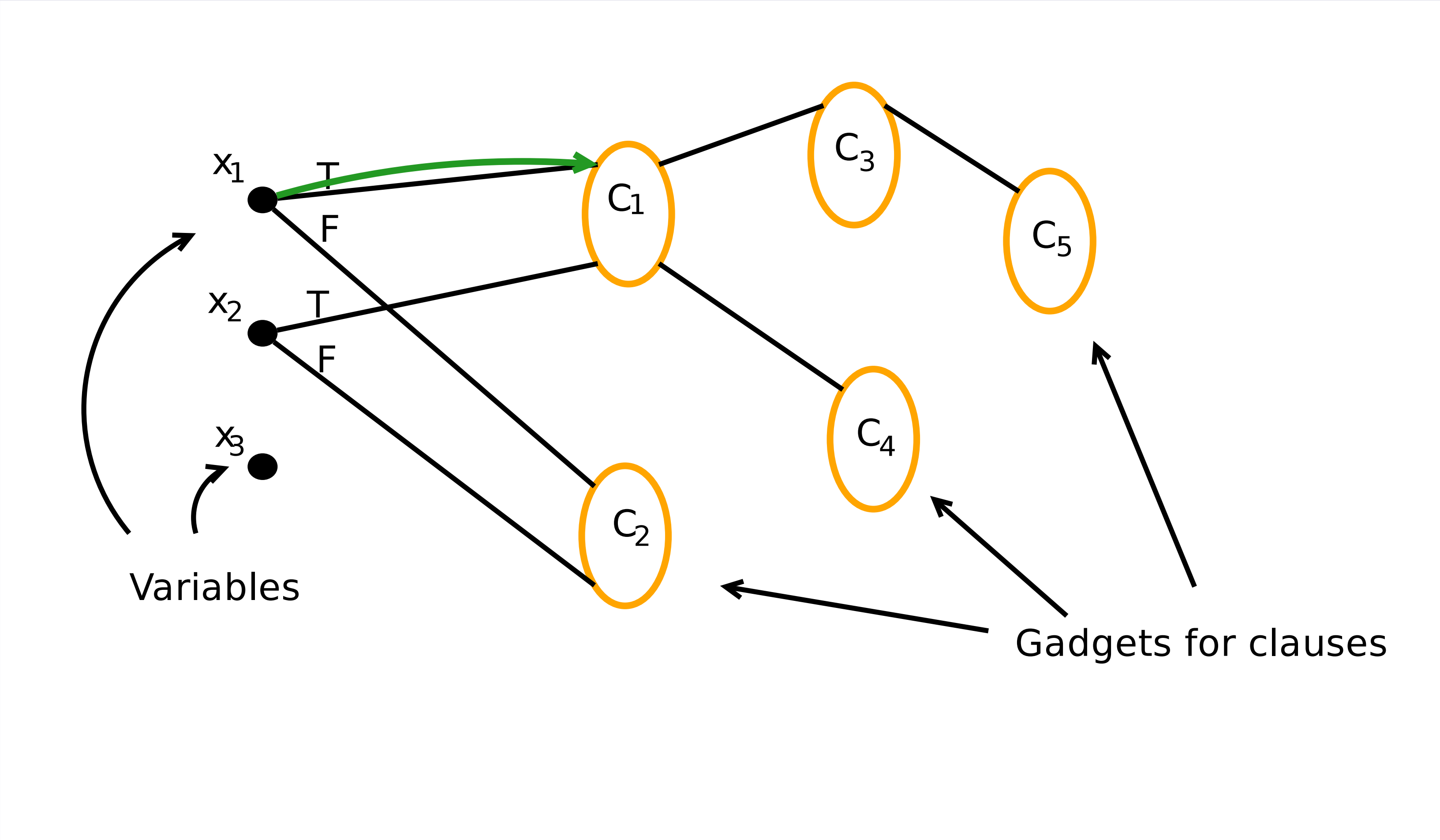


. . . and another path for its negative appearances

Reduction Technique

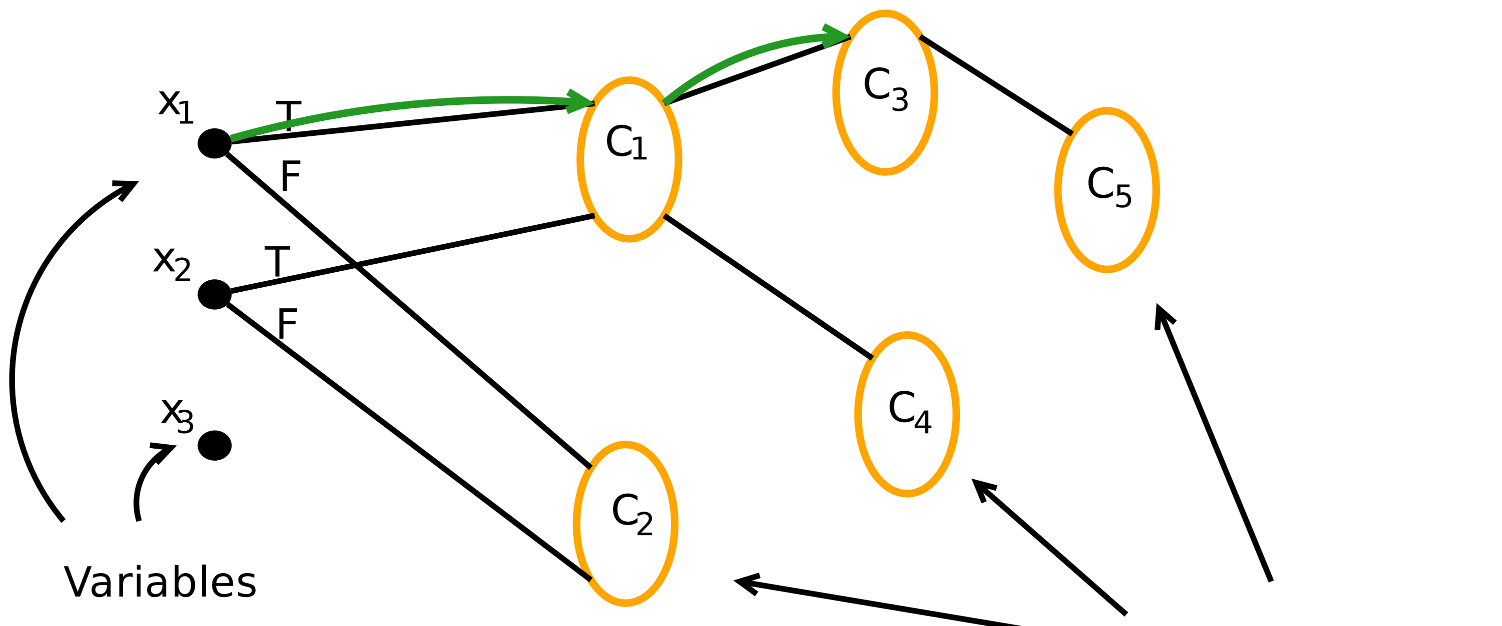


Reduction Technique

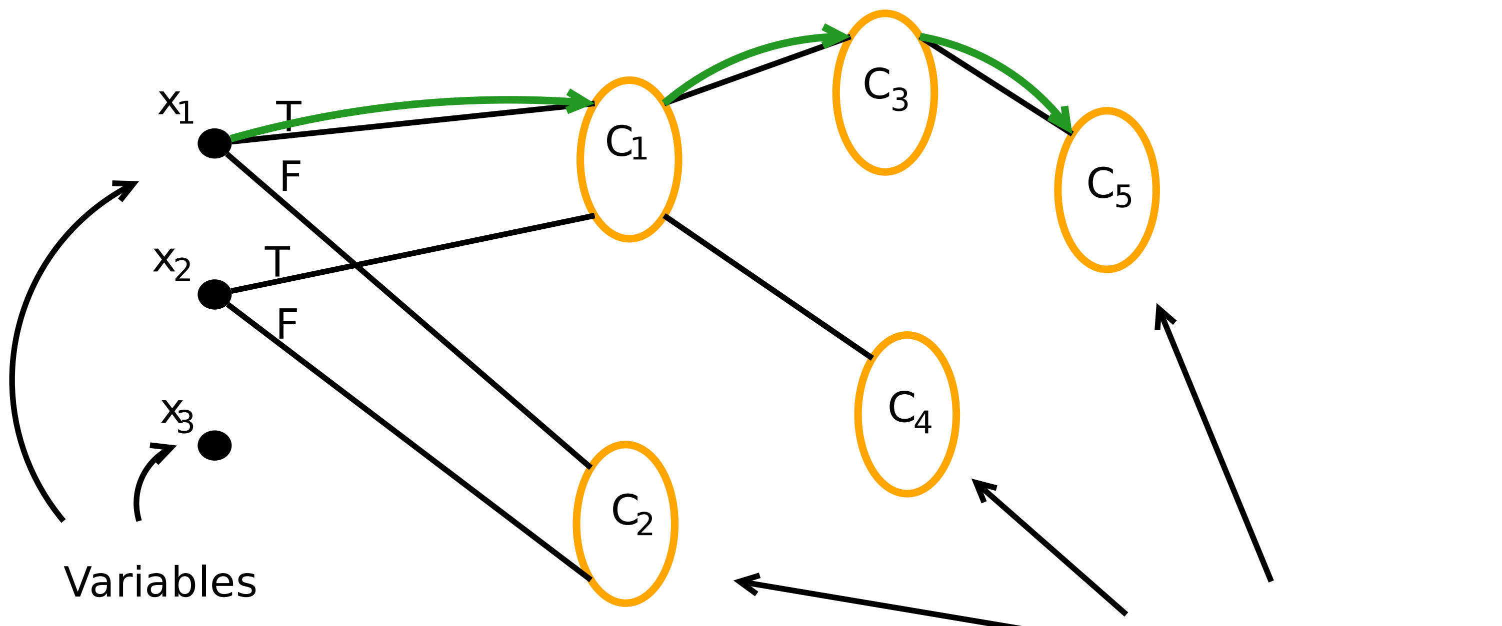


A truth assignment dictates a general path

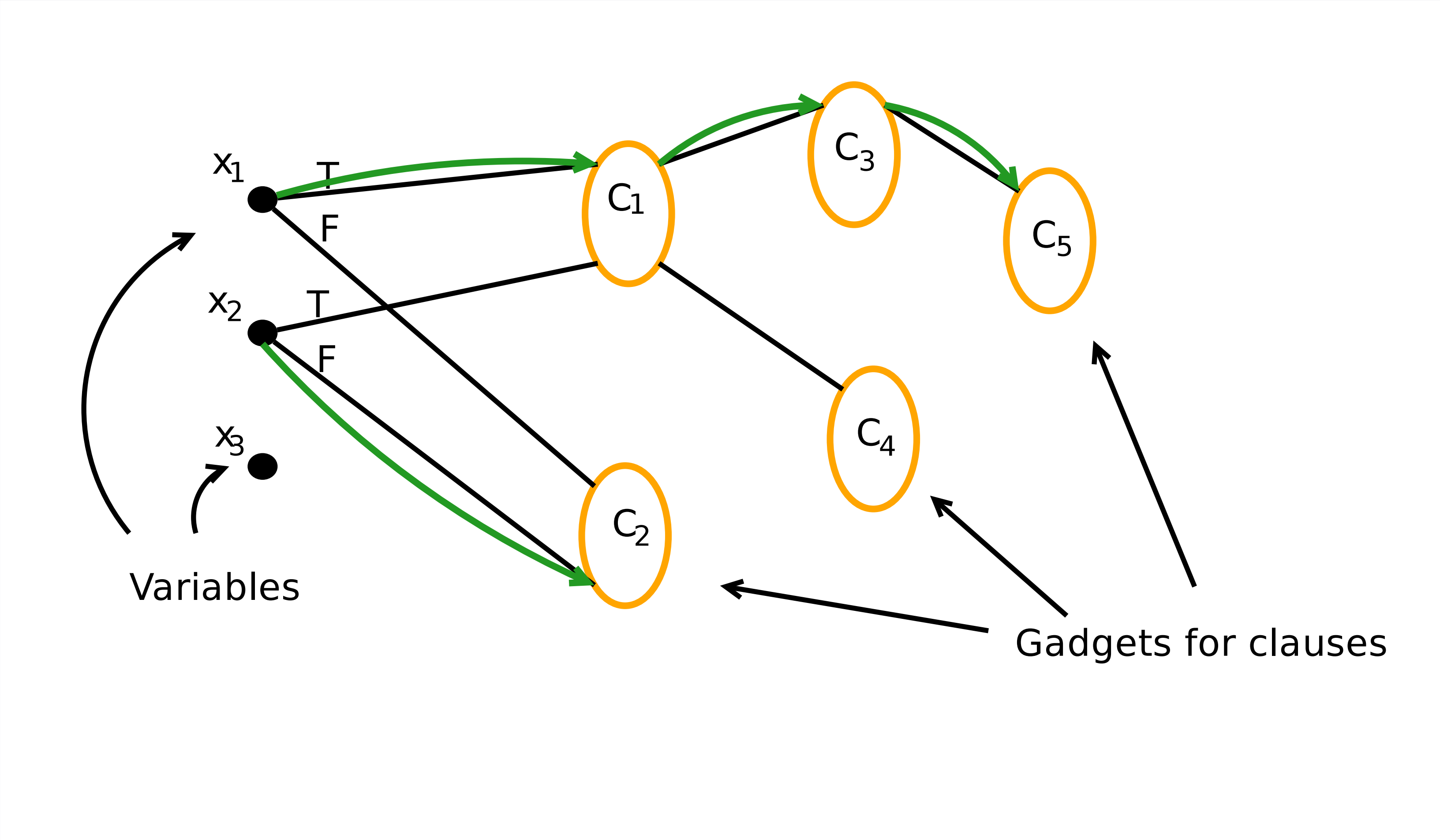
Reduction Technique



Reduction Technique

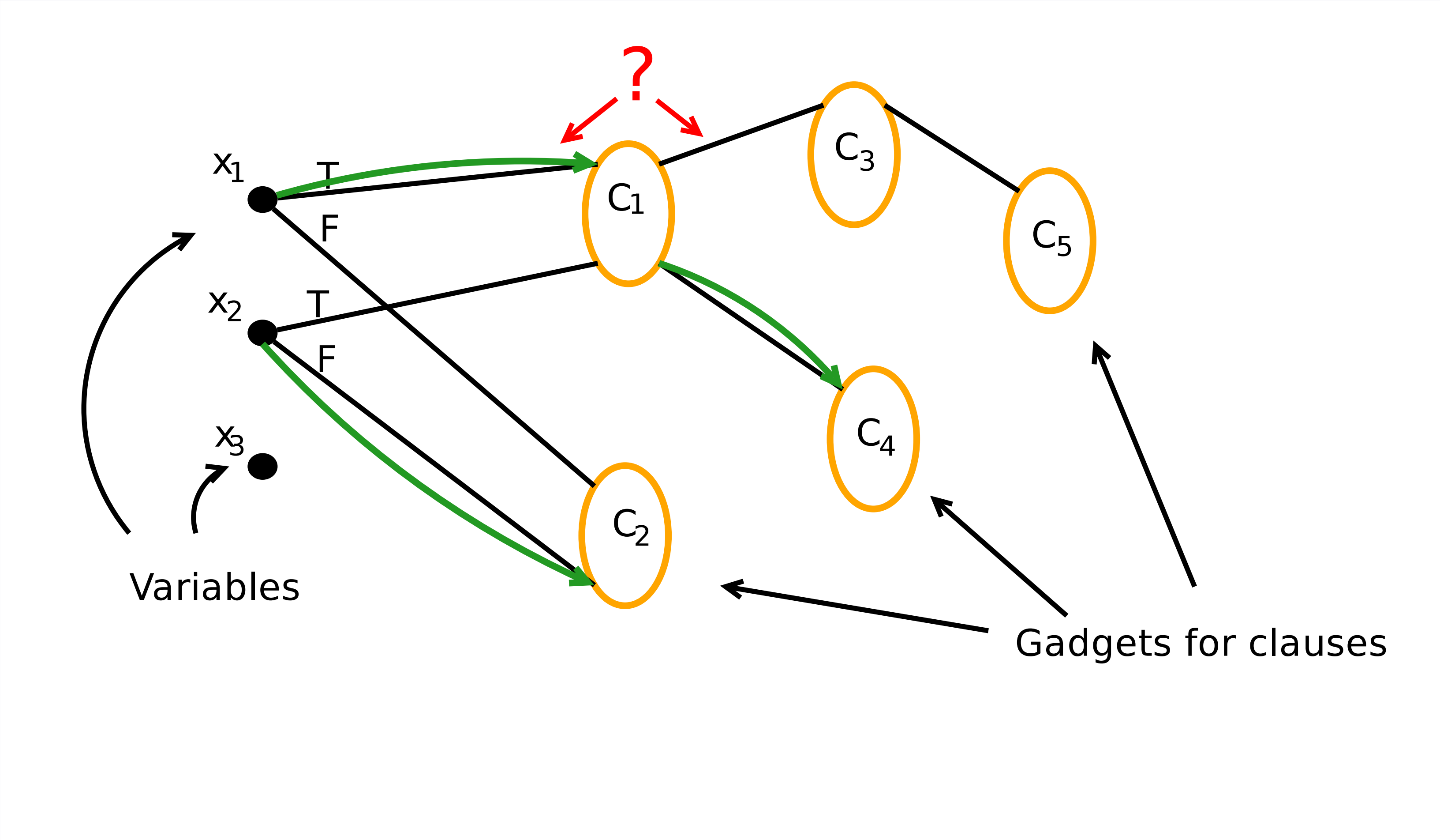


Reduction Technique



We must make sure that gadgets are cheaper to traverse if corresponding clause is satisfied

Reduction Technique



For the converse direction we must make sure that ”cheating” tours are not optimal!

How to ensure consistency

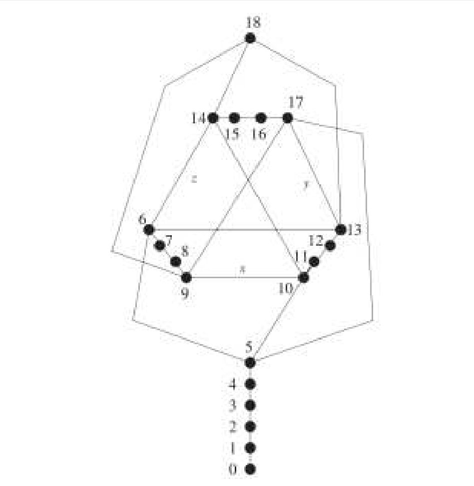


Figure 6. Equation gadget for the symmetric TSP

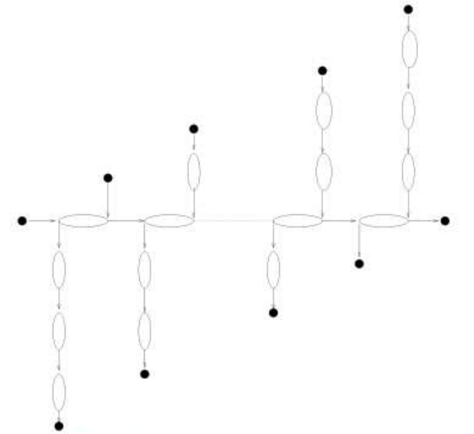


Figure 3. The neighborhood of an edge gadget

- Papadimitriou and Vempala design a gadget for Parity.
- They eliminate variable vertices altogether.
- Consistency is achieved by hooking up gadgets ”randomly”
  - In fact gadgets that share a variable are connected according to the structure dictated by a special graph
  - The graph is called a ”pusher”. Its existence is proved using the probabilistic method.

How to ensure consistency

- Basic idea here: consistency would be easy if each variable occurred at most c times, c a constant.
    - Cheating would only help a tour ”fix” a bounded number of clauses.

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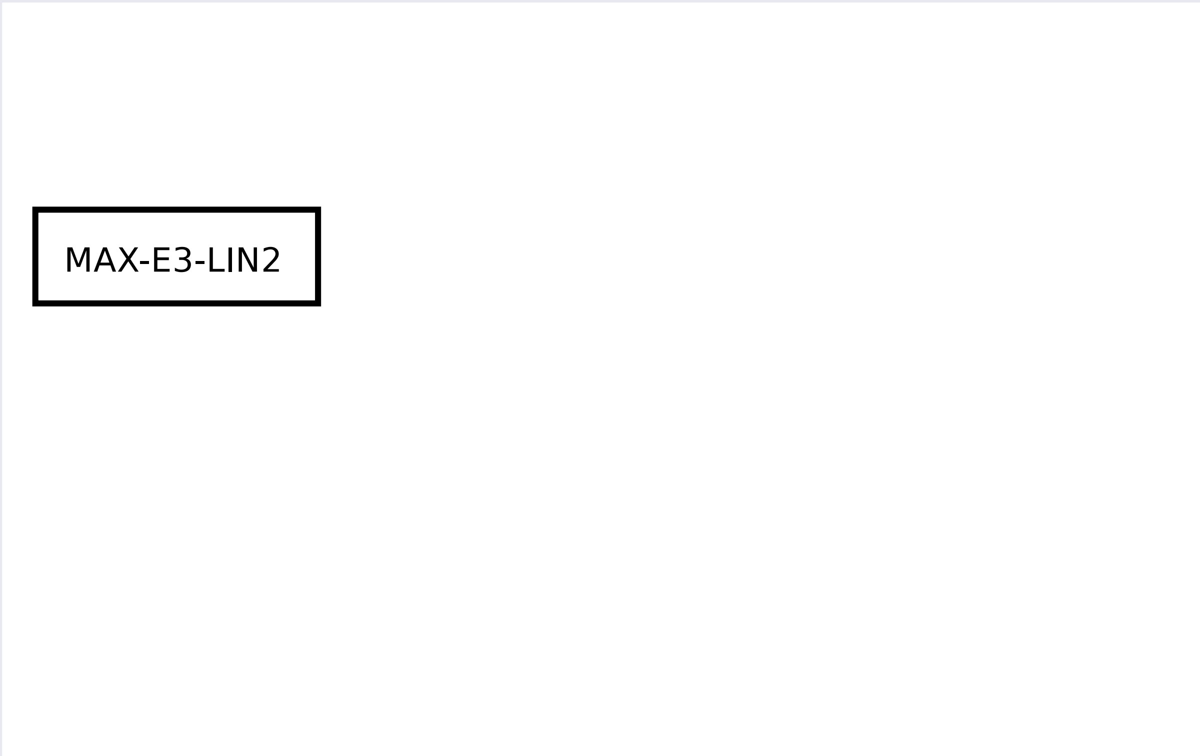
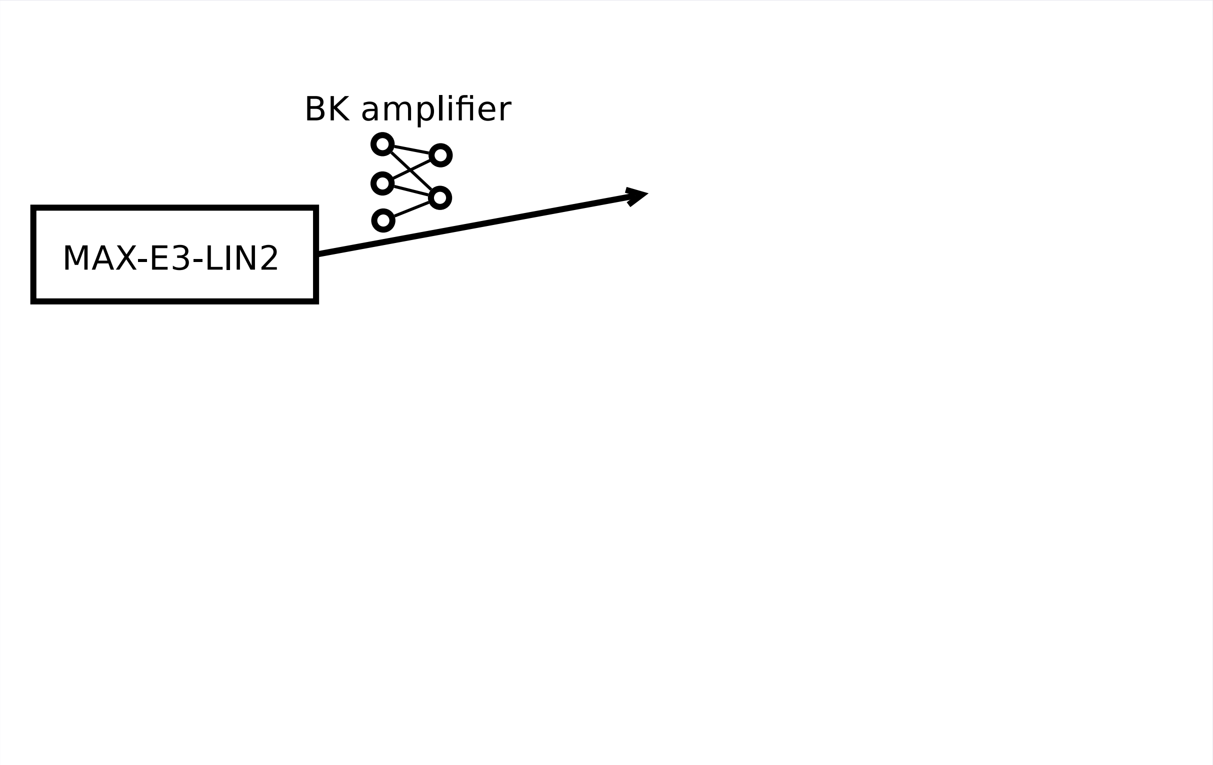
- We will rely on techniques and tools used to prove inapproximability for bounded-occurrence CSPs.
  - Main tool: an ”amplifier graph” construction due to Berman and Karpinski.
- Result: an easier hardness proof that can be broken down into independent pieces, and also gives an improved bound.

Overview



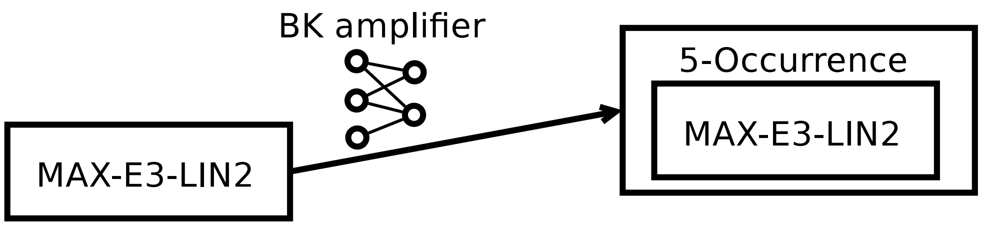
We start from an instance of MAX-E3-LIN2. Given a set of linear equations (mod 2) each of size three satisfy as many as possible. Known to be 2-inapproximable (Håstad).

Overview



We use the Berman-Karpinski amplifier construction to obtain an instance where each variable appears exactly 5 times (and most equations have size 2).

Overview



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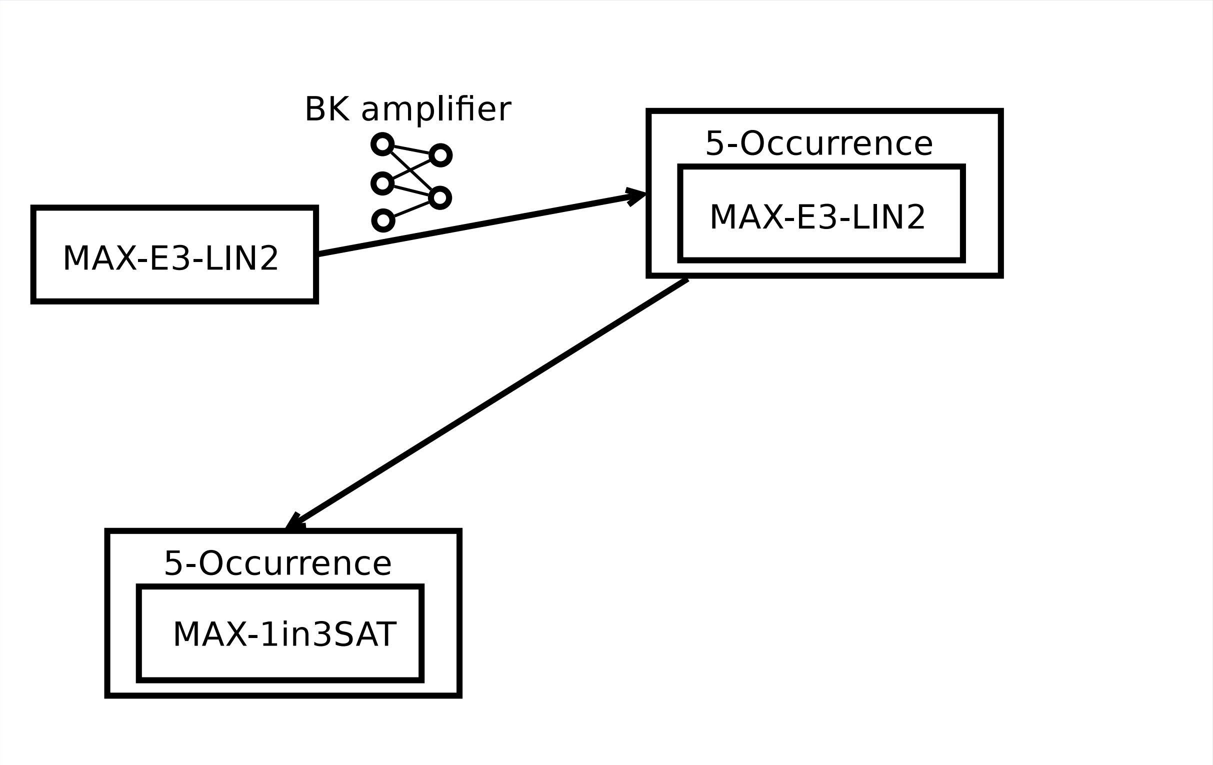
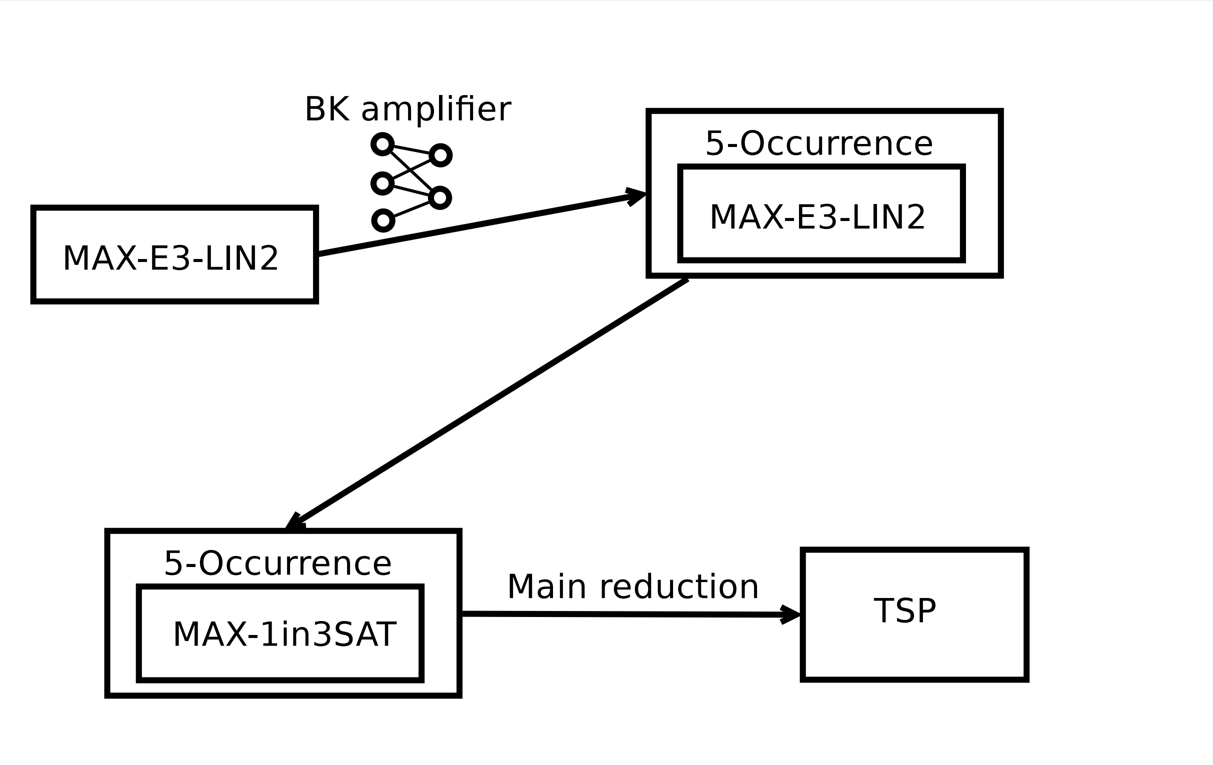


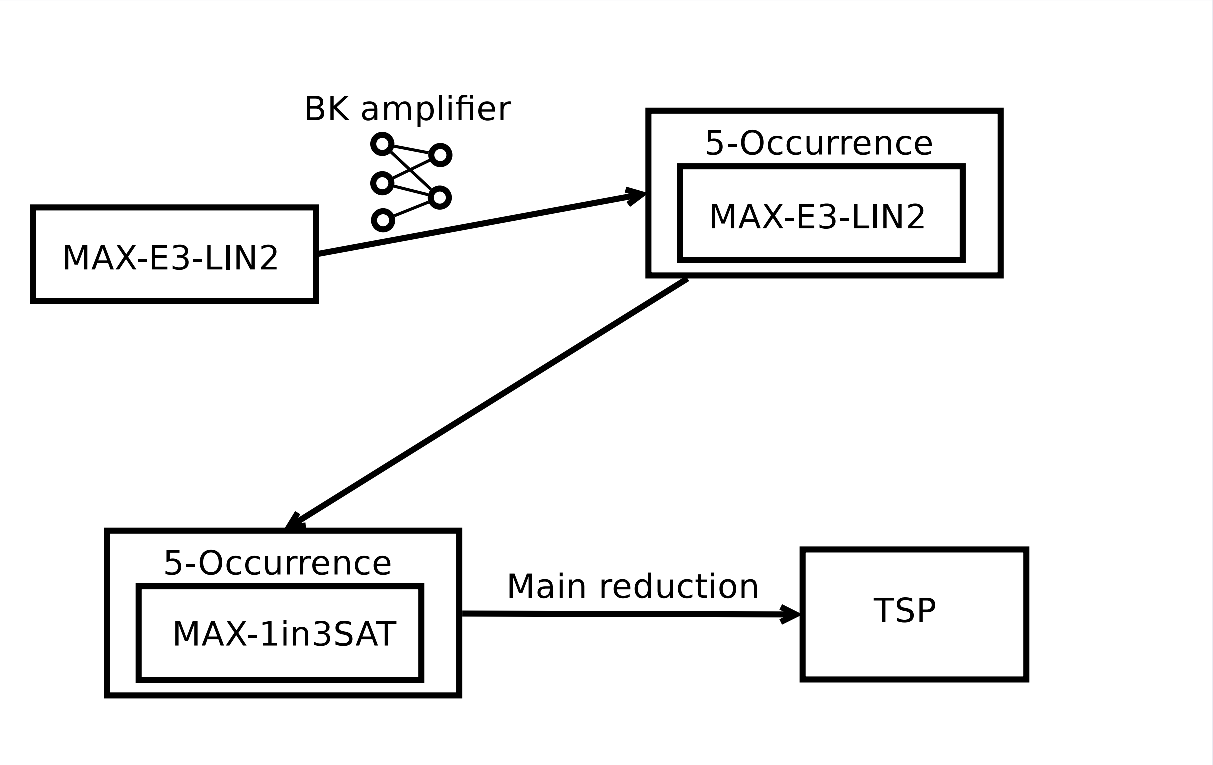
A simple trick reduces this to the 1in3 predicate.

Overview



From this instance we construct a graph.

Overview



From this instance we construct a graph.

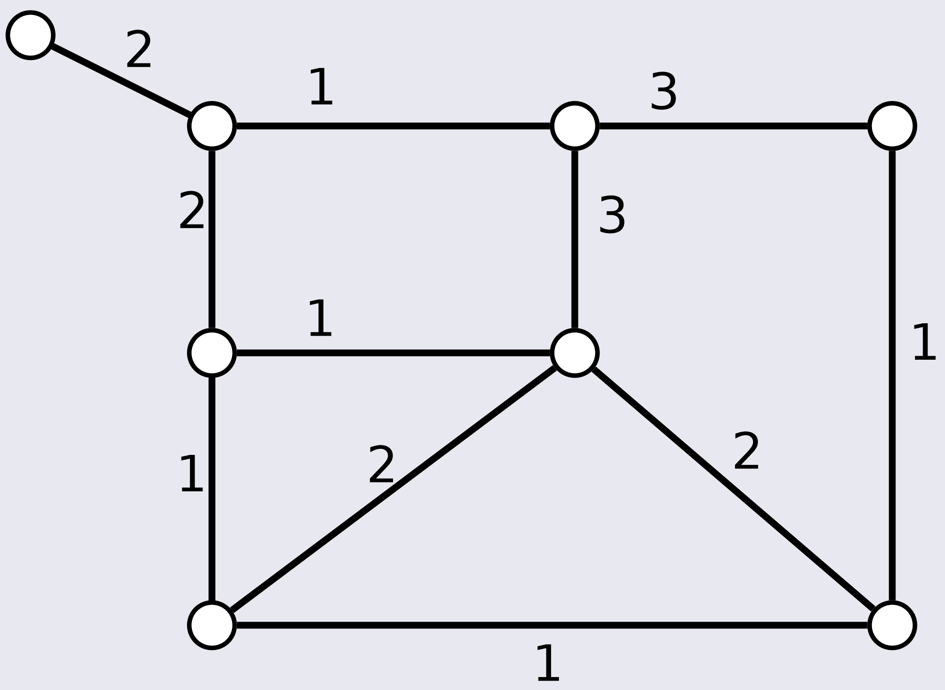
Rest of this talk: some more details about the construction.

1in3-SAT

Input:
A set of clauses $(l_1 \vee l_2 \vee l_3)$, l_1, l_2, l_3 literals.
Objective:
A clause is satisfied if exactly one of its literals is true. Satisfy as many clauses as possible.

- Easy to reduce MAX-LIN2 to this problem.
  - Especially for size two equations $(x + y = 1) \leftrightarrow (x \vee y)$.
- Naturally gives gadget for TSP
  - In TSP we'd like to visit each vertex at least once, but not more than once (to save cost)

TSP and Euler tours



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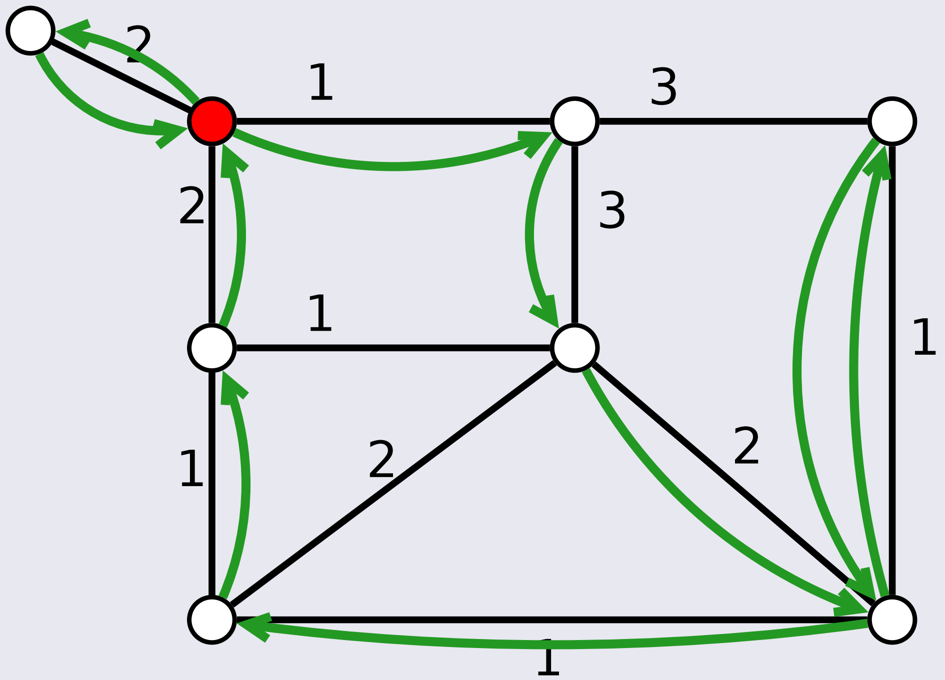
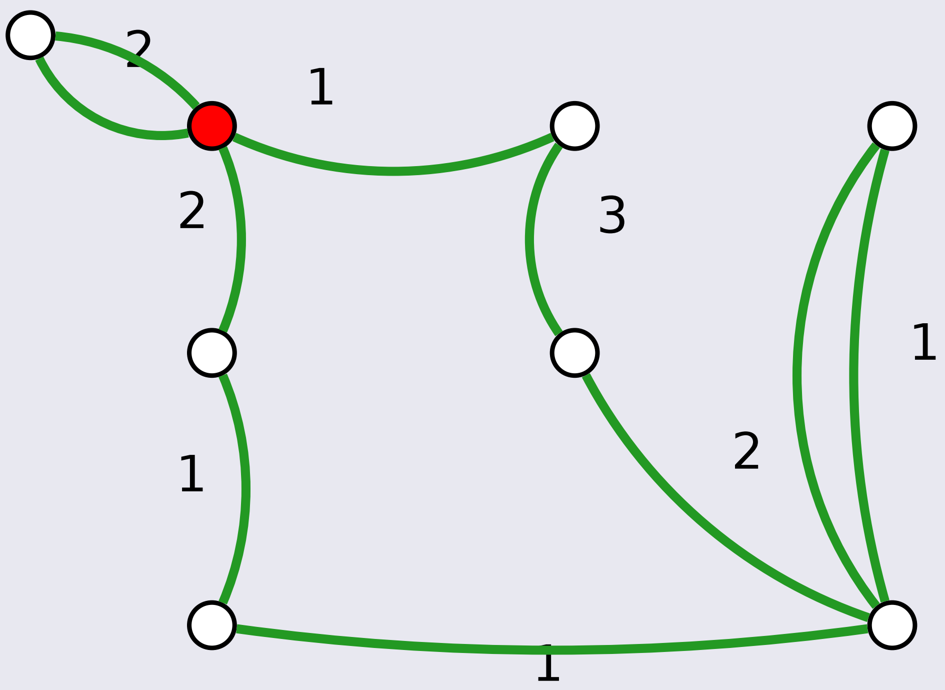


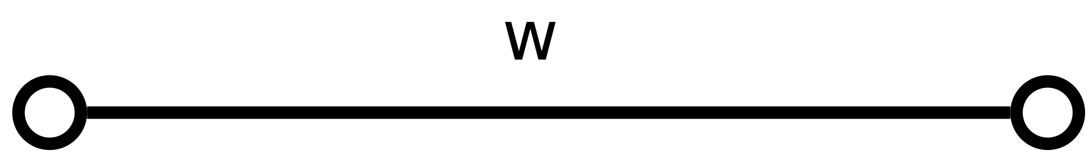
TSP and Euler tours



TSP and Euler tours

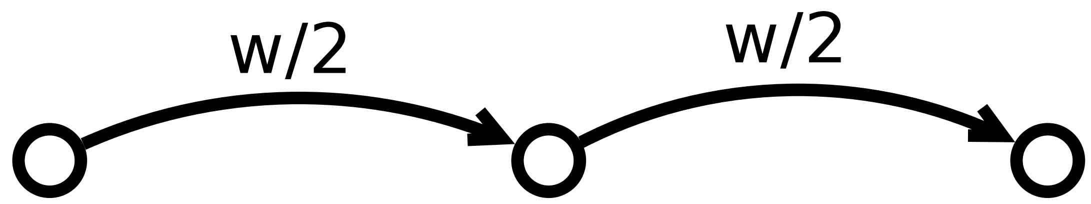
- A TSP tour gives an Eulerian multi-graph composed with edges of G.
- An Eulerian multi-graph composed with edges of G gives a TSP tour.
  - TSP $\equiv$ Select a multiplicity for each edge so that the resulting multi-graph is Eulerian and total cost is minimized
  - **Note**: no edge is used more than twice

Gadget – Forced Edges



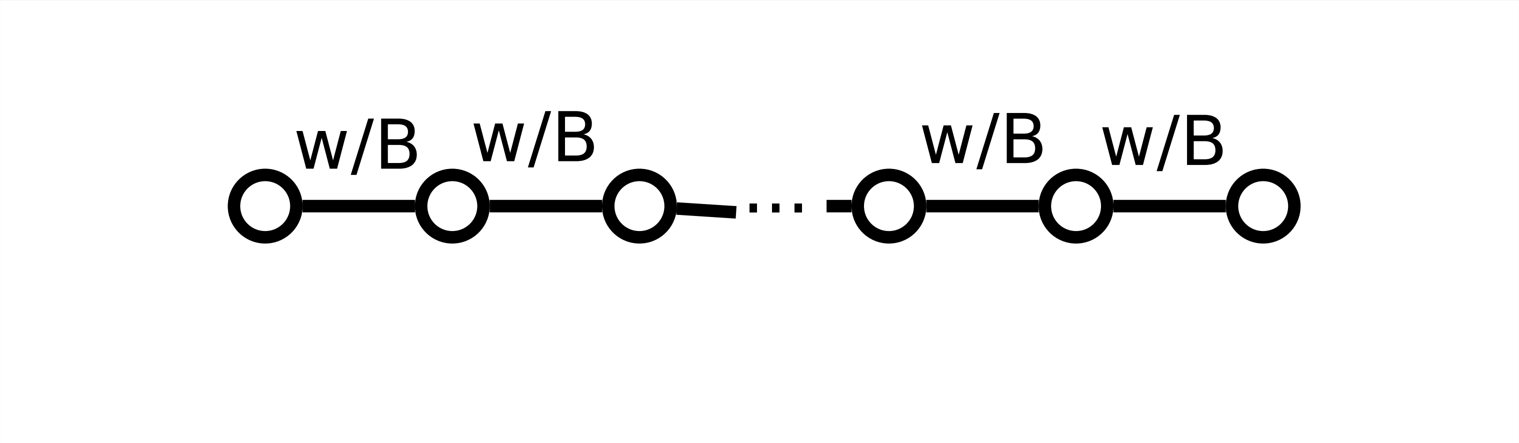
We would like to be able to dictate in our construction that a certain edge has to be used at least once.

Gadget – Forced Edges



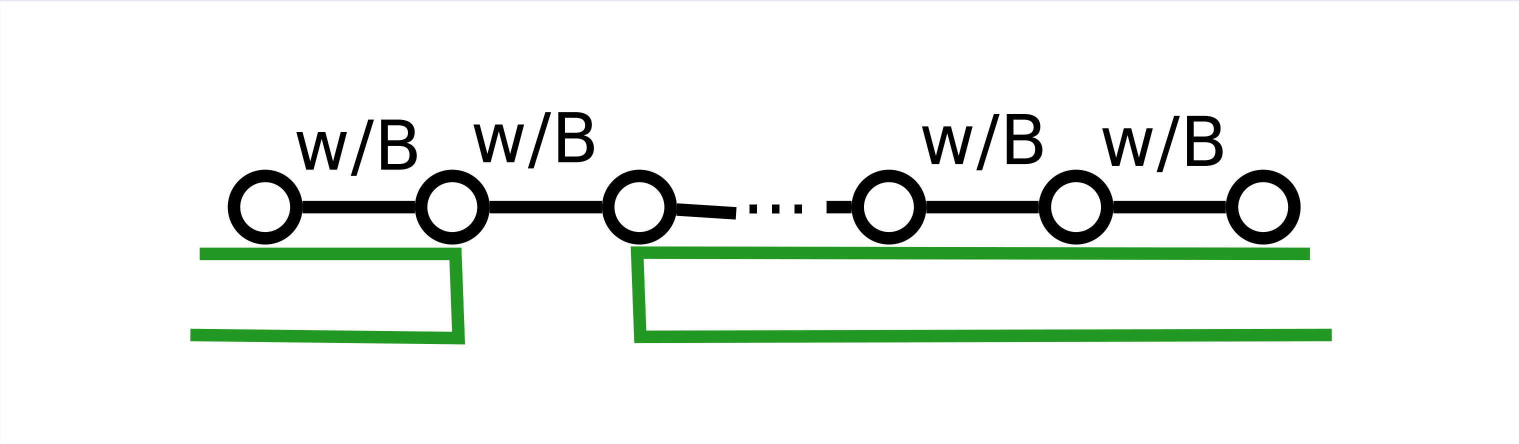
If we had directed edges, this could be achieved by adding a dummy intermediate vertex

Gadget – Forced Edges



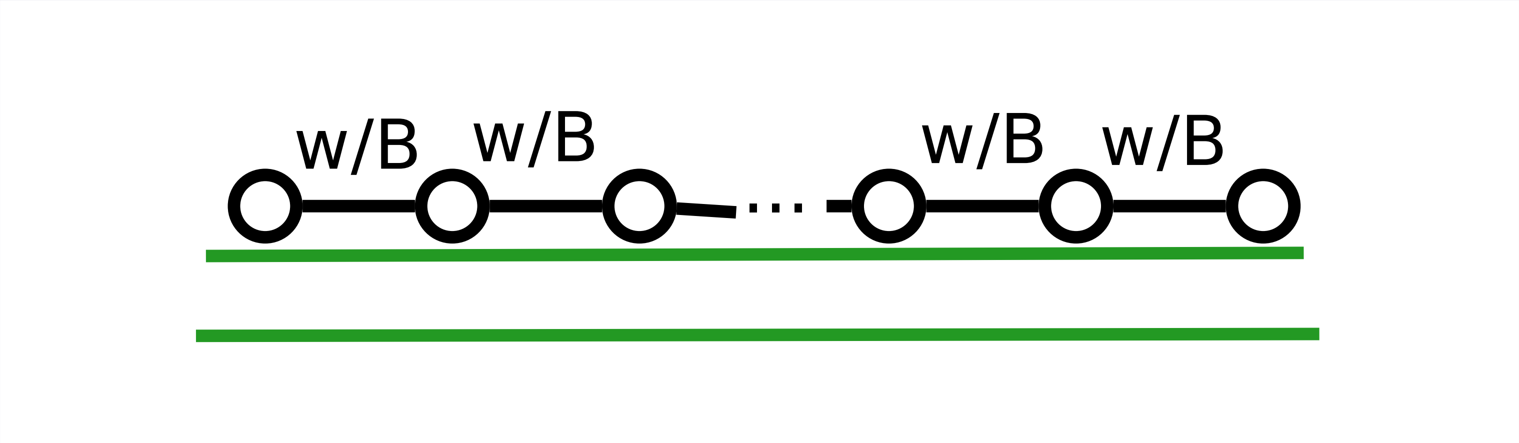
Here, we add many intermediate vertices and evenly distribute the weight w among them. Think of B as very large.

Gadget – Forced Edges



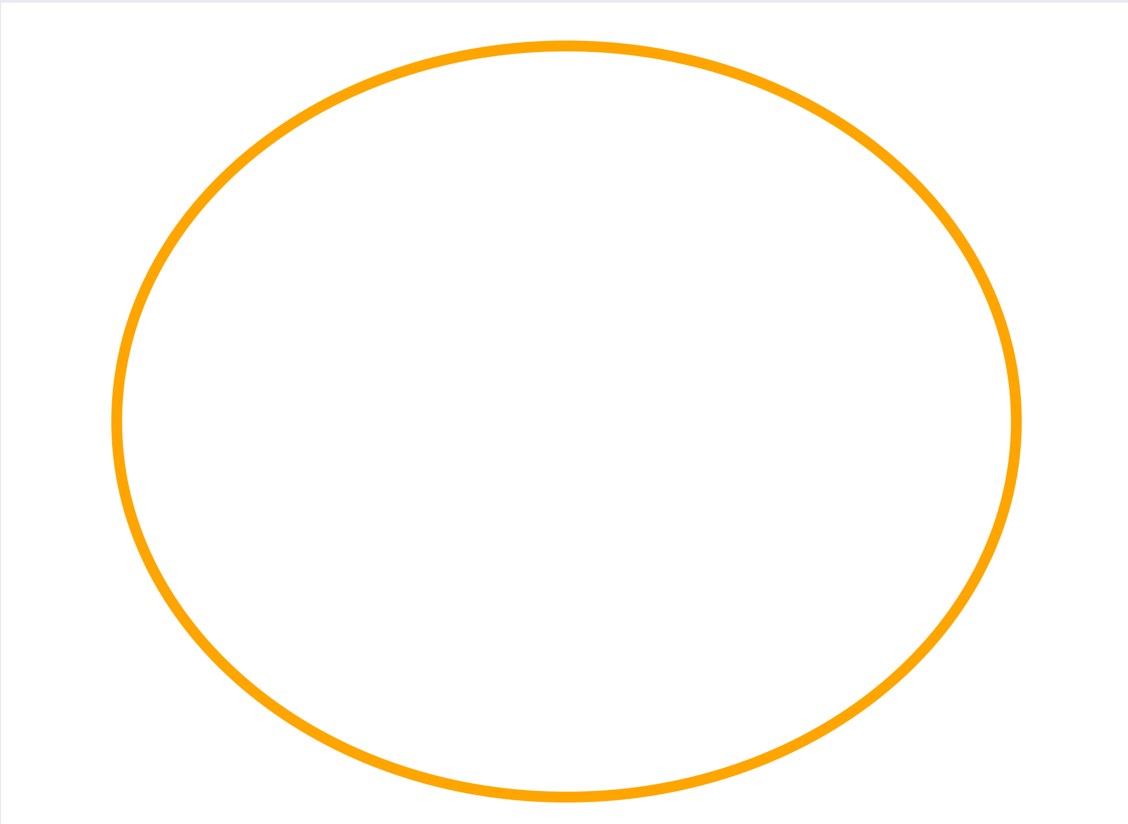
At most one of the new edges may be unused, and in that case all others are used twice.

Gadget – Forced Edges



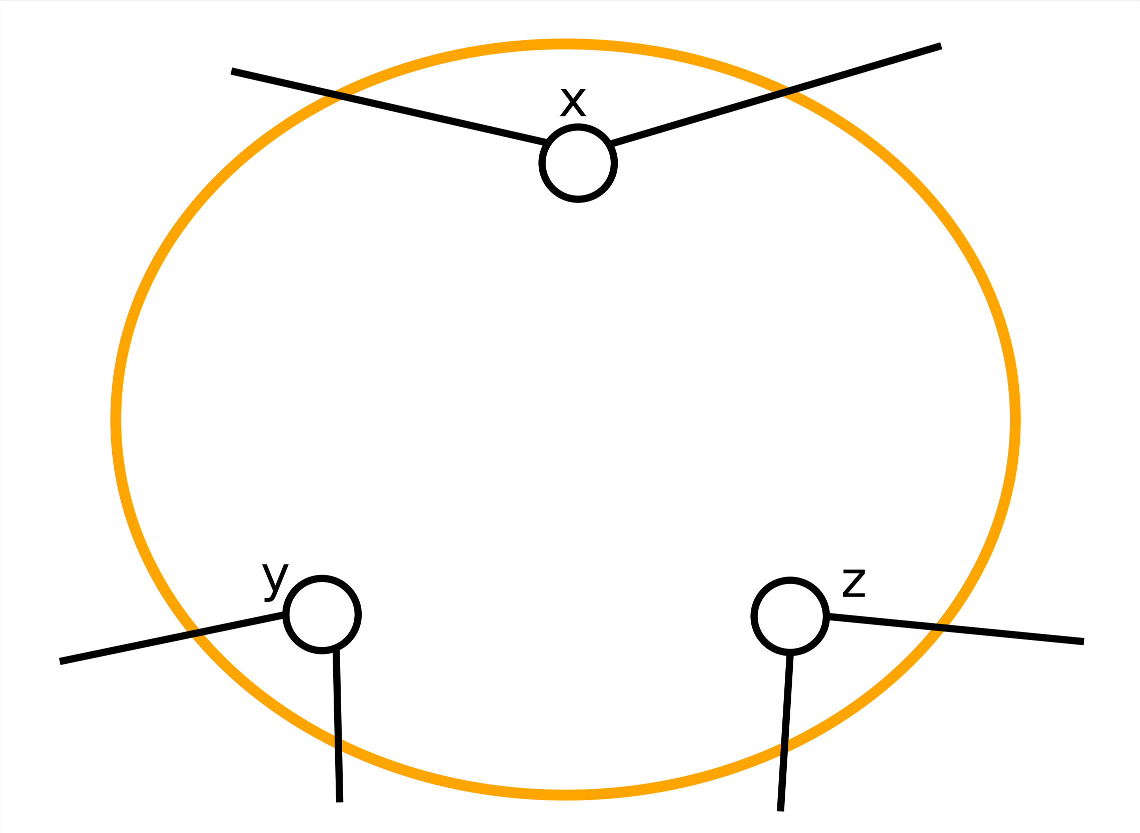
In that case, adding two copies of that edge to the solution doesn't hurt much (for B sufficiently large).

1in3 Gadget



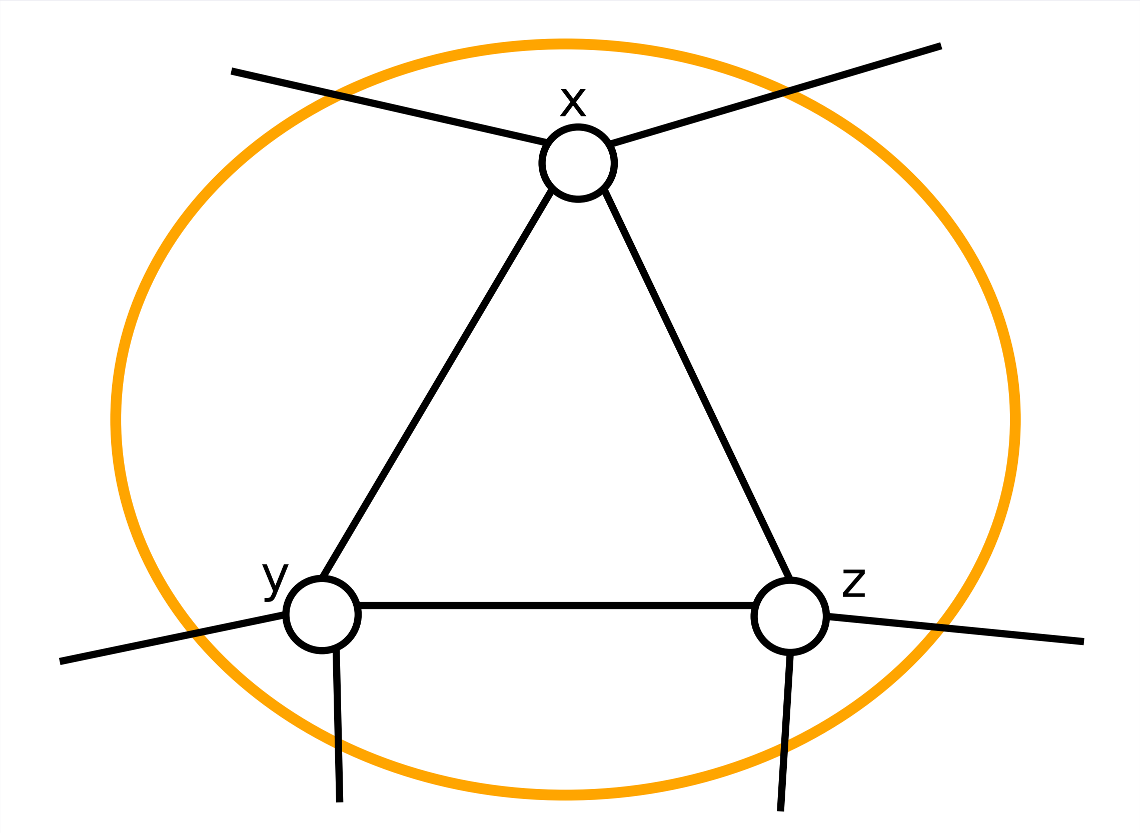
Let's design a gadget for $(x \vee y \vee z)$

1in3 Gadget



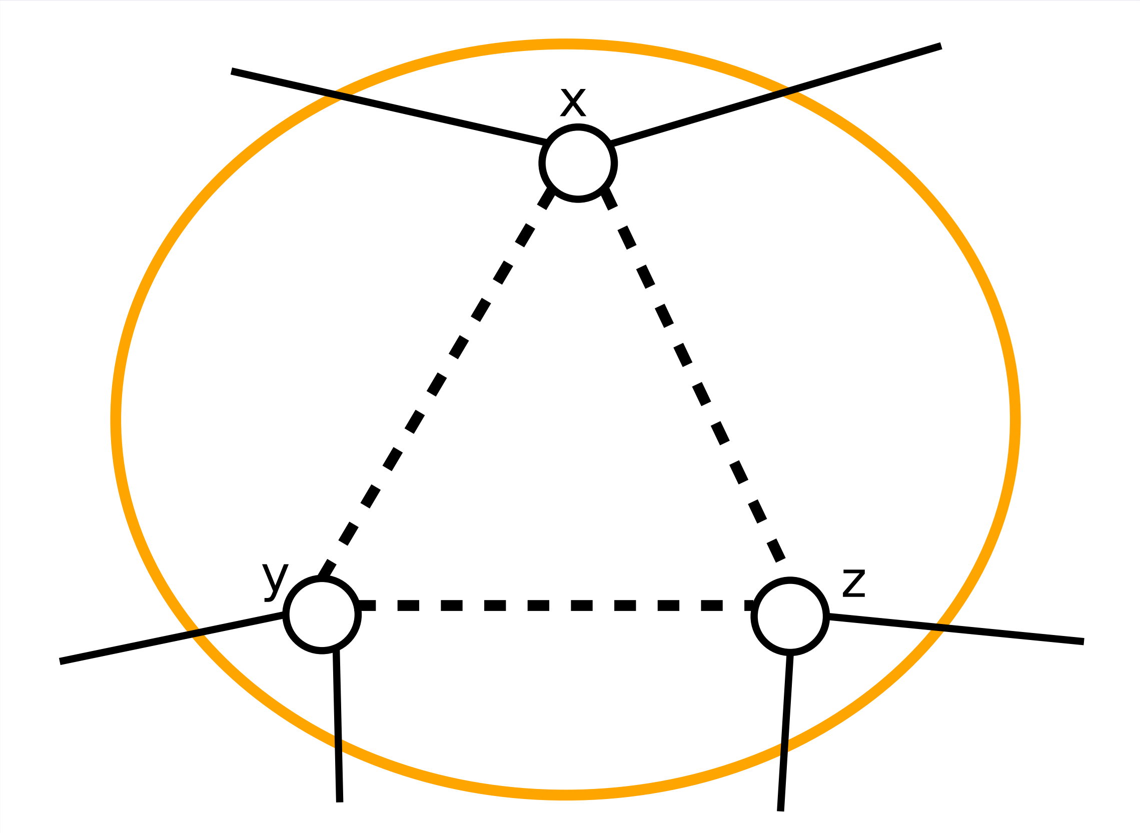
First, three entry/exit points

1in3 Gadget



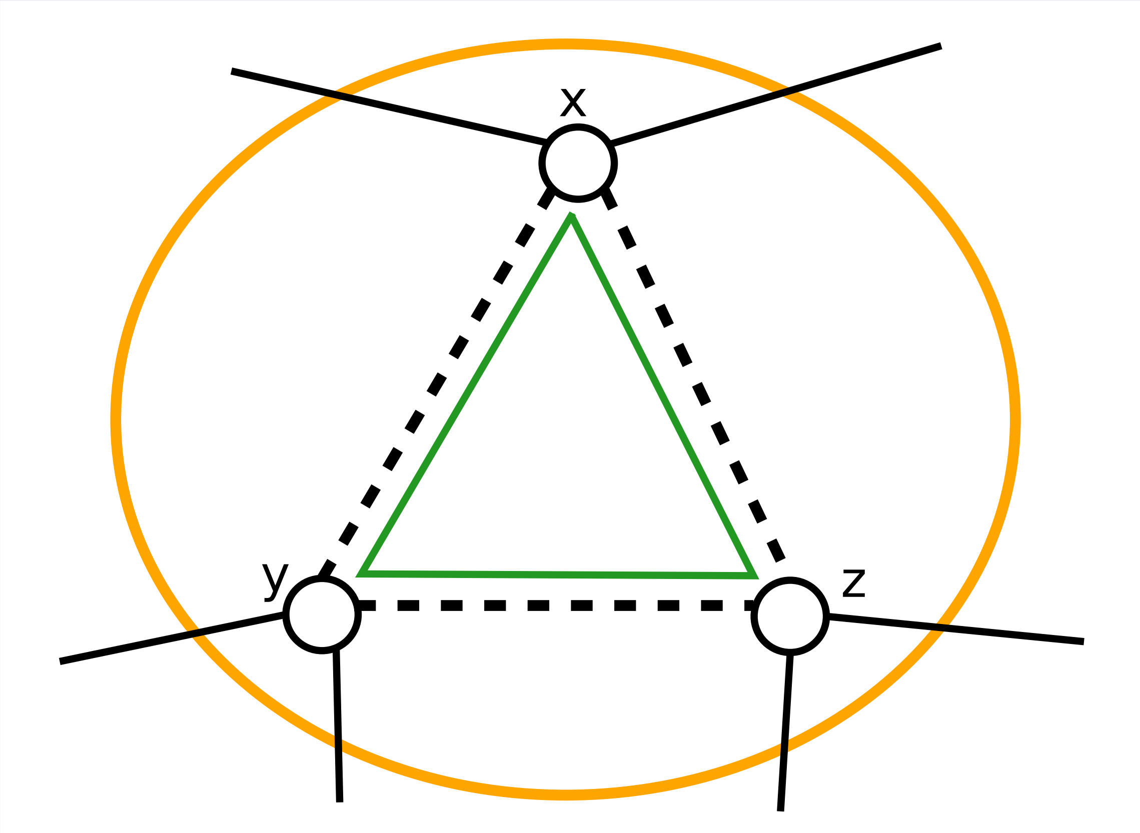
Connect them . . .

1in3 Gadget



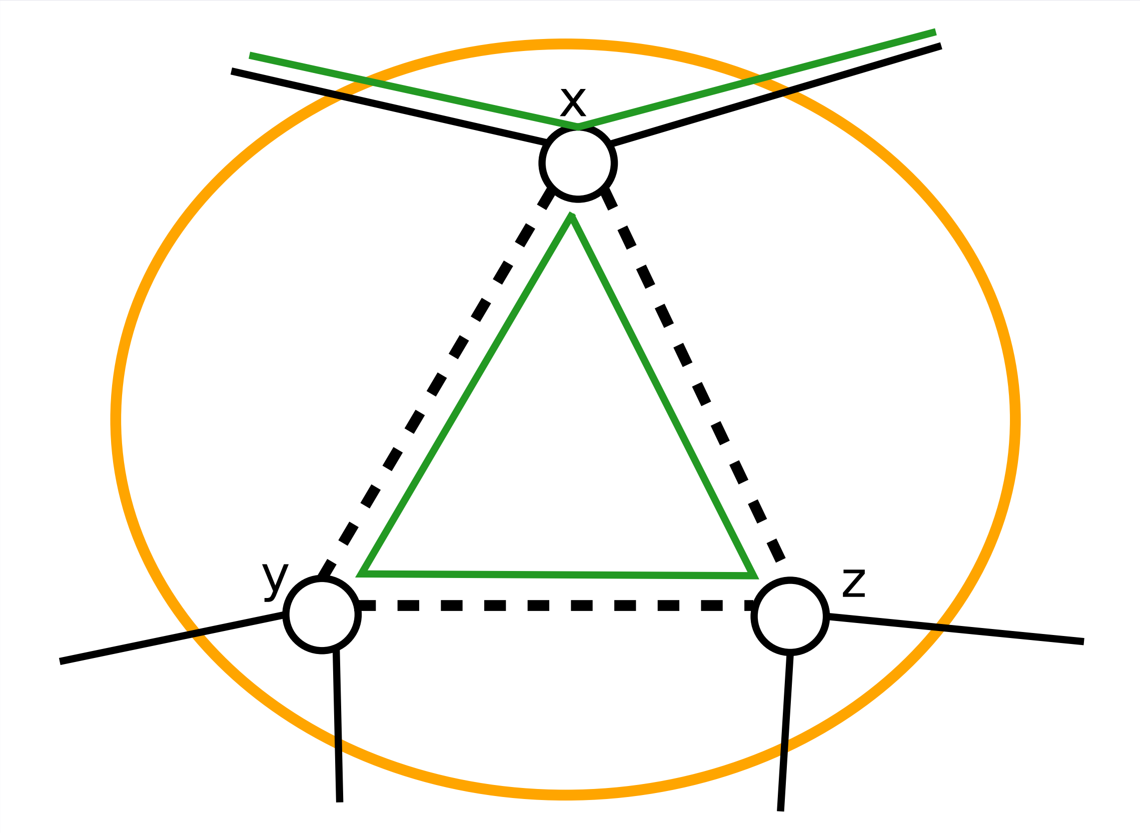
. . . with forced edges

1in3 Gadget



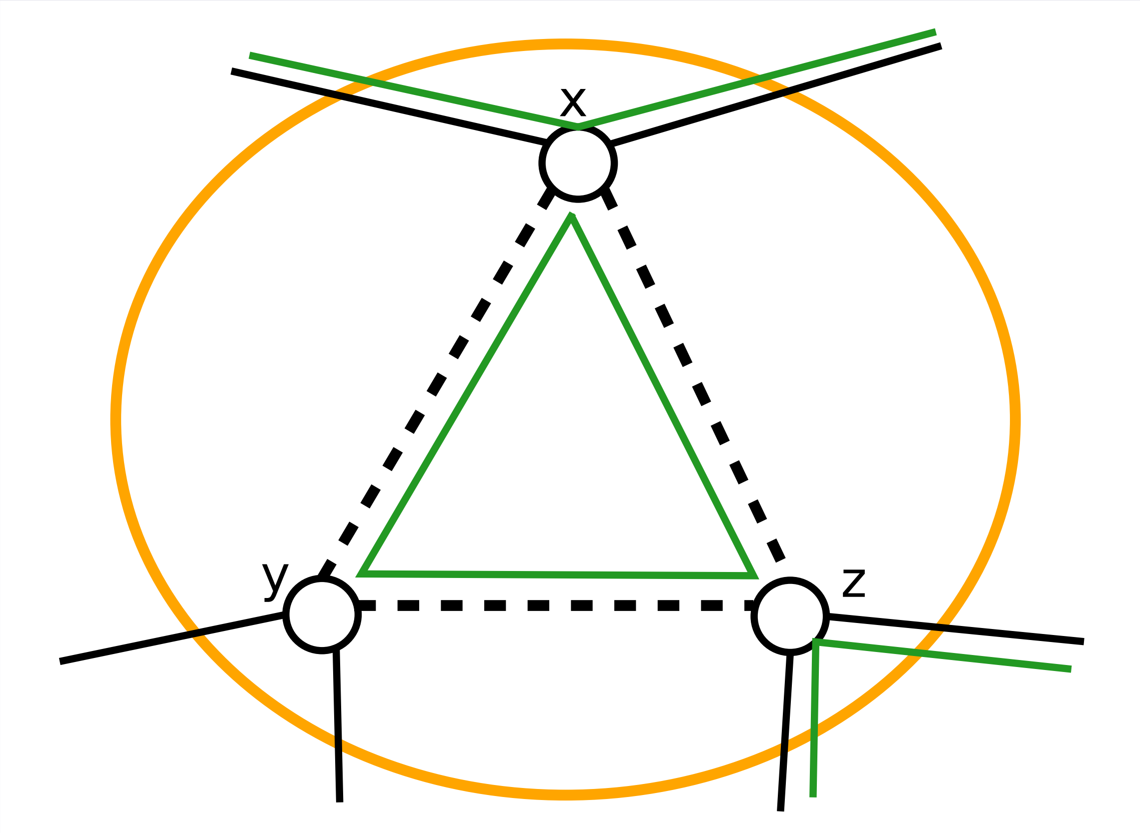
The gadget is a connected component. A good tour visits it once.

1in3 Gadget



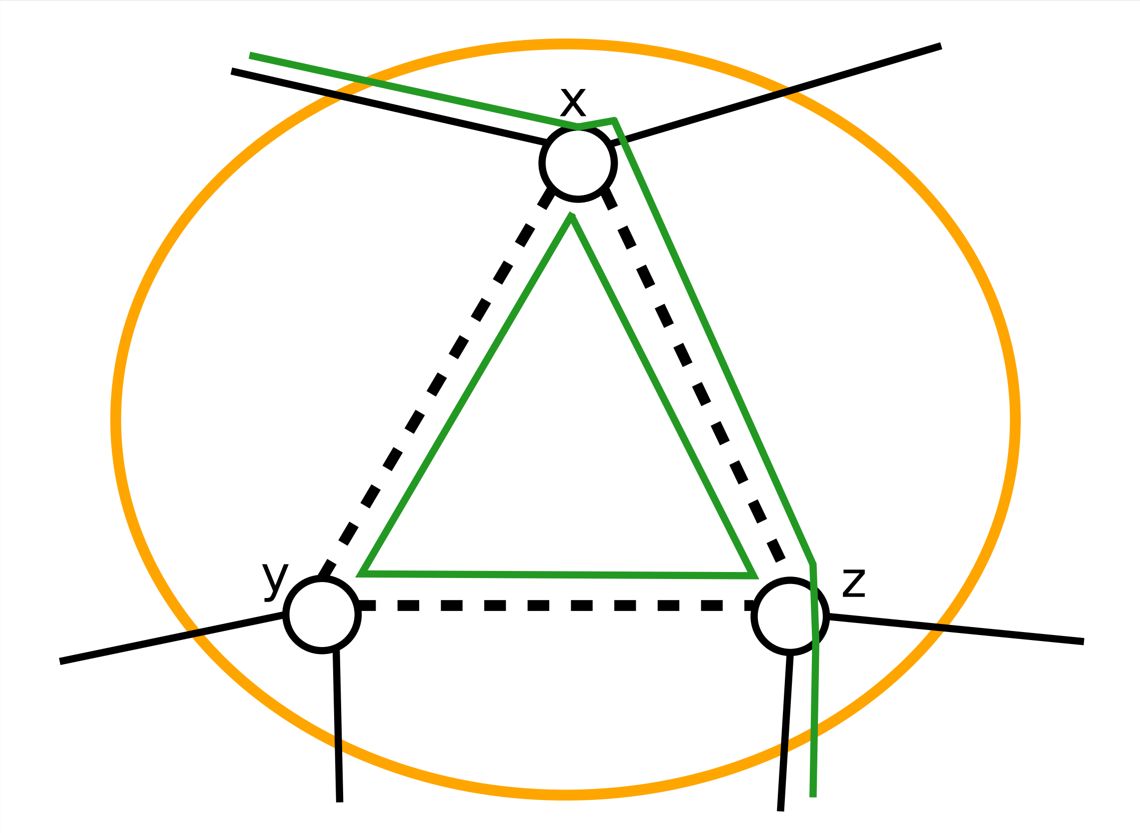
. . . like this

1in3 Gadget



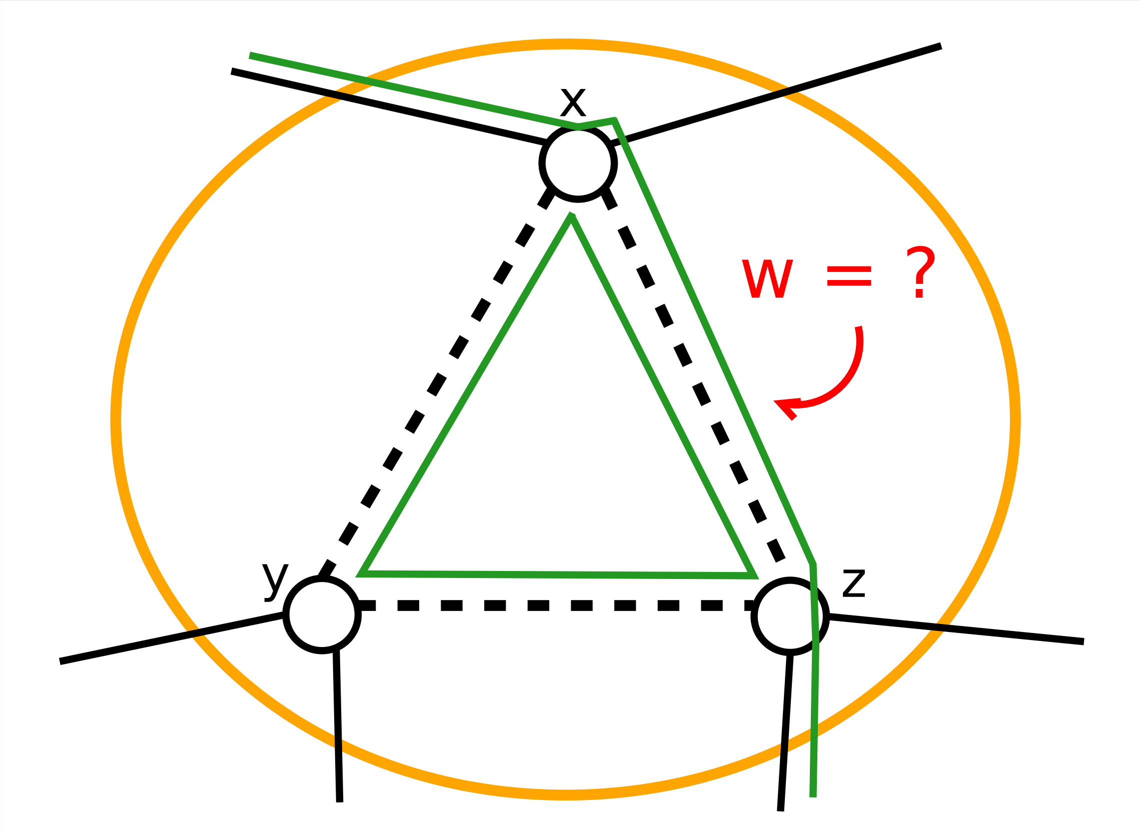
This corresponds to an unsatisfied clause

1in3 Gadget



This corresponds to a dishonest tour

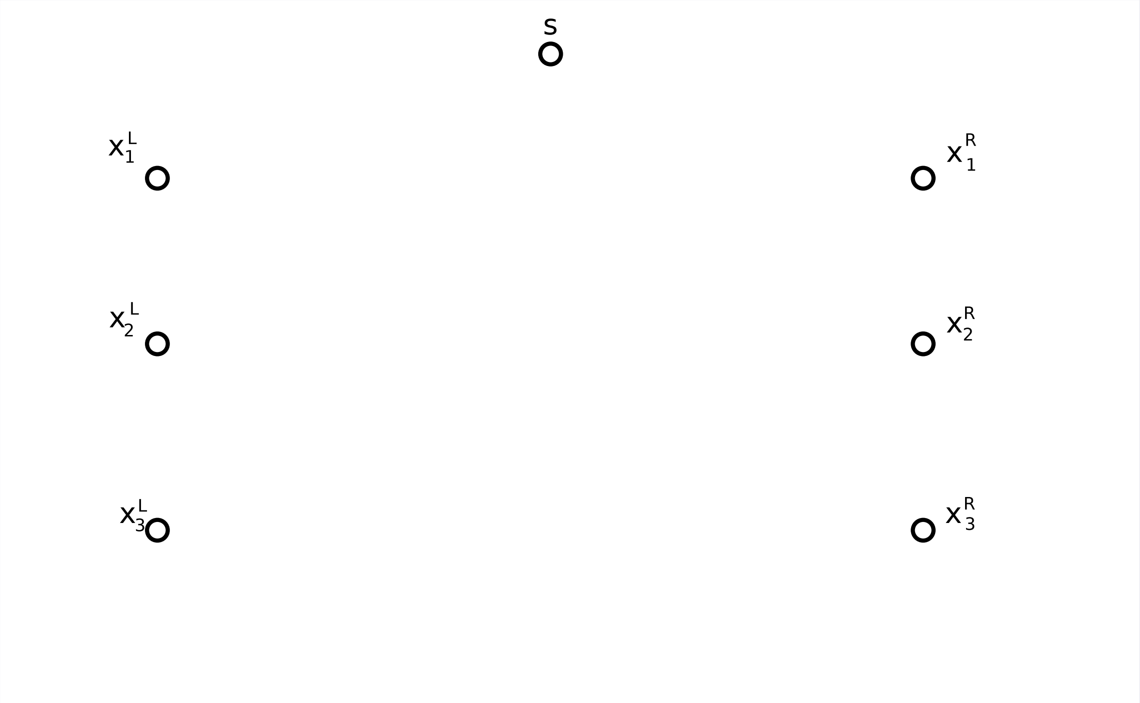
1in3 Gadget



The dishonest tour pays this edge twice. How expensive must it be before cheating becomes suboptimal?

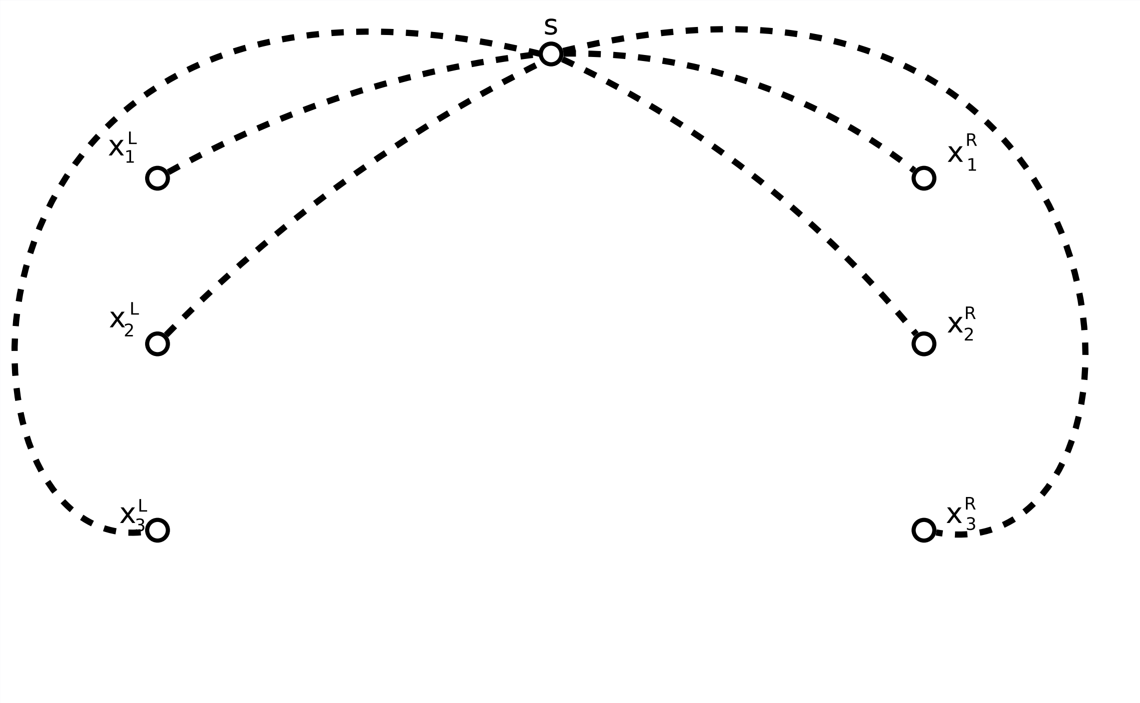
Note that $w = 10$ suffices, since the two cheating variables appear in at most 10 clauses.

Construction



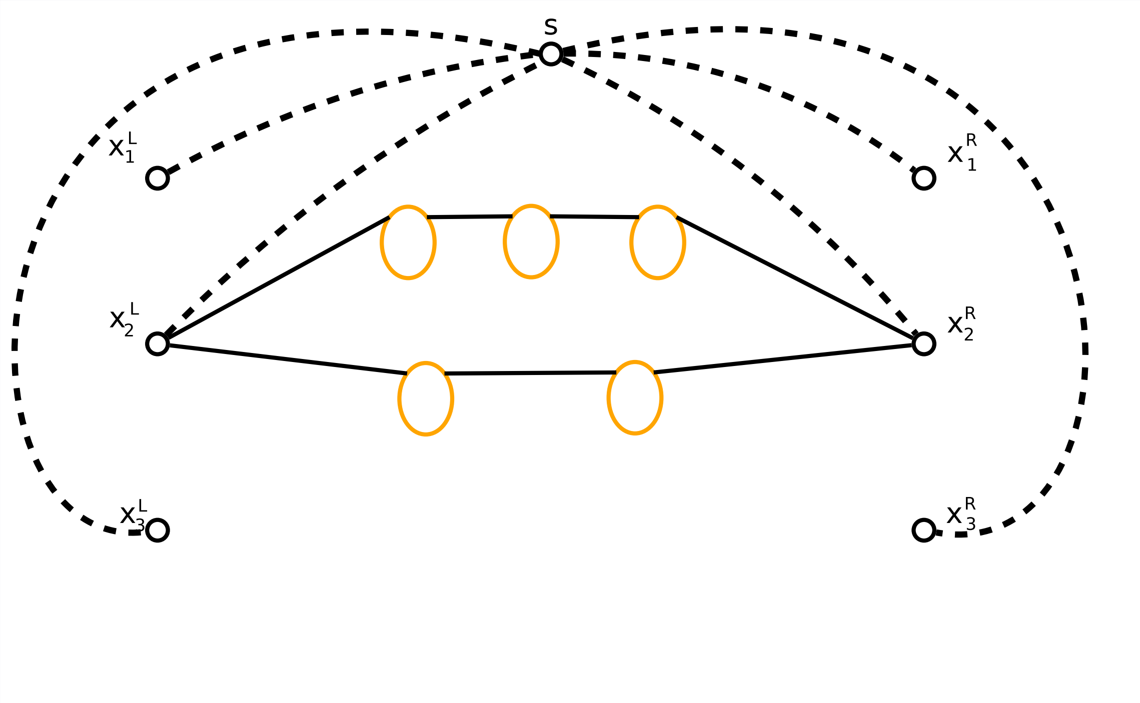
High-level view: construct an origin s and two terminal vertices for each variable.

Construction



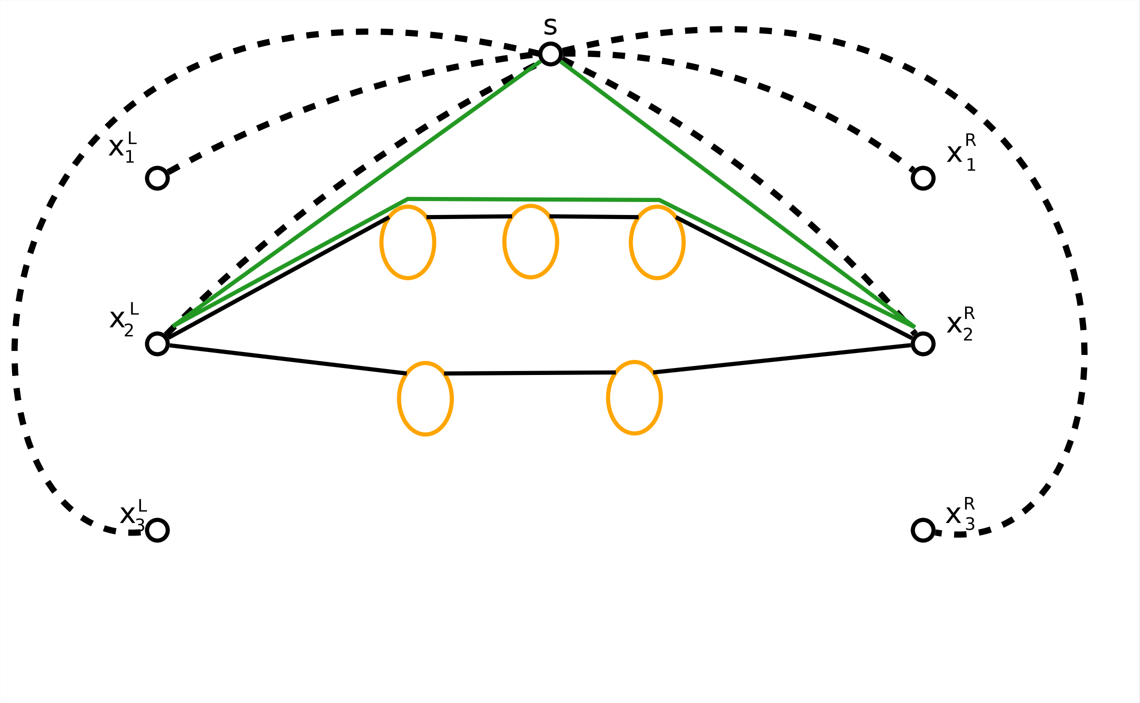
Connect them with forced edges

Construction



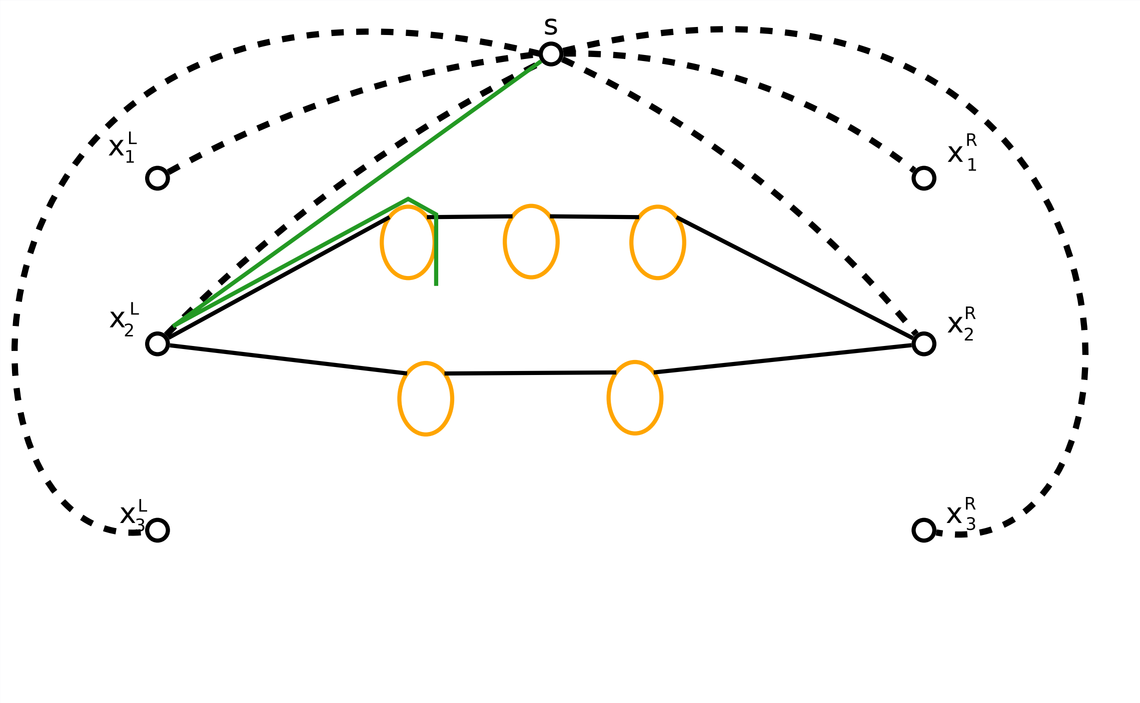
Add the gadgets

Construction



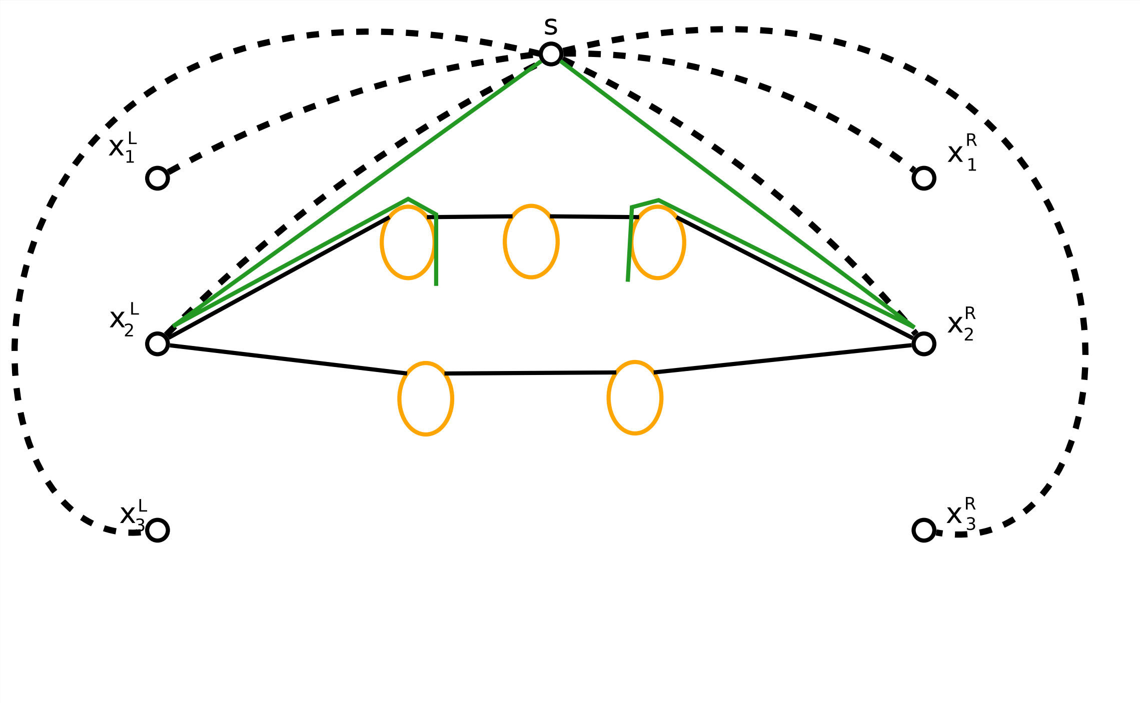
An honest traversal for x_2 looks like this

Construction



A dishonest traversal looks like this. . .

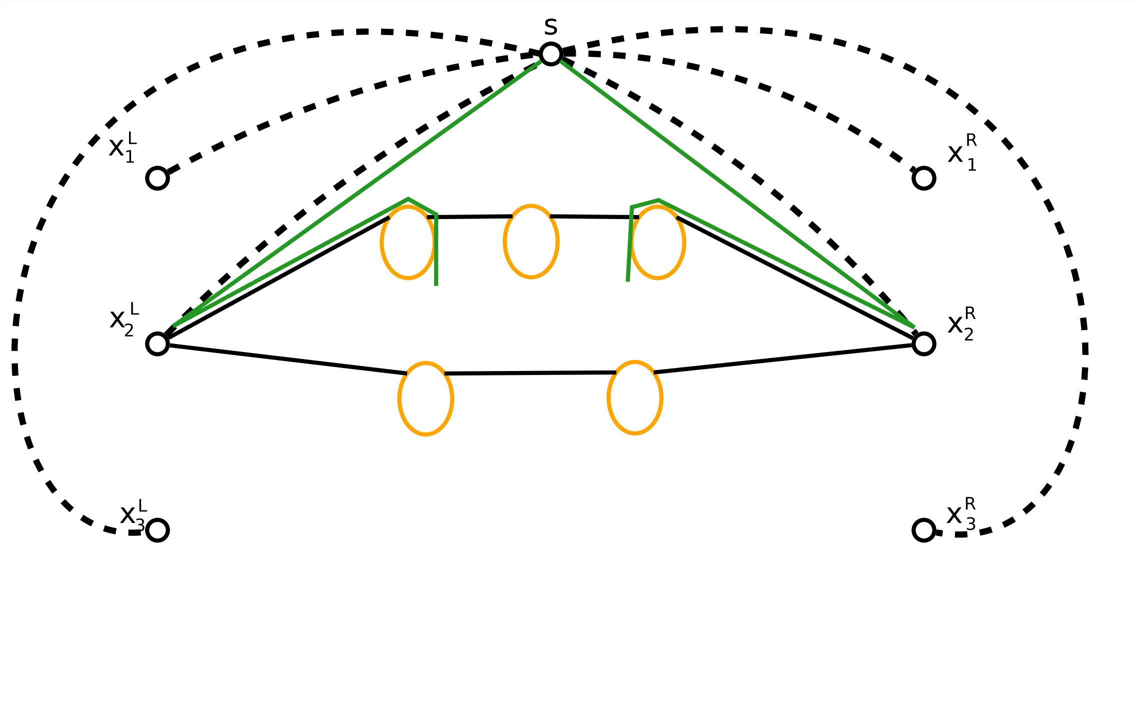
Construction



...but there must be cheating in two places

There are as many doubly-used forced edges as affected variables
$\rightarrow w \leq 5$

Construction



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$\rightarrow w \leq 5$

In fact, no need to write off affected clauses. Use random assignment for cheated variables and some of them will be satisfied

Under the carpet

- Many details missing
  - Dishonest variables are set randomly but not independently to ensure that some clauses are satisfied with probability 1.
  - The structure of the instance (from BK amplifier) must be taken into account to calculate the final constant.

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  - Dishonest variables are set randomly but not independently to ensure that some clauses are satisfied with probability 1.
  - The structure of the instance (from BK amplifier) must be taken into account to calculate the final constant.

Theorem:
There is no $\frac{185}{184}$ approximation algorithm for TSP, unless P=NP.

Conclusions – Open problems

- A simpler reduction for TSP and a better inapproximability threshold
  - But, constant still very low!

Future work

- Better amplifier constructions?
- Get rid of 1in3 SAT?
- ATSP

The end

Questions?