# *Model Checking Lower Bounds for Simple Graphs*

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# Algorithmic Meta-Theorems

Positive results

- Problem X is tractable.

Negative results

- Problem X is hard.

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Main objective of today's talk: barriers to meta-theorems:

"There exists a problem in class C that is hard"

# Good news so far

- Most famous meta-theorem: Courcelle's theorem

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  Meta-theorems for clique-width, local treewidth,. . .

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  This can be extended to optimization versions of MSO.

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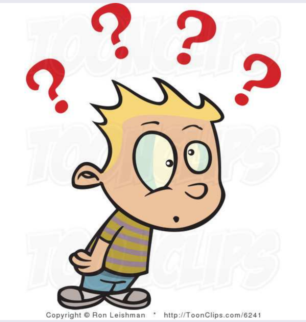

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This is the main question we are concerned with today.

# Some bad news

- Courcelle's theorem:

  There exists an algorithm which, given an MSO formula $\phi$ and a graph $G$ with treewidth $w$ decides if $G \models \phi$ in time $f(w, \phi)|G|$.

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- In fact, Frick and Grohe's lower bound applies to FO logic on trees!

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- For vertex cover, neighborhood diversity, max-leaf [L. ’10]
- For twin cover [Ganian ’11]
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What’s next?

# Let's destroy all hope!

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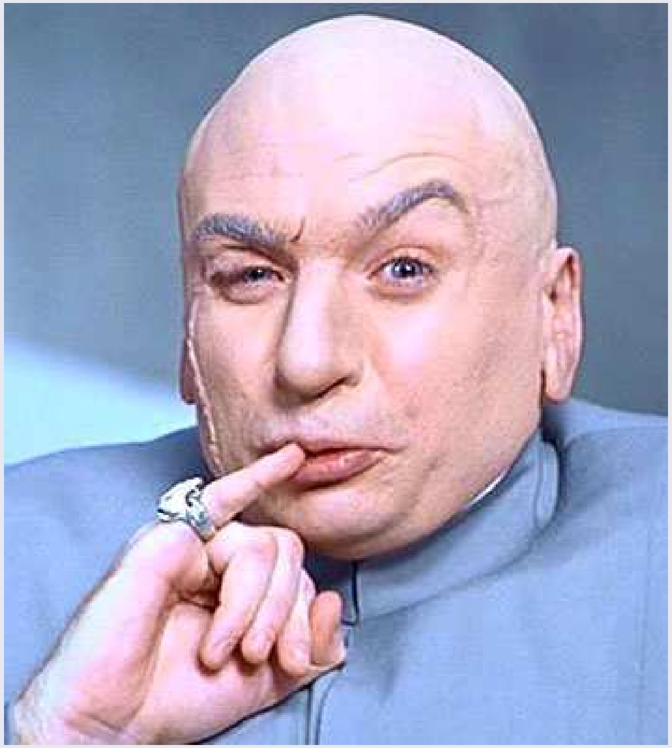
- Main goal: prove hardness results even more devastating than Frick& Grohe.
- Motivation: If we know what we can't do, we might find things we can do.



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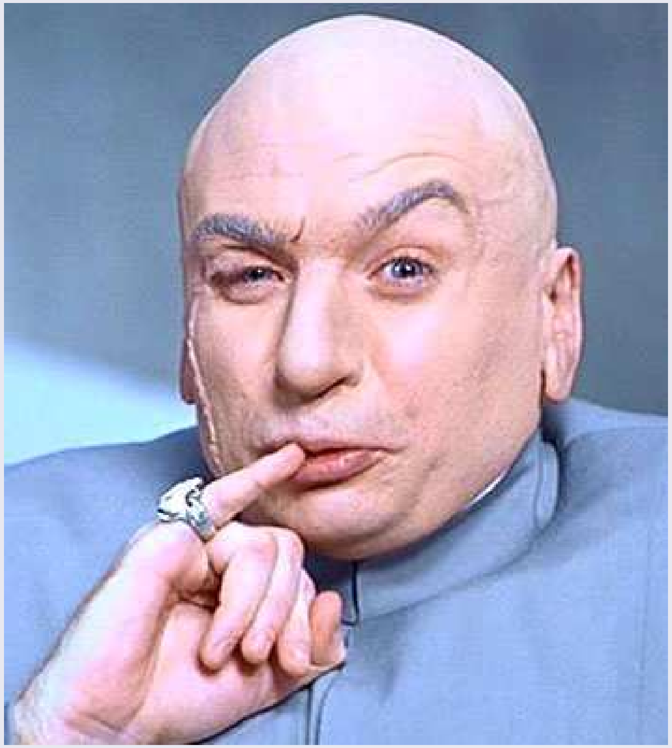

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Today: Three new hardness results.

- Threshold graphs
- Paths
- Bounded-height trees

An appetizer:

Threshold Graphs

# More background

Theorem:

- $\mathrm{MSO}_1$ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics '00]

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- $MSO_1$ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics '00]

A graph has clique-width $k$ if it can be constructed with the following operations using $\leq k$ labels

- Introduce a new vertex with label $i \in [k]$.
- Connect all vertices with label $i$ to all vertices with label $j$.
- Rename all vertices with label $i$ to label $j$.
- Take the disjoint union of two clique-width $k$ graphs.

# More background

Theorem:

- $\text{MSO}_1$ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]

An $\text{MSO}_1$ formula $\phi$ may contain:

- $\exists x, \forall x$ (quantifying over a graph’s vertices)
- $\exists X, \forall X$ (quantifying over a set of vertices)
- Relation $E(x, y)$ (edges), $x = y$
- Boolean connectives

# More background

Theorem:

- $\text{MSO}_1$ expressible properties can be decided in linear time on graphs of bounded clique-width [Courcelle, Makowsky, Rotics ’00]

- Trees have clique-width 3.

  Frick&Grohe $\rightarrow$ non-elementary dependence.

- Graphs with clique-width 1 are easy for $\text{MSO}_1$.

What about clique-width 2?

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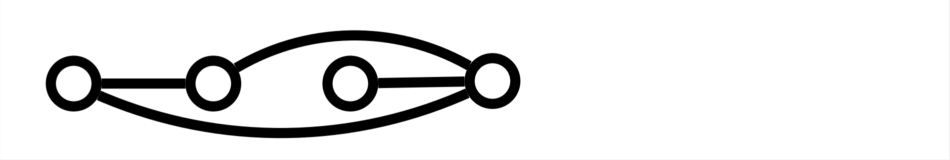

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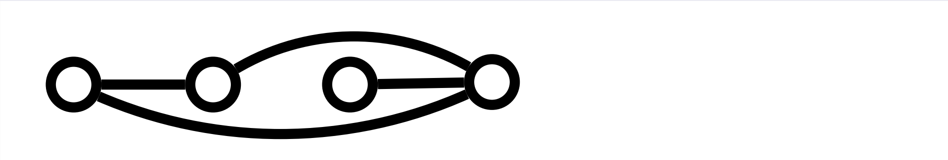

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Thm: Threshold graphs have clique-width 2.

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Input:

- String $w$, FO formula $\phi$:
  - $\exists x$, $\forall x$ ($x$ will correspond to a character in the string)
  - Relation $\prec$ ($x \prec y$ if $x$ comes before $y$ in the string)
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Example:

$$\forall x P_1(x) \to \exists y \neg P_1(y) \wedge x \prec y$$

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Given a string $w$ we construct a threshold graph $G$

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Idea: union vertices represent the characters

$$\begin{aligned} union(x) &:= \forall y \forall z (E(x,y) \wedge E(x,z) \wedge y \neq z) \rightarrow E(y,z) \\ main(x) &:= union(x) \wedge (\exists y \neg union(y) \wedge \neg E(x,y)) \end{aligned}$$

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This allows us to interpret $\exists x \psi(x)$ (in the string) to $\exists x(main(x) \wedge \psi^I(x))$ (in the graph).

## Hardness for threshold graphs

Interpretation continued:

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Thm: There is no elementary-dependence model-checking algorithm for FO logic on threshold graphs.

# Consequences

Recall some of the “good” graph classes we know

- Some are closed under complement (neighborhood diversity, shrub-depth)
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Recall some of the "good" graph classes we know

- Some are closed under complement (neighborhood diversity, shrub-depth)
- Some are closed under union (tree-depth)
- None are closed under both operations...

Any class of graph closed under both operations must* contain threshold graphs.

Main course:

Paths

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- MSO for max-leaf is open!

## Why would this be easy?

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- Reduction seems impossible…

“Normal” reduction:

- Start with $n$-variable 3-SAT
- Construct graph $G$ with $|G| = n^c$
- Construct formula $\phi$ with $|\phi| = \log^* n$
- Prove YES instance $\leftrightarrow G \models \phi$

Problem: New instance would be encodable with $O(\log n)$ bits. We are making a sparse NP-hard language!

## How the reduction can work

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- Formula will be somewhat larger, but still small enough.

# Counting with MSO

- The basic obstacle (as in Frick and Grohe) is counting efficiently.
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- Example: For independent sets, $q$ quantifiers work for size $2^q$.

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Today: $q$ quantifiers count up to size $tow(\Omega(\log q))$ on unary strings.

# Learning to count

Induction:

- We have a MSO formula $eq_L(P_1, P_2)$ which correctly compares sets up to size $L$.
- The formula is only true for equal sets (independent of size).

Use this to compare larger sets economically.

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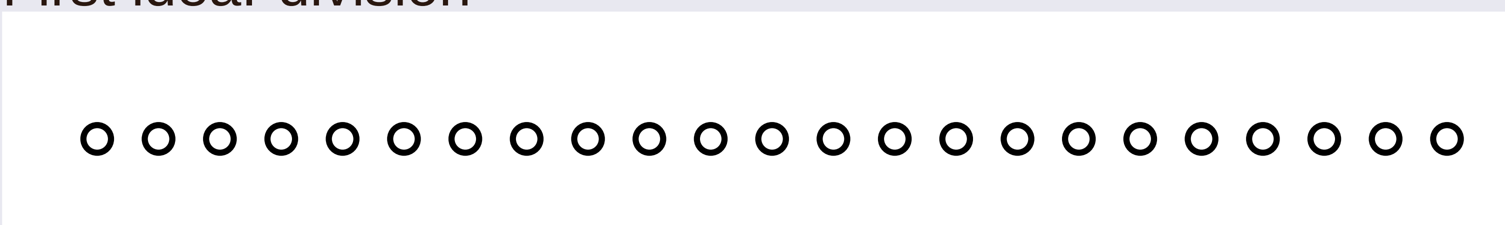

Given an ordered set of elements to compare with another, we first select a subset of it.

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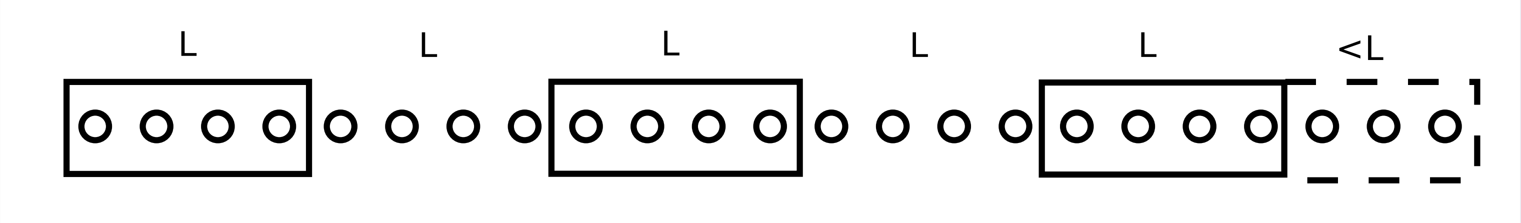


We can impose some structure: each “section” must have the same length ($\leq L$). We do this on both sets.

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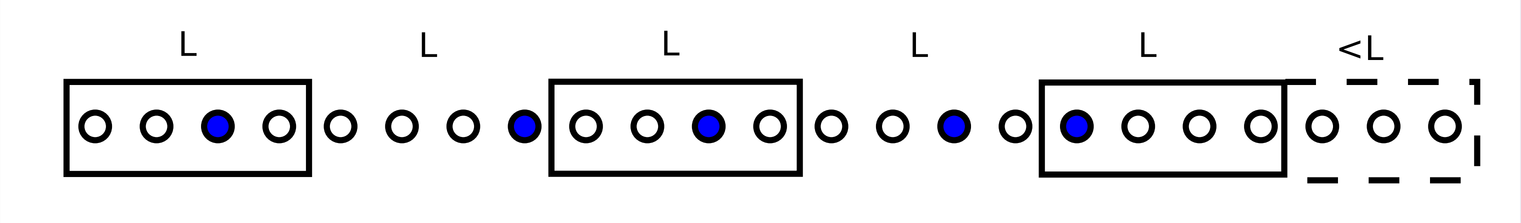


Now we need to count the number of sections. Select one representative from each. Compare the two sets of representatives.

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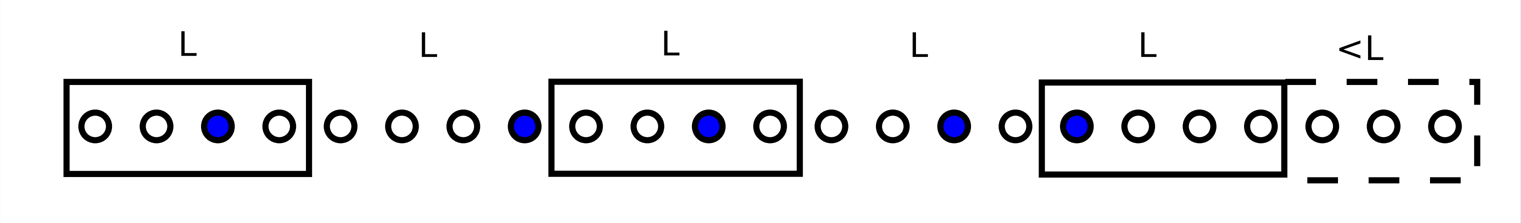


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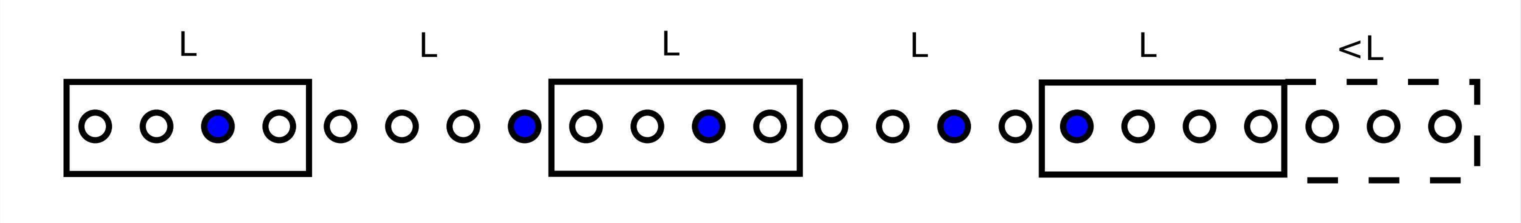


This allows us to go from $L$ to $L^2$ with $O(1)$ additional quantifiers (if done carefully).
Counting power: $2^{2^q}$. Not good enough, but we're moving.

# Learning to count better

- Good: a single set gives many sections.
- Bad: hard to count how many sections we have. Using induction not good enough.

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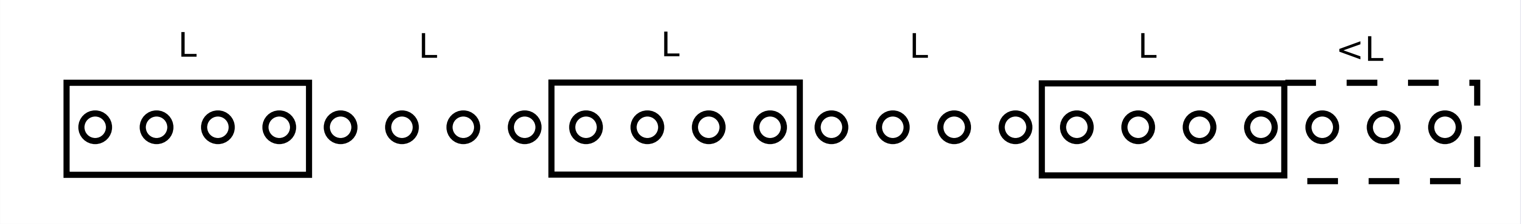


Select the same division into sections.

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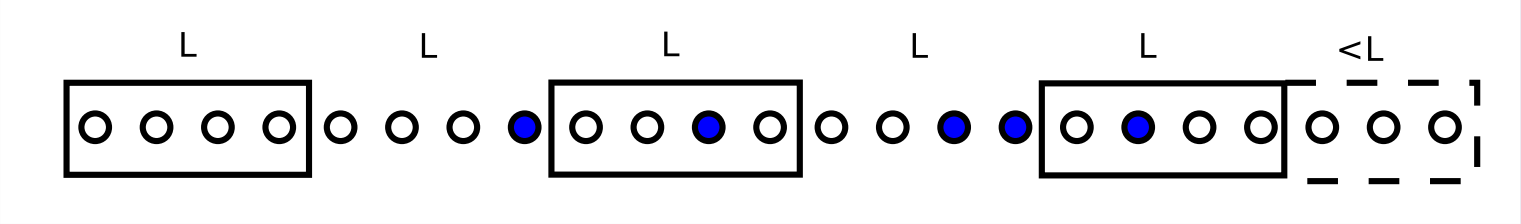


To count sections, select a subset that “writes” a binary number in each section.

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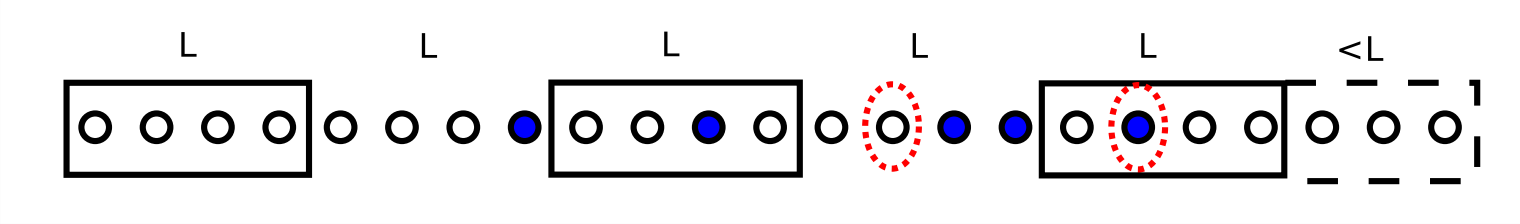


Demand that counting is correct for consecutive sections.

- Proof: hand-waving (but check the paper!)

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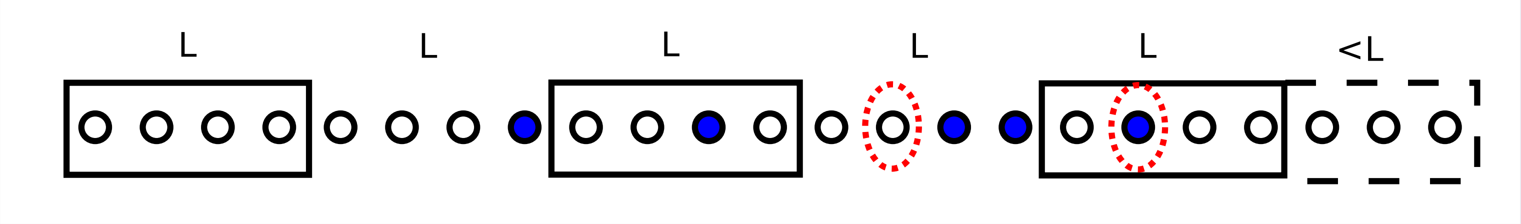


We went from $L$ to $L2^L$ using $eq_L$ $O(1)$ times.
$\rightarrow$ each level of exponentiation increases size by a constant factor.
$\rightarrow$ can compare sets of size $n$ with $2^{\log^* n}$ quantifiers.

# Are we done with the math?

Using $eq_L$ it's easy to do comparisons, div, mod, ...

- We will also need exponentiation. $exp_L(P_1, P_2)$ is true if $|P_2| = 2^{|P_1|}$.

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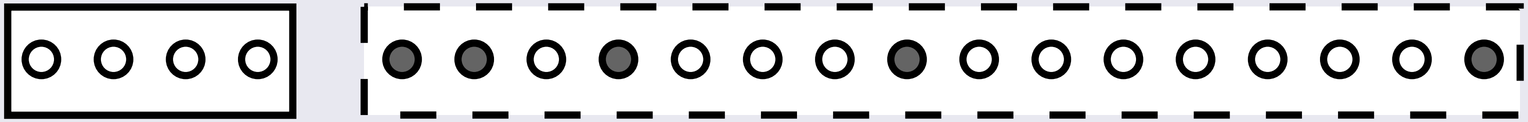

Idea: Find a set in $P_2$ with size $|P_1| + 1$. Ensure that consecutive distances are doubled.

DONE!

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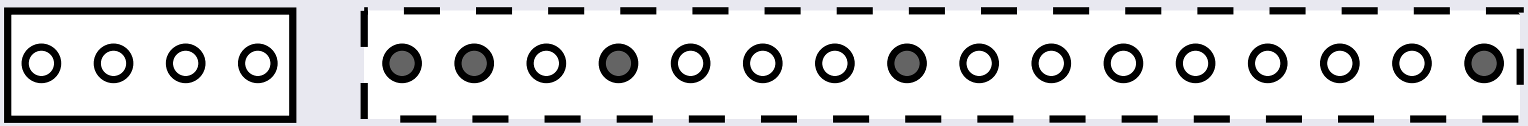
Using $eq_L$ it's easy to do comparisons, div, mod, ...

- We will also need exponentiation. $exp_L(P_1, P_2)$ is true if $|P_2| = 2^{|P_1|}$.



Idea: Find a set in $P_2$ with size $|P_1| + 1$. Ensure that consecutive distances are doubled.

DONE!

The hard part is over!

# Putting things together

- Reduction from NEXP Turing machine acceptance with $n$ input bits.
- Machine runs in $T = 2^{n^c}$ time. Input (read as binary number) is $I \leq 2^n$.
- Construct a path of length $T^2(2I + 1)$.
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The last one is the tricky part. But we now have the right tools.

- Locate a set of length $T^2$. Divide it into sections of size $T$. These will represent snapshots of the machine's tape.
- Locate a set of length $I$. Use $exp$ to "read" input bits from it.
- Guess the contents of the tape.
- Check that the computation is correct and accepting.

## Consequences

Unless EXP=NEXP:

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- Max-leaf is hard
- Graph classes closed under edge sub-divisions are hard
- Graph classes closed under induced subgraphs with unbounded (dense)$^*$ diameter are hard
- MSO$_2$ for cliques is very hard! (not in XP)

The last one was already known. But "easier" proof using that $eq_L$ has constant size on cliques with MSO$_2$.

Dessert:

Trees of bounded height

## Why trees of bounded height?

This class of graphs is important for two recent meta-theorems:

- Shrub-depth in "When trees grow low: Shrubs and fast MSO$_1$" [Ganian et al. MFCS '12]
- Tree-depth in "Faster deciding MSO properties of trees of fixed height, and some consequences" [Gajarský and Hliněný FSTTCS '12]

In both cases the main tool is the following:

MSO model-checking for $q$ quantifiers on trees of height $h$ colored with $t$ colors can be done in $\exp^{(h+1)}(O(q(t+q)))$ time.

# Lower bound

Goal: prove that $h+1$ levels of exponentiation are exactly necessary.

- Start from an $n$-variable 3-SAT instance.
- Construct a tree of height $h$. Use $t = \log^{(h)}(n)$ colors.
- Construct a formula with $q = O(h)$ quantifiers.
- Prove equivalence between instances.

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Argument: an algorithm running in $\exp^{(h+1)}(o(t))$ would run in $2^{o(n)}$ here, disproving ETH.

# Let's count some more!

Fix $h$. The main problem is again to count efficiently.

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# Let's count some more!

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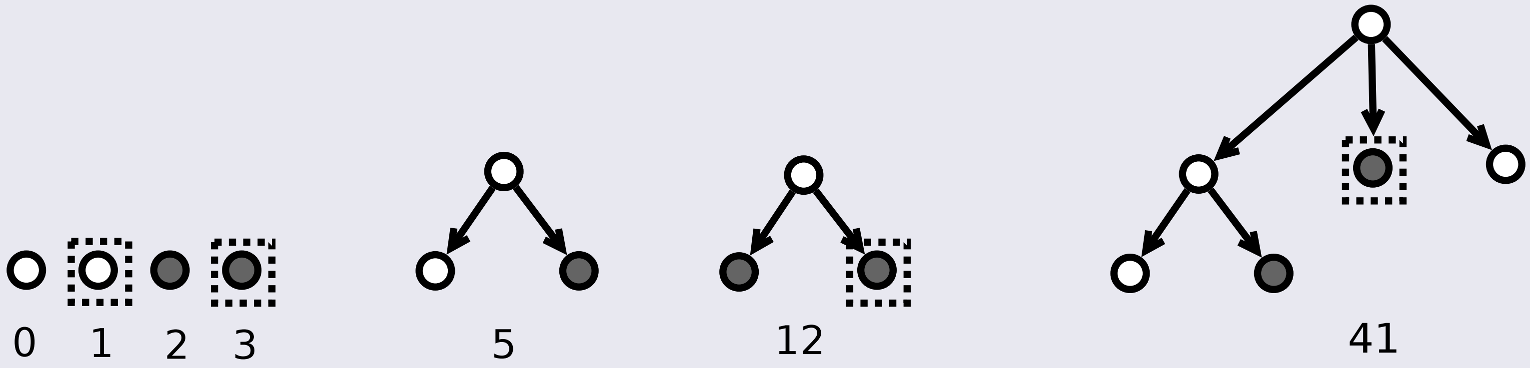
- Assuming we can do numbers up to $L$ with trees of height $i$. We do numbers up to $2^L$ with trees of height $i+1$ (Frick& Grohe).



# The rest is easy

- Construct a tree of height $h - 1$ for each variable, encoding its index.
- Construct a tree of height $h - 1$ for each clauses, encoding the indices of its three literals.
- Add a root.
- Express satisfiability with a constant quantifier-depth formula.

Essential idea: we are using the proof of Frick and Grohe for $h$ levels.

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Essential idea: we are using the proof of Frick and Grohe for $h$ levels.

Thm: There is no $\exp^{(h+1)}(o(t))$ algorithm for MSO logic on $t$-colored trees of height $h$ unless the ETH is false.

# Conclusions - Open problems

- Three natural barriers to future improvements.
- Paths are probably the toughest to work around.

Future work

- (Uncolored) tree-depth?
- Height of tower for paths?

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Future work

- (Uncolored) tree-depth?
- Height of tower for paths?
- Other logics?!?

# Thank you!