Algorithmic Meta-Theorems for Restrictions of Treewidth

Michael Lampis

Computer Science Dept.

Graduate Center, City University of New York

Algorithmic Meta-Theorems

- Algorithmic Theorems
  - Vertex Cover, Dominating Set, 3-Coloring are solvable in linear time on graphs of constant treewidth.
  - Vertex Cover, Feedback Vertex Set can be solved in sub-exponential time on planar graphs

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  - All MSO-expressible problems are solvable in linear time on graphs of constant treewidth.
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- Main uses: quick complexity classification tools, mapping the limits of applicability for specific techniques.
- This talk: Algorithmic Meta-Theorems where the class of problems is defined using logic.

First Order Logic on graphs

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Example:

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Example: Dominating Set of size 2

$$\exists x_1 \exists x_2 \forall y E(x_1, y) \vee E(x_2, y) \vee x_1 = y \vee x_2 = y$$

(Monadic) Second Order Logic

- MSO logic: we add set variables $S_1, S_2, \ldots$ and a $\in$ predicate. We are now allowed to quantify over sets.
  - MSO_1 logic: we can quantify over sets of vertices only
  - MSO_2 logic: we can quantify over sets of edges

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Example: 2-coloring

$$\exists V_1 \exists V_2 \forall x \forall y E(x,y) \rightarrow (x \in V_1 \leftrightarrow y \in V_2)$$

The model checking problem

Problem: **p-Model Checking**
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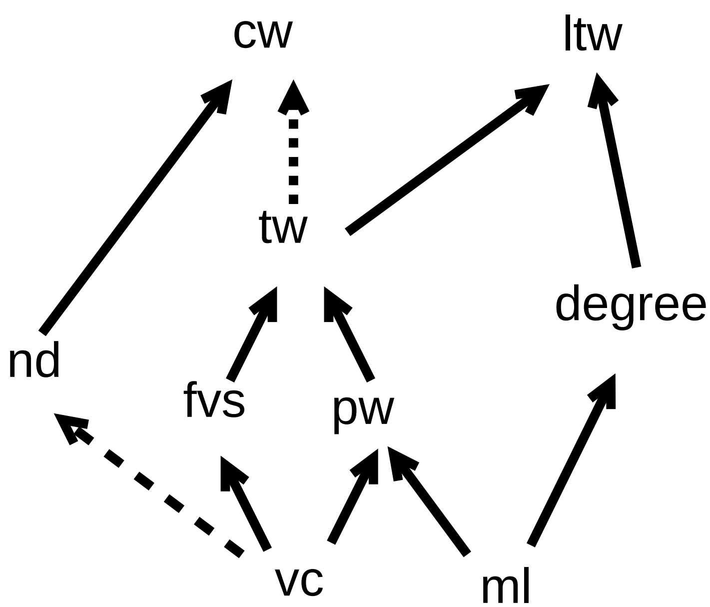
- For general graphs, this problem is W-hard even for FO logic
- We are interested in finding tractable, i.e. FPT, cases for more restricted classes of graphs.
- The most famous such result is Courcelle's theorem which states that p-Model Checking for MSO_2 logic is FPT when also parameterized by the graph's treewidth.

Lower Bounds

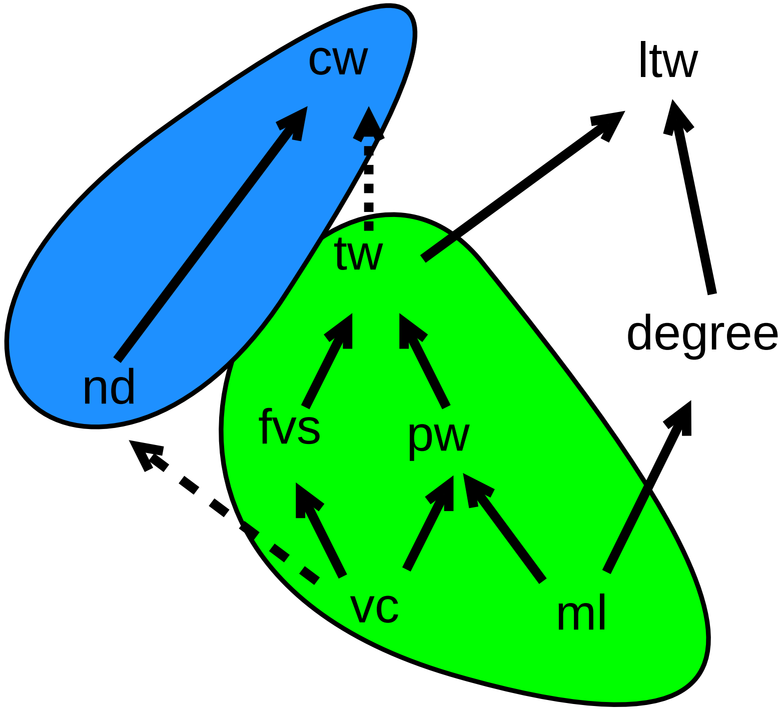
- Courcelle's theorem states that deciding if $G \models \phi$ can be done in time $f(tw(G), \phi) \cdot |G|$, for some function f.
- Unfortunately, in the worst case this function is horrible!
  - [Frick and Grohe 2004]: There is no algorithm which solves p-Model Checking on trees in time $O(f(\phi) \cdot n)$ for any elementary function f unless P=NP.
  - The lower bound applies also to FO logic, under the stronger assumption FPT$\neq$AW[*]
- Motivation: see if things improve when one looks at more restricted classes of graphs.

Graph classes



Some popular graph classes

Graph classes



Some popular graph classes

- FO logic is FPT for all, MSO_1 for the blue area, MSO_2 for the green area.
- Lower bounds:
  - FO logic is non-elementary for trees, triply exponential for binary trees.

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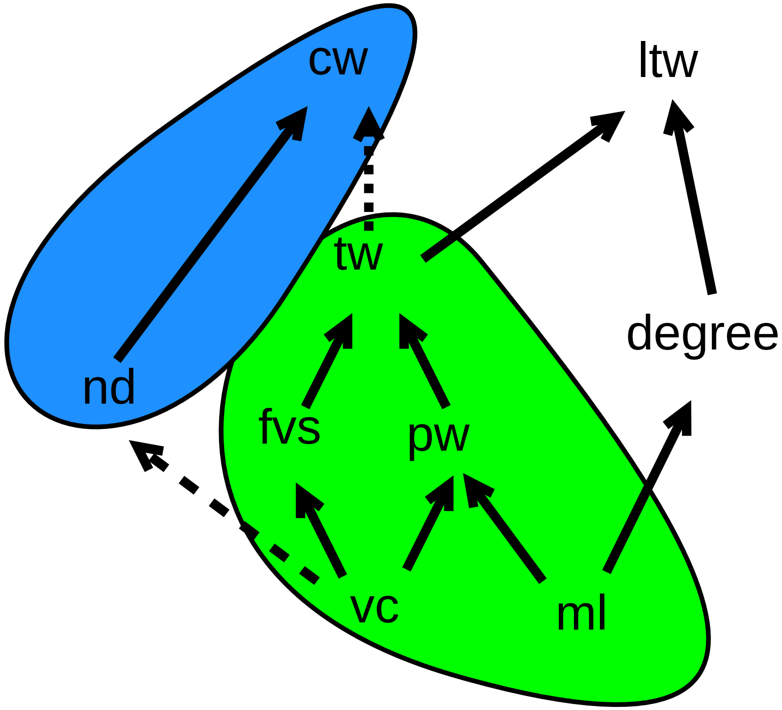

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Our focus is on improving on the bottom.

Summary of results

- FO logic for graphs of bounded vertex cover is singly exponential
- FO logic for graphs of bounded max-leaf number is singly exponential
- MSO logic for graphs of bounded vertex cover is doubly exponential
- Tight lower bounds (under the ETH) for vertex cover
- Generalize FO and MSO_1 results to neighborhood diversity

Graphs with small Vertex Cover

- A vertex cover is a set of vertices whose removal makes the graph an independent set.
- Usually viewed as just an optimization problem, but the existence of a small vertex cover gives a graph a very special form.
- Small vertex cover trivially implies small treewidth.
- It makes sense to study problems hard for treewidth parameterized by vertex cover
  - Good example: Bandwidth

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- Intuition:
  - Model checking FO logic on general graphs is in XP: each time we see a quantifier, we try all possible vertices.
  - The existence of a vertex cover of size k partitions the remainder of the graph into at most 2^k sets of vertices, depending on their neighbors in the vertex cover.
  - Crucial point: Trying all possible vertices in a set is wasteful. One representative suffices.

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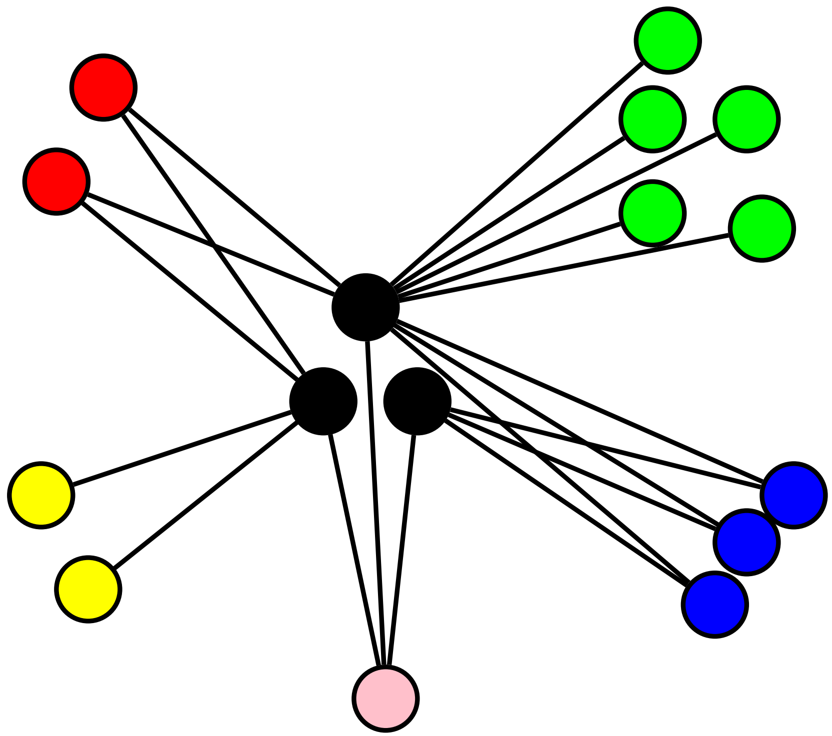

Vertex cover - A warm-up

- Model checking FO logic on graphs of bounded vertex cover is singly exponential.
- Algorithm: For each of the q quantified vertex variables in the formula try the following
  - Each of the vertices of the vertex cover (k choices)
  - Each of the previously selected vertices (q choices)
  - An arbitrary representative from each type (2^k choices)
- Total time: $O^*(k+q+2^k)^q = O^*(2^{kq+q\log q})$

Max-Leaf Number

- The max-leaf number of graph $ml(G)$ is the maximum number of leaves of any sub-tree of G.
- Again, small max-leaf number implies a special structure
  - Trivially, small degree and small treewidth
  - [Kleitman and West] A graph of max-leaf number k is a sub-division of a graph of at most $O(k)$ vertices.
- Again, it makes sense to study problems hard for treewidth parameterized by max-leaf number
  - Good example: Bandwidth

FO logic on paths

- Let us first try to solve this basic problem: Given a path on n vertices and a FO sentence ϕ, decide if ϕ holds on that path.
- This is an important special case of max-leaf number graphs. We cannot use the previous technique since the vertex cover is high.

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- Key intuition: if the path is very long, its precise length does not matter.

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- By applying the lemma, any path can be shortened to size 2^q. Applying the trivial algorithm for FO logic gives a time bound of $O^*(2^{q^2})$
- This is a classic idea related to Ehrenfaucht-Fraisse games in logic.

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- Generalize this idea to graphs of small max-leaf number.
- Lemma: If a topo-edge has length at least 2^q it can be shortened without affecting the truth value of any FO sentence with at most q quantifiers.
- The graph can be reduced to size $O(k^2 2^q)$ so the trivial FO algorithm runs in $2^{O(q^2+q\log k)}$

MSO logic for vertex cover (sketch)

- Trivial algorithm: for each set variable, try all 2^n subsets.
- Use types: complexity comes down to $n^{f(k,q)}$, not good enough!
  - Intuition: when selecting a set only the number of vertices of each type matters.
- Basic idea: prove that a lot of sets are equivalent for MSO sentences with at most q quantifiers.
  - Intuition: the exact number of vertices from each type matters only if it's $< 2^q$ (or the complement has size $< 2^q$).
- End result: $2^{2^{O(k+q)}}$ (doubly exponential) algorithm.

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- Our results will rely on the ETH
- ETH: There is no $2^{o(n)}$ algorithm for 3SAT.

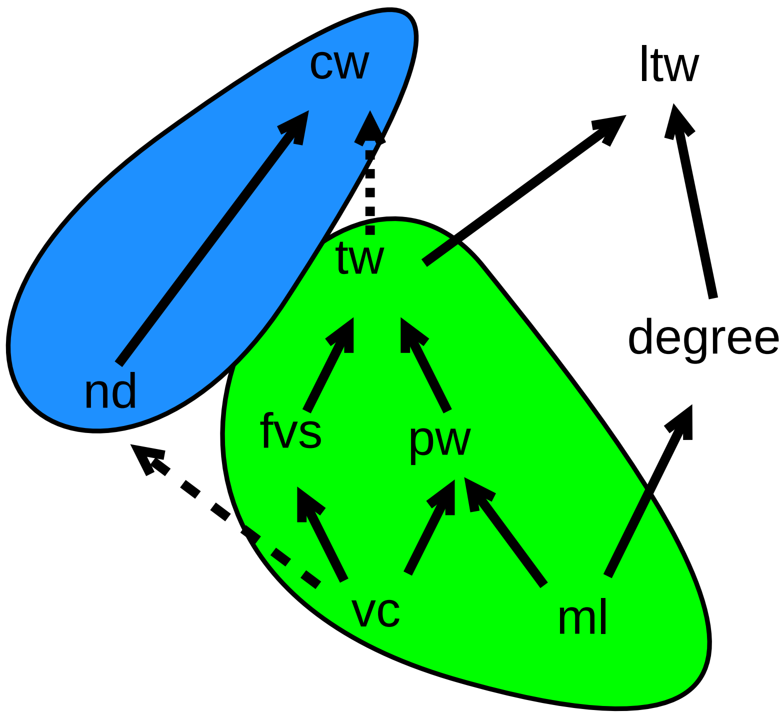
Reduction (sketch)

- Reduction from 3-SAT to model checking.
- Create a graph G to encode a propositional formula with n variables.
- G will have vertex cover $O(\log n)$. The MSO formula will have constant size.
  - Each vertex of the vertex cover encodes one of the bits in the index of the propositional variables.
- A $2^{2^{o(k+q)}}$ algorithm would then give $2^{2^{o(\log n)}} = 2^{o(n)}$ algorithm for 3SAT.
- Same reduction works for FO logic, starting from weighted 3-SAT.

Neighborhood diversity

- We have seen that we can prove stronger meta-theorems for bounded vertex cover than we can for bounded treewidth.
- However, we are essentially only using one property of bounded vertex cover graphs: the fact that vertices can be partitioned into a small number of types.
- This motivates the following definition:
  - The neighborhood diversity of a graph is the minimum number $nd(G)$ s.t. the vertices of G can be partitioned in $nd(G)$ sets with all vertices in each set having the same type.
- Observe that this is a strict superset! Example: complete bipartite graphs.

Graph classes



- Neighborhood diversity is a special case of clique-width but incomparable to treewidth.
- Our results for FO logic and MSO_1 logic can trivially be extended to nd.
- MSO_2 is FPT for vertex cover (Courcelle) but W-hard for clique-width. What about nd?

Conclusions - Open problems

- Stronger meta-theorems (and some lower bounds) for restrictions of treewidth.
  - MSO is doubly exponential for vc (upper and lower bound).
  - FO is singly exponential for vc (upper and lower bound) and for ml.
- Interesting to continue this line of work for other such graph classes or for other logics.
- More concrete open problems:
  - MSO_2 for nd
  - Lower bound for FO on max-leaf
  - MSO for max-leaf

Thank you!