ERROR ESTIMATES FOR GAUSSIAN BEAMS AT A FOLD CAUSTIC

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ABSTRACT. In this work we show an error estimate for a first order Gaussian beam at a fold caustic, approximating time-harmonic waves governed by the Helmholtz equation. For the caustic that we study the exact solution can be constructed using Airy functions and there are explicit formulae for the Gaussian beam parameters. Via precise comparisons we show that the pointwise error on the caustic is of the order $O(k^{-5/6})$ where k is the wave number in Helmholtz.

1. Introduction

Gaussian beam superpositions is a high frequency asymptotic approximation for solutions of wave equations [8]. It is used in numerical methods to simulate waves in the high frequency regime. Unlike standard geometrical optics, the Gaussian beam approximation does not break down at caustics, which is one of its main advantages.

In this paper we consider error estimates for the approximation in terms of the wave number $k > 0$. Error estimates for Gaussian beams are known in a number of settings. See for instance [4, 5] and the references therein. The main result is that, in L^2 and Sobolev norms, the relative error of first order beams decays as $O(k^{-1/2})$, independently of dimension and regardless of the presence of caustics. This has been shown for general strictly hyperbolic partial differential equations and the Schrödinger equation [4, 5] as well as the Helmholtz equation [2]. The better rate $O(k^{-1})$ is typically observed in numerical computations and has been shown in L^2 for the Schrödinger equation [9], and also in L^∞ for the Schrödinger and the acoustic wave equation on sets strictly away from caustics [5]. Similar estimates have also been derived for higher order beams. For p-th order beams the rates are $O(k^{-p/2})$ and $O(k^{-\lceil p/2\rceil})$ respectively. There are, however, no precise, pointwise, error estimates for the solution at a caustic. In particular, for first order beams it has not been shown that this error vanishes as $k \to \infty$, although there is ample numerical evidence to this effect; see for instance [3].

The purpose of this paper is to show such an error estimate for a typical fold caustic in two dimensions. More precisely, we consider the Helmholtz equation

$$\Delta u + k^2(1-x)u = 0. \tag{1}$$

We assume there is an incident wave u_{inc} from $x = -\infty$ making an angle $\Theta \in (0, \pi/2)$ with the x-axis. Moreover, at $x = 0$ it has the amplitude envelope $A(y)$, so that

$$u_{\text{inc}}(0, y) = A(y)e^{ik\sin\Theta}.$$

This wave will generate a fold caustic at the line $x = x_c$ where

$$x_c = \cos^2(\Theta).$$

Figure 1 shows a representative solution. In Section 3 we make this physical situation precise and derive an exact solution using Airy functions on $\mathbb{R}^2$. We subsequently study the solution in the region $0 \le x \le x_c$ and compare it at $x = x_c$ to an approximation using Gaussian beams, denoted by u_{GB}. (Note that $0 < 1 - x \le 1$ in this region; we do not make comparisons elsewhere, as the equation then no longer models the physical situation.) The main result is the following theorem.

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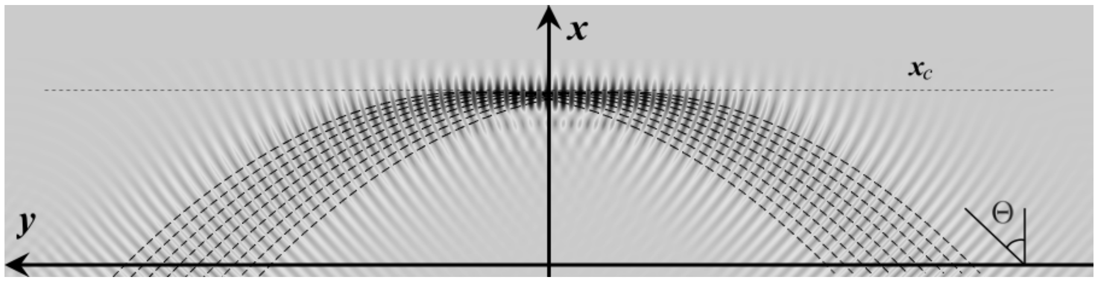

FIGURE 1. The fold caustic: Example of solution with ray tracing picture for $0 \leq x \leq 1$.

Theorem 1.1. *Suppose A is a Schwartz class function, $A \in \mathcal{S}(\mathbb{R})$, and that $0 < \bar{\Theta}_0 \leq \Theta \leq \bar{\Theta}_1 < \pi/2$. Let u be the exact solution of (1) as defined in Section 3 and u_{GB} a first order Gaussian beam approximation detailed in Section 4. Then there is a constant C independent of k such that*

$$||u_{GB}(x_c,\cdot) - u(x_c,\cdot)||_{L^\infty} \leq Ck^{-5/6}.$$

This result hence confirms that first order Gaussian beams do converge pointwise at the caustic. Moreover, since the solution itself grows as $O(k^{1/6})$ at this caustic [6], the relative error is $O(k^{-1})$, the same as away from the caustic. We conjecture that this will be the case also for more general caustics.

The paper is organized as follows. In Section 2 notations are established and some preliminary results are discussed. In Section 3 the exact solution is defined and a formula for it is derived. In Section 4, the corresponding Gaussian beam approximation is introduced. Sections 5, 6 and 7 contain estimates of the Gaussian beam parameters, the phase, various oscillatory integrals, as well as the exact solution and the Gaussian beam approximation. In Section 8 the proof of the main result in Theorem 1.1 is carried out. Finally, in Section 9 some properties of Airy functions are presented.

2. Preliminaries

In the analysis we use a k-scaled Fourier transform and indicate it with a hat mark on the function,

$$\hat{f}(\eta) := \mathcal{F}_k(f)(\eta) := \sqrt{k}\mathcal{F}(f)(k\eta) = \sqrt{\frac{k}{2\pi}}\int f(x)e^{-ikx\eta}dx. \tag{2}$$

The corresponding scaled inversion formula reads

$$f(x) = \sqrt{\frac{k}{2\pi}}\int \hat{f}(\eta)e^{ikx\eta}d\eta =: \mathcal{F}_k^{-1}(\hat{f})(x) = \sqrt{k}\mathcal{F}^{-1}(\hat{f})(kx).$$

We also have

$$\widehat{f_x}(\eta) = ik\eta\hat{f}(\eta), \qquad ||f||_{L^\infty} \leq \sqrt{\frac{k}{2\pi}}||\hat{f}||_{L^1}, \qquad \mathcal{F}_k(f*g)(\eta) = \sqrt{\frac{2\pi}{k}}\hat{f}(\eta)\hat{g}(\eta)$$

We will frequently make use of a smooth, even, cut-off function which we denote $\psi \in C_c^\infty(\mathbb{R})$. It is defined as

$$\psi(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2, \\ \in (0,1), & 1 < |x| < 2, \end{cases} \qquad \psi(-x) = \psi(x).$$

This is used to divide integrals into subdomains and to regularize the Fourier transform of functions in $\mathcal{S}'$, the space of tempered distributions. For example, if f is in $L^\infty(\mathbb{R})$, but not in $L^1(\mathbb{R})$, the definition (2) must be interpreted in distributional sense. We then let $\psi_t := \psi(x/t)$ and consider instead the Fourier transform of the compactly supported function $f\psi_t$, which is well-defined by (2) for all $t > 0$. The following Lemma shows that the limit as $t \to \infty$ gives us the Fourier transform in $\mathcal{S}'$.

Lemma 2.1. *Let $f \in L^\infty$ and set $\psi_t = \psi((b_0x+b_1)/t)$ for any fixed real numbers $b_0 \neq 0$ and b_1. Then, with $\mathcal{F}_k$ as defined above,*

$$\lim_{t\to\infty} \mathcal{F}_k(f\psi_t) = \mathcal{F}_k(f),$$

in $\mathcal{S}'$. Moreover, if $g \in \mathcal{S}$ then, again in $\mathcal{S}'$,

$$\lim_{t\to\infty} \mathcal{F}_k((f*g)\psi_t) = \lim_{t\to\infty} \sqrt{\frac{2\pi}{k}}\mathcal{F}_k(f\psi_t)\mathcal{F}_k(g) = \sqrt{\frac{2\pi}{k}}\mathcal{F}_k(f)\mathcal{F}_k(g).$$

The short proof is found in the Appendix. In particular, if $\hat{f}$ is defined pointwise, the Lemma shows that

$$\hat{f}(\eta) = \lim_{t\to\infty} \sqrt{\frac{k}{2\pi}} \int \psi(x/t) f(x) e^{-ikx\eta} dx. \tag{3}$$

We also introduce some notation that will prove useful later on in the paper. We let

$$\xi_0 = \cos\Theta, \qquad \eta_0 = \sin\Theta, \tag{4}$$

so that $(\xi_0,\eta_0)^T$ is the unit vector pointing in the propagation direction of the incident wave. Following Theorem 1.1 we will assume, throughout the paper, that $0 < \bar{\Theta}_0 \le \Theta \le \bar{\Theta}_1 < \pi/2$. This translates to bounds on ξ_0 and η_0 of the form

$$0 < \bar{\xi}_0 \le \xi_0 \le \bar{\xi}_1 < 1, \qquad 0 < \bar{\eta}_0 \le \eta_0 \le \bar{\eta}_1 < 1. \tag{5}$$

for some $\bar{\xi}_j$ and $\bar{\eta}_j$. Moreover, we let

$$\delta = x_c - x = \xi_0^2 - x, \tag{6}$$

be the distance to the caustic. Finally, we introduce the polynomial q, which is related to the geometrical spreading of the rays,

$$q(s) = 1 + 2is - s^2\beta, \qquad \beta = 1 + 2i\xi_0. \tag{7}$$

It will be used frequently in the analysis.

3. Expression of the exact Helmholtz equation solution

In this section we define an exact solution to the Helmholtz equation for the physical setup described in the introduction. Using a property of the Airy function we deduce a decomposition of the solution into forward and backward going waves.

We consider a solution u to (1) which is a tempered distribution on $\mathbb{R}^2$, i.e. $u \in \mathcal{S}'(\mathbb{R}^2)$. The solution then has a k-scaled Fourier transform in y which we denote $\hat{u}(x,\eta)$. Upon Fourier transforming also (1) in y, we obtain an ODE for $\hat{u}(x,\eta)$,

$$\hat{u}_{xx} + k^2(1 - x - \eta^2)\hat{u} = 0. \tag{8}$$

The only tempered distribution solution to this ODE is given by

$$\hat{u}(x,\eta) = a(\eta)\mathrm{Ai}(k^{2/3}(x+\eta^2-1)),$$

where Ai is the Airy function of the first kind, and $a(\eta)$ is a function to be determined. This solution is thus a C^∞ bounded solution. The Airy function in this expression contains waves going both forward and backward. In the sequel, we will choose the function $a(\eta)$ as $k^{1/6}P(k,\eta)$ defined in (10). To arrive at this choice, we first note that when

$$\alpha = \exp(i\pi/3),$$

it holds for all z that [7, Eq. 9.2.14],

$$\mathrm{Ai}(z) = \alpha\mathrm{Ai}\left(-\alpha z\right) + \bar{\alpha}\mathrm{Ai}\left(-\bar{\alpha}z\right). \tag{9}$$

This follows since $\mathrm{Ai}(-\alpha z)$ and $\mathrm{Ai}(-\bar{\alpha}z)$ solve the same ODE as $\mathrm{Ai}(z)$ given that $\alpha^3 = -1$. We then introduce the scaled variables

$$\zeta(x) = k^{\frac{2}{3}}(x+\eta^2-1), \qquad \zeta_+(x) = -\alpha\zeta, \qquad \zeta_-(x) = -\bar{\alpha}\zeta,$$

such that

$$\hat{u}(x,\eta) = a(\eta)\mathrm{Ai}(\zeta(x)) = a(\eta)\alpha\mathrm{Ai}(\zeta_+(x)) + a(\eta)\bar{\alpha}\mathrm{Ai}(\zeta_-(x)).$$

To further understand this decomposition, we note that the asymptotics of the Airy function (in the angular sector $|\arg(z)| < 2\pi/3$) is

$$\mathrm{Ai}(z) \simeq \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp\left(-\frac{2}{3} z^{\frac{3}{2}}\right).$$

Therefore, upon defining the phase $\phi(x) = \frac{2}{3}(1-\eta^2-x)^{2/3}$ we have, when $x < 1-\eta^2$,

$$\mathrm{Ai}(\zeta_+(x)) \simeq Ck^{-1/6} \frac{e^{-ik\phi(x)}}{(1-\eta^2-x)^{\frac{1}{4}}}, \qquad \mathrm{Ai}(\zeta_-(x)) \simeq Ck^{-1/6} \frac{e^{+ik\phi(x)}}{(1-\eta^2-x)^{\frac{1}{4}}}.$$

Thus, the phases of the two expressions, and their gradients, have opposite signs, meaning that the two terms in (9) represent a decomposition into forward and backward going waves in the region $x < 1-\eta^2$. More precisely, the solution is fully known when $a(\eta)$ is given, and one assumes that this solution of $\Delta u + k^2(1-x)u = 0$ for all x (including $x < 0$ where the velocity grows) is the sum of an incoming and an outgoing wave of the form $a(\eta)\alpha\mathrm{Ai}(\zeta_+(x))$ and $a(\eta)\bar{\alpha}\mathrm{Ai}\left(\zeta_-(x)\right)$, respectively. This leads us to define

$$T_0(x,\eta) = \frac{\mathrm{Ai}\left(\zeta(x)\right)}{\mathrm{Ai}\left(\zeta(0)\right)}, \qquad T_+(x,\eta) = \frac{\mathrm{Ai}\left(\zeta_+(x)\right)}{\mathrm{Ai}\left(\zeta_+(0)\right)}, \qquad T_-(x,\eta) = \frac{\mathrm{Ai}\left(\zeta_-(x)\right)}{\mathrm{Ai}\left(\zeta_-(0)\right)}.$$

which are three particular solutions of (8), since each term in (9) solve the ODE. These solutions are normalized such that they equal one for $x = 0$. The solutions T_+ and T_- represent forward and backward going waves. Among the three solutions, only T_0 is bounded, since $\mathrm{Ai}(z)$ is bounded, while $T_\pm$ are not even in $\mathcal{S}'$, since $\mathrm{Ai}(\alpha z)$ includes also the unbounded second kind Airy function $\mathrm{Bi}(z)$, cf. (72).

We are looking for the solution

$$\hat{u}(x,\eta) = T_0(x,\eta)\hat{u}(0,\eta).$$

We do not know $\hat{u}(0,\eta)$, just that the incoming part of $\hat{u}(0,\eta)$ represents the incident plane wave. We therefore write it as a sum of an incoming and an outgoing part

$$\hat{u}(0,\eta) = \hat{u}_+(0,\eta) + \hat{u}_-(0,\eta),$$

where we define $\hat{u}_+(0,\eta)$ as the k-scaled Fourier transform in y of the incoming wave with amplitude A and direction Θ (recall $\eta_0 = \sin\Theta$),

$$u_+(0,y) = u_{\mathrm{inc}}(0,y) = A(y)e^{ik\eta_0 y}.$$

We then want to find $\hat{u}_-(0,\eta)$ such that

$$T_0(x,\eta)\hat{u}(0,\eta) = T_+(x,\eta)\hat{u}_+(0,\eta) + T_-(x,\eta)\hat{u}_-(0,\eta).$$

To achieve this, it is necessary and sufficient that the values of the functions and their derivatives agree at $x = 0$, since both sides satisfy the same second order ODE. This gives us the linear relations

$$\begin{aligned}\hat{u}(0,\eta) &= \hat{u}_+(0,\eta) + \hat{u}_-(0,\eta),\\ T_0'(0,\eta)\hat{u}(0,\eta) &= T_+'(0,\eta)\hat{u}_+(0,\eta) + T_-'(0,\eta)\hat{u}_-(0,\eta),\end{aligned}$$

from which we can deduce

$$\hat{u}_-(0,\eta) = \frac{T_+'(0,\eta) - T_0'(0,\eta)}{T_0'(0,\eta) - T_-'(0,\eta)}\hat{u}_+(0,\eta) =: T(\eta)\hat{u}_+(0,\eta).$$

It follows that

$$\hat{u}(x,\eta) = T_0(x,\eta)\hat{u}(0,\eta) = T_0(x,\eta)(\hat{u}_+(0,\eta) + \hat{u}_-(0,\eta)) = T_0(x,\eta)(1+T(\eta))\hat{u}_+(0,\eta).$$

The decomposition is not valid at the roots of $\mathrm{Ai}(\zeta(0)) = \mathrm{Ai}(k^{\frac{2}{3}}(\eta^2-1))$. Another form is available, which is valid at all points. It is given in the following Lemma.

Lemma 3.1. *One can express $T(\eta)$ as follows*

$$T(\eta) = -\alpha\frac{\mathrm{Ai}(\zeta_-(0))}{\mathrm{Ai}(\zeta_+(0))}, \qquad 1+T(\eta) = \bar{\alpha}\frac{\mathrm{Ai}(\zeta(0))}{\mathrm{Ai}(\zeta_+(0))},$$

Proof. To simplify notation we write $\mathrm{Ai}_{+,-,0}(x) := \mathrm{Ai}(\zeta_{+,-,0}(x))$. Then, using (9),

$$\begin{aligned} T_\pm - T_0 &= \frac{\mathrm{Ai}_\pm(x)}{\mathrm{Ai}_\pm(0)} - \frac{\mathrm{Ai}_0(x)}{\mathrm{Ai}_0(0)} = \frac{\mathrm{Ai}_\pm(x)\mathrm{Ai}_0(0) - \mathrm{Ai}_\pm(0)\mathrm{Ai}_0(x)}{\mathrm{Ai}_\pm(0)\mathrm{Ai}_0(x)} \\ &= \frac{\mathrm{Ai}_\pm(x)(\alpha\mathrm{Ai}_+(0) + \bar{\alpha}\mathrm{Ai}_-(0)) - \mathrm{Ai}_\pm(0)(\alpha\mathrm{Ai}_+(x) + \bar{\alpha}\mathrm{Ai}_-(x))}{\mathrm{Ai}_\pm(0)\mathrm{Ai}_0(x)} \\ &= \pm\alpha^{\mp 1}\frac{\mathrm{Ai}_-(0)\mathrm{Ai}_+(x) - \mathrm{Ai}_+(0)\mathrm{Ai}_-(x)}{\mathrm{Ai}_\pm(0)\mathrm{Ai}_0(x)}. \end{aligned}$$

Hence,

$$T_+ - T_0 = (T_- - T_0)\frac{\mathrm{Ai}_-(0)}{\mathrm{Ai}_+(0)}(-\bar{\alpha}^2) = (T_- - T_0)\frac{\mathrm{Ai}_-(0)}{\mathrm{Ai}_+(0)}\alpha.$$

The first identity follows upon differentating this expression with respect to x. The second identity is then given by another application of (9). ☐

It follows now that

$$T_0(x,\eta)(1 + T(\eta)) = \bar{\alpha}\frac{\mathrm{Ai}(\zeta(x))}{\mathrm{Ai}(\zeta(0)}\frac{\mathrm{Ai}(\zeta(0))}{\mathrm{Ai}(\zeta_+(0))} = \bar{\alpha}\frac{\mathrm{Ai}(\zeta(x))}{\mathrm{Ai}(\zeta_+(0))}.$$

Since we know $\hat{u}_+(0,\eta)$ we can thus express the full solution as

$$\hat{u}(x,\eta_0+\eta) = \bar{\alpha}\frac{\mathrm{Ai}(k^{\frac{2}{3}}(x-X))}{\mathrm{Ai}(\alpha k^{\frac{2}{3}}X)}\hat{u}_+(0,\eta_0+\eta),$$

where we defined

$$X(\eta) = 1 - (\eta_0+\eta)^2.$$

In this expression, one notices that the denominator never vanish because all the roots of the Airy function are on the negative real axis.

Finally, since

$$u_+(0,y) = A(y)e^{ik\eta_0 y} \quad \Rightarrow \quad \hat{u}_+(0,\eta) = \hat{A}(\eta-\eta_0),$$

we get

$$\hat{u}(x,\eta_0+\eta) = \bar{\alpha}\frac{\mathrm{Ai}(k^{\frac{2}{3}}(x-X))}{\mathrm{Ai}(\alpha k^{\frac{2}{3}}X)}\hat{A}(\eta).$$

We write this as

$$\hat{u}(x,\eta_0+\eta) = \hat{v}(\eta,x,k)\hat{A}(\eta),$$

where

$$\hat{v}(\eta,x,k) = k^{1/6}P(k,\eta)\mathrm{Ai}(k^{\frac{2}{3}}(x-X)), \qquad P(k,\eta) = \frac{\bar{\alpha}k^{-1/6}}{\mathrm{Ai}(\alpha k^{2/3}X)}. \tag{10}$$

4. Construnction of the Gaussian beam approximation

In this section we derive expressions for a first order Gaussian beam approximation to the solution of (1). A Gaussian beam is a high frequency asymptotic solution to the Helmholtz equation. To model a general solution of (1), superpositions of Gaussian beams are used. We give the general form of a Gaussian beam and their superposition in $\mathbb{R}^2$ below. The derivations of the expressions can be found in [2].

The Helmholtz equation with a general index of refraction $n(x)$ reads

$$\Delta u + k^2 n(x)^2 u = 0. \tag{11}$$

When $n(x)$ is real, the equation models wave propagation, but it has a well-defined solution also when $n(x)$ is imaginary. However, Gaussian beams can only be defined for real $n(x)$.

A first order Gaussian beam for (11) has the form

$$v_b(\boldsymbol{x}) = a(s)e^{ik(S(s)+(\boldsymbol{x}-\gamma(s))\cdot\boldsymbol{p}(s)+\frac{1}{2}(\boldsymbol{x}-\gamma(s))^T M(s)(\boldsymbol{x}-\gamma(s))}, \qquad s = s^*(\boldsymbol{x}), \tag{12}$$

where $a(s) \in \mathbb{C}$ is the amplitude, $S(s) \in \mathbb{R}$ the reference phase, $\boldsymbol{p}(s) \in \mathbb{R}^2$ the phase gradient and $M(s) \in \mathbb{C}^{2\times 2}$ the phase Hessian. Moreover, $\gamma(s) \in \mathbb{R}^2$ is the *central ray*, which agrees with the rays

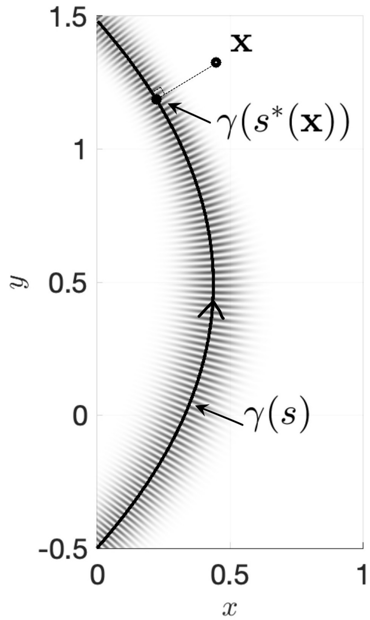

FIGURE 2. A Gaussian beam starting at $(x, y) = (0, -0.5)$ with direction $\eta_0 = \sin\Theta = 3/4$. The central ray γ is indicated with a solid line.

of geometrical optics. An example of a Gaussian beam is shown in Figure 2. In (12) the parameter s depends on the point of evaluation $\boldsymbol{x}$. Normally one takes the s-value for the point on the central ray that is closest to $\boldsymbol{x}$, as indicated in Figure 2. However, in the analysis below we make a simpler choice. By a result in [8] the Hessian M will always have a positive definite imaginary part. The solution v_b will therefore be a "fattened" version of the central ray, with a Gaussian profile normal to the ray with a width determined by M.

The s-dependent parameters in the Gaussian beam are all given by ODEs [2], as follows

$$\begin{aligned} &\frac{d\gamma}{ds} = 2\boldsymbol{p}, \qquad \frac{d\boldsymbol{p}}{ds} = \nabla n^2(\gamma), \qquad \frac{dM}{ds} = D^2(\nabla n^2)(\gamma) - 2M^2, \\ &\frac{dS}{ds} = 2n^2(\gamma), \qquad \frac{da}{ds} = -\mathrm{tr}(M)a. \end{aligned} \tag{13}$$

The initial data for γ and $\boldsymbol{p}$ is given by the starting point (x_0, y_0) and direction (ξ_0, η_0) of the beam,

$$\gamma(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \qquad \boldsymbol{p}(0) = \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}.$$

In order to form an admissible Gaussian beam, $M(0)$ must always satisfy

$$M(0)^T = M(0), \qquad M(0)\gamma'(0) = \boldsymbol{p}'(0), \qquad \boldsymbol{a}^T(\mathrm{Im}M(0))\boldsymbol{a} > 0, \quad \text{when } \boldsymbol{a} \perp \gamma'(0). \tag{14}$$

The choice of $M(0)$ and the precise form of the incoming wave finally determine the initial data for S and a. We come back to this issue below.

To build more general solutions we use superpositions of Gaussian beams. We assume that the incoming wave is known along a curve Γ in $\mathbb{R}^2$, which we can parameterize with the parameter z, so that $\Gamma(z) = (x_0(z), y_0(z))$. For each point on Γ we launch one Gaussian beam in the direction $\theta_0(z)$ of the wave at that point. The parameters of the beams then also depend on z and we write $\gamma = \gamma(s; z)$, $\boldsymbol{p} = \boldsymbol{p}(s; z)$, etc. This gives the beams $v_b = v_{\text{beam}}(\boldsymbol{x}; z)$, from which we finally construct the Gaussian beam superposition

$$u_{GB}(\boldsymbol{x}) = \sqrt{\frac{k}{2\pi}} \int v_{\text{beam}}(\boldsymbol{x}; z) dz. \tag{15}$$

See [2] for more details. In a numerical scheme the z-variable is discretized and for each discrete value, the ODEs (13) are solved with a numerical ODE method. The superposition (15) is subsequently computed using numerical quadrature.

4.1. **Expressions for Gaussian beam parameters.** In (1) we have the index of refraction $n(x) = \sqrt{1-x}$. The ODEs (13) can be solved explicitly and we get analytic formulae for all parameters in the Gaussian beam. To show this we let $\boldsymbol{p} = (\xi, \eta)$ and $\gamma = (x, y)$ and also recall that $\boldsymbol{p}(0) = (\xi_0, \eta_0)$, where $\xi_0^2 + \eta_0^2 = 1$. Since we let the beam start at $x_0 = 0$, we set $\gamma(0) = (0, y_0)$. With the particular choice of n it follows from (13) that $\xi'(s) = -1$ and $\eta'(s) = 0$. Hence,

$$\xi(s) = \xi_0 - s, \qquad \eta(s) = \eta_0.$$

For the positions, we get $x'(s) = 2\xi(s) = 2\xi_0 - 2s$ and $y'(s) = 2\eta(s) = 2\eta_0$. By also using the initial data we obtain

$$x(s) = 2s\xi_0 - s^2, \qquad y(s) = y_0 + 2s\eta_0.$$

The caustic $x = x_c$ is located at the point where the ray turns back, i.e. where $x'(s) = 0$, which gives $s = \xi_0$ and

$$x_c = x(\xi_0) = \xi_0^2.$$

Note that all the rays are confined to the region $x \le x_c < 1$, where the index of refraction is real-valued. The fact that $n(x)$ is complex-valued for $x > 1$ therefore does not affect the Gaussian beams.

We also need to compute the coefficients corresponding to the phase S, the second derivative of the phase M and the amplitude a. We have

$$\frac{dS}{ds} = 2n^2(x(s)) = 2(1 - x(s)) = 2[1 - 2s\xi_0 + s^2], \qquad S(0) = S_0,$$

so the phase is a third order polynomial,

$$S(s) = S_0 + 2s - 2s^2\xi_0 + \frac{2}{3}s^3.$$

For M we have the Riccati equation

$$\frac{dM}{ds} = D^2(n^2)(x(s)) - 2M(s)^2 = -2M(s)^2, \qquad M(0) = M_0,$$

with the solution

$$M(s) = (I + 2sM_0)^{-1}M_0.$$

The matrix M_0 must satisfy the conditions in (14). We pick

$$M_0 = Q + iP, \qquad P = \begin{pmatrix} \eta_0^2 & -\eta_0\xi_0 \\ -\eta_0\xi_0 & \xi_0^2 \end{pmatrix}, \qquad Q = \frac{1}{2}\begin{pmatrix} -\xi_0 & -\eta_0 \\ -\eta_0 & \xi_0 \end{pmatrix}.$$

Note that P, Q are symmetric, P is the orthogonal projection on $\boldsymbol{p}_0^\perp$, and $2Q\boldsymbol{p}_0 = -\boldsymbol{e}_1 = \boldsymbol{p}'(0)$. Moreover, $(\boldsymbol{p}_0^\perp)^T Q \boldsymbol{p}_0^\perp = \xi_0 > 0$.

Next, one checks that

$$M(s) = (I + 2sM_0)^{-1}M_0 = \frac{1}{q(s)}(I + 2s(iI - M_0))M_0,$$

where

$$\det(I + 2sM_0) = 1 + 2s\mathrm{Tr}M_0 + (2s)^2\det M_0 = 1 + 2is - s^2\beta = q(s), \qquad \beta = 1 + 2i\xi_0.$$

We note that q is related to the geometrical spreading of the beams. Further manipulations, using the facts that $P^2 = P$, $4Q^2 = I$ and $2(PQ + QP - Q) = \xi_0 I$ reveals that $M(s)$ can be written simply as

$$M(s) = \frac{1}{q(s)}\left(M_0 - \frac{1}{2}s\beta I\right).$$

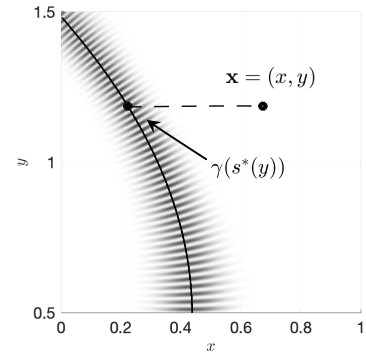

FIGURE 3. The simplified map $s^*(y)$.

We let m_{ij} be the elements of M and deduce that

$$m_{11}(s) = -\frac{1}{2q(s)}(\xi_0 - 2i\eta_0^2 + s\beta) = -\frac{1}{2q(s)}(-2i + (\xi_0 + s)\beta), \tag{16}$$

$$m_{22}(s) = \frac{1}{2q(s)}(\xi_0 + 2i\xi_0^2 - s\beta) = \frac{1}{2q(s)}(\xi_0 - s)\beta. \tag{17}$$

Finally, for a,

$$\frac{da}{ds} = -\mathrm{tr}(M(s))a, \qquad a(0) = a_0.$$

We note that if λ_0 and λ_1 are eigenvalues of M_0, then $q(s) = \det(I + 2sM_0) = (1 + 2s\lambda_0)(1 + 2s\lambda_1)$ and

$$\mathrm{tr}(M(s)) = \frac{\lambda_0}{1 + 2s\lambda_0} + \frac{\lambda_1}{1 + 2s\lambda_1} = \frac{1}{2}\frac{d}{ds}\log q(s).$$

It follows that

$$a(s) = \frac{a_0}{\sqrt{q(s)}}. \tag{18}$$

The last thing needed to make the expression (12) for the Gaussian beam well defined, is to decide which s-value to use for a given $\boldsymbol{x}$, i.e. the function $s^*(\boldsymbol{x})$. As mentioned above, this is normally taken to be the s-value for the point on the central ray that is closest to $\boldsymbol{x}$. Here, however, to simplify, we just take the s-value for the point of the curve that has the same y-coordinate; see Figure 3.

With $\boldsymbol{x} = (x, y)$ this leads to

$$s^*(\boldsymbol{x}) = s^*(y) = \frac{y - y_0}{2\eta_0}.$$

Then $\boldsymbol{x} - \gamma(s^*(\boldsymbol{x})) = (x - x(s^*),\, 0)^T$ and (12) becomes

$$v_b(x, y) = a\,(s^*) \exp\left[ik\left(S(s^*) + (x - x(s^*))\xi(s^*) + \frac{1}{2}m_{11}(s^*)(x - x(s^*))^2\right)\right],$$

with $x(s)$, $\xi(s)$, $S(s)$, $m_{11}(s)$, $a(s)$ and $s^*(y)$ given above.

Remark 4.1. The value of $q(s) = 1 + 2si - s^2\beta$ crosses the negative real axis when $s = 1/\xi_0$. To find a better branch cut for the square root in the expression (18) for $A(s)$ we note first that the equation

$$\Im\frac{q(s)}{\beta} = 0 \quad \Leftrightarrow \quad (\Im\beta)(\Re q(s)) = (\Re\beta)(\Im q(s)) \quad \Leftrightarrow \quad 2\xi_0(1 - s^2) = 2s(1 - \xi_0 s)$$

has the unique solution $s = \xi_0$. Therefore, the equation $q(s) = t\beta$ with $t \in \mathbb{R}$ only has a solution if $t = q(\xi_0)/\beta = (1 - \xi_0^2) > 0$. Hence, $q(s)$ never crosses the line $\{-t\beta : t \geq 0\}$, which we therefore use as branch cut. This guarantees a smooth dependence of the Gaussian beam on s for all $s \geq 0$. It can be written as $\sqrt{z\beta^*}/\sqrt{\beta^*}$ if $\sqrt{\cdot}$ is the usual square root with branch cut along the negative real axis.

4.2. **Gaussian beam superposition.** We will now prepare the superposition. The initial curve Γ is simply the y-axis so that $\Gamma(z) = \gamma(0; z) = (x_0(z), y_0(z)) = (0, z)$. By assumption the incoming wave propagates in the same direction (ξ_0, η_0) for all z. Moreover, the same initial data for M is used for all z. This means that x, ξ, η, M and the geometrical spreading parameter q are all independent of z. Only y, S and a depend on z. We obtain from (12),

$$v_{\text{beam}}(x, y; z) = a(s; z)e^{ik(S(s;z)+(\boldsymbol{x}-\gamma(s;z))\cdot\boldsymbol{p}(s)+\frac{1}{2}(\boldsymbol{x}-\gamma(s;z))^T M(s)(\boldsymbol{x}-\gamma(s;z))}, \qquad s = s^*(\boldsymbol{x}).$$

To derive the initial data $a_0(z)$, $S_0(z)$ for $a(s; z)$ and $S(s; z)$ we consider the trace of v_{beam} and u_{GB} on $x = 0$. To get explicit formulae we also let $s^*(\boldsymbol{x})$ be the s-value for the point on the curve that has the same x-value, i.e. $s^*(0, y) = 0$. That gives

$$\begin{aligned} v_{\text{beam}}(0, y; z) &= a_0(z)e^{ik[S_0(z)+((0,y)-\gamma(0;z))\cdot\boldsymbol{p}_0+\frac{1}{2}((0,y)-\gamma(0;z))^T M(0)((0,y)-\gamma(0;z))]} \\ &= a_0(z)e^{ik[S_0(z)+(y-z)\eta_0+\frac{1}{2}(y-z)^2 m_{22}(0)]}. \end{aligned}$$

and for u_{GB},

$$\begin{aligned} u_{GB}(0, y) &= e^{iky\eta_0}\sqrt{\frac{k}{2\pi}}\int a_0(z)e^{ik[S_0(z)-z\eta_0]+\frac{1}{2}ik(y-z)^2 m_{22}(0)}dz \\ &= a_0(y)e^{iky\eta_0}\sqrt{\frac{k}{2\pi}}\int e^{ik[S_0(z+y)-(z+y)\eta_0]+\frac{1}{2}ikz^2 m_{22}(0)}dz + O(k^{-1}). \end{aligned}$$

To match this with the incoming wave on $x = 0$, i.e. $u_{\text{inc}}(0, y) = A(y)\exp(ik\eta_0 y)$, we take

$$S_0(z) = \eta_0 z,$$

and

$$a_0(y) = A(y)\left(\sqrt{\frac{k}{2\pi}}\int e^{\frac{1}{2}ikz^2 m_{22}(0)}dz\right)^{-1} = A(y)\sqrt{-im_{22}(0)}.$$

Thus the expressions for the Gaussian beam coefficients are

$$x(s) = 2s\xi_0 - s^2, \tag{19a}$$

$$y(s; z) = z + 2s\eta_0, \tag{19b}$$

$$\xi(s) = \xi_0 - s, \tag{19c}$$

$$\eta(s) = \eta_0, \tag{19d}$$

$$S(s; z) = \eta_0 z + 2s - 2s^2\xi_0 + \frac{2}{3}s^3, \tag{19e}$$

$$m_{11}(s) = \frac{2i - (\xi_0 + s)\beta}{2q(s)}, \tag{19f}$$

$$a(s; z) = \frac{A(z)}{\sqrt{q(s)}}\sqrt{-im_{22}(0)}. \tag{19g}$$

This gives us the simplified expression for v_{beam},

$$v_{\text{beam}}(x, y; z) = a\left(s^*; z\right)e^{ik\left(S(s^*;z)+(x-x(s^*))\xi(s^*)+\frac{1}{2}m_{11}(s^*)(x-x(s^*))^2\right)}, \quad s^*(y; z) = \frac{y - z}{2\eta_0}, \tag{20}$$

which, together with (19), define u_{GB} via

$$u_{GB}(x, y) = \sqrt{\frac{k}{2\pi}}\int v_{\text{beam}}(x, y; z)dz. \tag{21}$$

We will continue now to simplify (20) and compute the k-scaled Fourier transform of u_{GB} in y. Since $x - x(s^*) = x - 2s^*\xi_0 - {s^*}^2$ and $\delta = \xi_0^2 - x$ by (6) we have

$$v_{\text{beam}}(x, y; z) = A(z)f(y - z)e^{ik\eta_0 y}, \qquad f(2\eta_0 s^*) = \sqrt{-im_{22}(0)}\frac{e^{ikS_g(s^*;\delta)}}{\sqrt{q(s^*)}}$$

where

$$
\begin{aligned}
S_g(s^*;\delta) &:= S(s^*;z) + (x-x(s^*))\xi(s^*) + \frac{1}{2}m_{11}(s^*)(x-x(s^*))^2 - \eta_0 y \\
&= \eta_0(z-y) + 2s^* - 2s^{*2}\xi_0 + \frac{2}{3}s^{*3} + (x - 2s^*\xi_0 + s^{*2})(\xi_0 - s^*) + \frac{1}{2}m_{11}(s^*)(x - 2s^*\xi_0 + s^{*2})^2 \\
&= 2\xi_0^2 s^* - 2s^{*2}\xi_0 + \frac{2}{3}s^{*3} + ((s^*-\xi_0)^2 - \delta)(\xi_0 - s^*) + \frac{1}{2}m_{11}(s^*)((s^*-\xi_0)^2-\delta)^2 \\
&= \frac{2}{3}\xi_0^3 + \frac{2}{3}(s^*-\xi_0)^3 + ((s^*-\xi_0)^2-\delta)(\xi_0 - s^*) + \frac{1}{2}m_{11}(s^*)((s^*-\xi_0)^2-\delta)^2.
\end{aligned}
$$

Then we can write

$$
u_{GB}(x,y) = \sqrt{\frac{k}{2\pi}}(A * f)(y)e^{ik\eta_0 y}.
$$

Since $A \in \mathcal{S}$ the beam v is always integrable in z, so that u_{GB} in (21) is well-defined. However, there is no guarantee that $u_{GB}(x,\cdot)$ is in L^1; in general it is not. When we compute its Fourier transform we therefore use Lemma 2.1 and (3). By Lemma 5.1 and 5.2 below, q is bounded away from zero and $\Im m_{11}$ is strictly positive. Hence, $f \in L^\infty$. Therefore,

$$
\begin{aligned}
\hat{u}_{GB}(x,\eta_0+\cdot) &= \lim_{t\to\infty}\mathcal{F}_k\left(\psi_t u_{GB}(x,\cdot)\exp(-ik\eta_0\cdot)\right) \\
&= \lim_{t\to\infty}\sqrt{\frac{k}{2\pi}}\mathcal{F}_k\left(\psi_t(A*f)\right) = \lim_{t\to\infty}\mathcal{F}_k(f\psi_t)\mathcal{F}_k(A),
\end{aligned}
$$

where we can choose $\psi_t(w) := \psi((w/2\eta_0 - \xi_0)/t)$, that is, $b_0 = 1/2\eta_0 \neq 0$ and $b_1 = -\xi_0$ in Lemma 2.1. We then compute

$$
\begin{aligned}
\mathcal{F}_k(f\psi_t)(\delta,\eta) &= \sqrt{\frac{k}{2\pi}}\int \psi_t(w)f(w)e^{-ik\eta w}dw = \sqrt{\frac{k}{2\pi}}\int \psi_t(2\eta_0 s^*)f(2\eta_0 s^*)e^{-2ik\eta\eta_0 s^*}2\eta_0 ds^* \\
&= \sqrt{-im_{22}(0)}\sqrt{\frac{k}{2\pi}}\int \psi_t(2\eta_0 s^*)\frac{e^{ikS_g(s^*;\delta)}}{\sqrt{q(s^*)}}e^{-2ik\eta\eta_0 s^*}2\eta_0 ds^* \\
&= \sqrt{-im_{22}(0)}\sqrt{\frac{k}{2\pi}}\int \psi(\theta/t)\frac{e^{ikS_g(\theta+\xi_0;\delta)}}{\sqrt{q(\theta+\xi_0)}}e^{-2ik\eta\eta_0(\theta+\xi_0)}2\eta_0 d\theta
\end{aligned}
$$

Here we made the change of variables $w = 2\eta_0 s^*$ and $\theta = s^* - \xi_0$. Moreover,

$$
\begin{aligned}
S_g\left(\theta+\xi_0;\delta\right) &= \frac{2}{3}\xi_0^3 + \frac{2}{3}\theta^3 - (\theta^2-\delta)\theta + \frac{1}{2}m_{11}(\theta+\xi_0)(\theta^2-\delta)^2 \\
&= \frac{2}{3}\xi_0^3 - \frac{1}{3}\theta^3 + \delta\theta + \frac{1}{2}m_{11}(\theta+\xi_0)(\theta^2-\delta)^2 \\
&= \frac{2}{3}\xi_0^3 + \phi_g(\delta,\theta,\eta) + 2\eta_0\eta\theta,
\end{aligned}
$$

where we have introduced the Gaussian beam phase ϕ_g as,

$$
\phi_g(\delta,\theta,\eta) = -\frac{1}{3}\theta^3 + \theta(\delta - 2\eta_0\eta) + \frac{1}{2}m_{11}(\xi_0+\theta)(\theta^2-\delta)^2. \tag{22}
$$

Since $m_{22}(0) = \xi_0\beta/2 =$ by (17) and $q(\xi_0) = \beta\eta_0^2$, this finally gives

$$
\begin{aligned}
\mathcal{F}_k(f\psi_t)(\delta,\eta) &= \sqrt{-im_{22}(0)}\sqrt{\frac{k}{2\pi}}\int \psi(\theta/t)\frac{e^{ik(\frac{2}{3}\xi_0^3 + \phi_g(\delta,\theta,\eta) - 2\eta\eta_0\xi_0)}}{\sqrt{q(\theta+\xi_0)}}2\eta_0 d\theta \\
&= 2\eta_0\sqrt{-im_{22}(0)}\sqrt{\frac{k}{2\pi}}e^{ik(\frac{2}{3}\xi_0^3 - 2\eta\eta_0\xi_0)}\int \psi(\theta/t)\frac{e^{ik\phi_g(\delta,\theta,\eta)}}{\sqrt{q(\theta+\xi_0)}}d\theta \\
&= \sqrt{-2i\xi_0 q(\xi_0)}\sqrt{\frac{k}{2\pi}}e^{ik(\frac{2}{3}\xi_0^3 - 2\eta\eta_0\xi_0)}\int \psi(\theta/t)\frac{e^{ik\phi_g(\delta,\theta,\eta)}}{\sqrt{q(\theta+\xi_0)}}d\theta \\
&=: k^{1/6}P_{GB}(k,\eta)I_t(\eta,\delta,k),
\end{aligned}
$$

where

$$P_{GB}(k,\eta) = 2(\pi\xi_0)^{1/2} e^{-i\pi/4} e^{ik(\frac{2}{3}\xi_0^3 - 2\eta\eta_0\xi_0)}, \tag{23}$$

and

$$I_t(\eta,\delta,k) = \frac{k^{1/3}\sqrt{q(\xi_0)}}{2\pi} \int \psi(\theta/t) \frac{e^{ik\phi_g(\delta,\theta,\eta)}}{\sqrt{q(\theta+\xi_0)}} d\theta. \tag{24}$$

Then

$$\hat{u}_{GB}(x,\eta_0+\eta) = \hat{v}_{GB}(\eta,x,k)\hat{A}(\eta), \tag{25}$$

with

$$\hat{v}_{GB}(\eta,x,k) = \lim_{t\to\infty} \; k^{1/6} P_{GB}(k,\eta) I_t(\eta,\delta,k).$$

5. Properties of the amplitude and phases

In this section we collect a series of estimates that we will need for the final proof of the magnitude of the Gaussian beam error.

5.1. **Geometrical spreading.** Here we show some properties of the q-polynomial in (7) that relates to the geometrical spreading, repeated here for convenience,

$$q(s) = 1 + 2is - \beta s^2, \qquad \beta = 1 + 2i\xi_0.$$

We have

Lemma 5.1. *There are positive constants q_0 and q_1, independent of ξ_0 and $\theta \in \mathbb{R}$, such that*

$$0 < q_0(1+\theta^2) \le |q(\xi_0+\theta)| \le q_1(1+\theta^2).$$

Furthermore, there are constants b_n independent of θ and ξ_0 such that

$$\left| \frac{d^n}{d\theta^n} \frac{1}{\sqrt{q(\xi_0+\theta)}} \right| \le \frac{b_n}{1+|\theta|^{n+1}}.$$

Proof. We first show that q has no real root for the considered values of ξ_0. Suppose therefore that q has a real root $s = r$. Then the real and imaginary parts of $q(r) = 0$ reads

$$1 - r^2 = 0, \qquad 2r - 2\xi_0 r^2 = 0.$$

The only solutions to this system are $r = \xi_0 = \pm 1$, which are both ruled out by (5). Let $\tilde{q}(\theta) := |q(\xi_0+\theta)|/(1+\theta^2)$, which is then continuous and non-zero for all θ. For large θ it is bounded from below and above since $\lim_{\theta\to\pm\infty} \tilde{q}(\theta) = |\beta|$. In fact, there is a a constant θ_0 such that

$$|\tilde{q}(\theta) - |\beta|| \le \frac{|\beta|}{2}, \qquad |\theta| > \theta_0,$$

uniformly in ξ_0, because of the bound (5). We can then take q_0 and q_1 as

$$q_0 = \min\left(\inf_{|\theta|\le\theta_0} \tilde{q}(\theta), \frac{|\beta|}{2} \right), \qquad q_1 = \max\left(\sup_{|\theta|\le\theta_0} \tilde{q}(\theta), \frac{3|\beta|}{2} \right).$$

The stated bound then follows.

For the second statement, we observe that there exists a sequence of polynomials p_n of degree n such that

$$\frac{d^n}{d\theta^n} \frac{1}{\sqrt{q(\theta)}} = \frac{p_n(\theta)}{q(\theta)^{n+1/2}},$$

given by the recursion

$$p_{n+1}(\theta) = p_n'(\theta) q(\theta) - (n+1/2) p_n(\theta) q'(\theta),$$

thanks to

$$\frac{d}{d\theta} \frac{p_n(\theta)}{q(\theta)^{n+1/2}} = \frac{p_n'(\theta)q(\theta) - (n+1/2)p_n(\theta)q'(\theta)}{q(\theta)^{n+3/2}}.$$

Then, by the lower bound on $|q|$ and (5), there are constants C_n and b_n depending on n but independent of θ and ξ_0 such that

$$\left|\frac{d^n}{d\theta^n}\frac{1}{\sqrt{q(\xi_0+\theta)}}\right| \le \frac{|p_n(\xi_0+\theta)|}{q_0^{n+1/2}(1+\theta^2)^{n+1/2}} \le C_n\frac{1+|\xi_0+\theta|^n}{1+|\theta|^{2n+1}} \le C_n\frac{1+2^{n-1}(|\xi_0|^n+|\theta|^n)}{1+|\theta|^{2n+1}} \le b_n\frac{1}{1+|\theta|^{n+1}}.$$

This shows the lemma. ☐

5.2. **Phase.** In this section we define the two phase functions ϕ_a and ϕ_g that turn up in our analysis and show a few properties of them. The first one, ϕ_a, is defined as

$$\phi_a(\delta,\theta,\eta) = -\frac{1}{3}\theta^3 + \theta(\delta - 2\eta_0\eta). \tag{26}$$

This has a close connection to the Airy function and we call it the Airy phase. Indeed, $\exp(i\theta^3/3)$ is the Fourier transform of $\mathrm{Ai}(z)$ in $\mathcal{S}'(\mathbb{R})$ and therefore, using the regularization of Lemma 2.1,

$$\lim_{s\to\infty}\frac{k^{1/3}}{2\pi}\int \psi(\theta/s)e^{-ik\phi_a(\delta,\theta,\eta)}d\theta = \mathrm{Ai}(k^{2/3}(2\eta_0\eta-\delta)).$$

The second phase is the Gaussian beam phase (22) derived in the previous section. It can be written as a sum of ϕ_a and a complex correction, by (16),

$$\phi_g(\delta,\theta,\eta) = \phi_a(\delta,\theta,\eta) + \frac{1}{2}m_{11}(\xi_0+\theta)(\theta^2-\delta)^2, \qquad m_{11}(s) = \frac{2i-(\xi_0+s)\beta}{2q(s)}. \tag{27}$$

We start by looking at the m_{11} part of the Gaussian beam phase.

Lemma 5.2. *For m_{11} it holds that*

$$\lim_{s\to\pm\infty} m_{11}(s)s = \lim_{s\to\pm\infty} -m_{11}'(s)s^2 = \frac{1}{2}, \tag{28}$$

$$\Im m_{11}(s) = \frac{\eta_0^2}{|q(s)|^2}, \tag{29}$$

$$\left|\frac{d^n m_{11}(\xi_0+\theta)}{d\theta^n}\right| \le \frac{d_n}{1+|\theta|^{n+1}}, \tag{30}$$

$$|m_{11}(\xi_0+\theta)| \ge \frac{D_0}{1+|\theta|}, \tag{31}$$

for some constants d_n, D_0 independent of θ and ξ_0.

Proof. From (27) we have

$$sm_{11}(s) = s\frac{2i-(\xi_0+s)\beta}{2q(s)} = \frac{(2i-\xi_0\beta)s-\beta s^2}{2(1+2i-\beta s^2)} = \frac{(-2i+\xi_0\beta)s^{-1}+\beta}{-2(1+2i)s^{-1}+2\beta} \to \frac{1}{2},$$

showing the first limit in (28). Moreover,

$$m_{11}'(s) = -\frac{1}{2}\frac{\beta(1+2is-s^2\beta)+(2i-2s\beta)(2i-(\xi_0+s)\beta)}{q(s)^2} = -\frac{1}{2}\frac{\beta-4-2i\beta\xi_0+2s\beta\Big(\xi_0\beta-2i\Big)+s^2\beta^2}{q(s)^2}$$

which similarly implies the second limit in (28). For (29) we have by (16),

$$\begin{aligned}\Im m_{11}(s) &= \frac{\Im\left[(2i-(\xi_0+s)\beta)(1-2is-s^2\beta^*)\right]}{2|q(s)|^2} = \frac{(\xi_0+s)(2s-2s^2\xi_0)+2(\eta_0^2-s\xi_0)(1-s^2)}{2|q(s)|^2}\\ &= \frac{\eta_0^2+s\left[(\xi_0+s)(1-s\xi_0)-\eta_0^2 s-\xi_0(1-s^2)\right]}{|q(s)|^2} = \frac{\eta_0^2}{|q(s)|^2} > 0.\end{aligned}$$

To show (31) with $D_0 = \min(1/2, 1-\bar{\xi}_1)/\sqrt{2}q_1$ we observe that,

$$\begin{aligned}|m_{11}(\xi_0+\theta)| &= \frac{|2i-(\theta+2\xi_0)(1+2i\xi_0)|}{|q(\theta+\xi_0)|} \ge \frac{|\theta+2\xi_0|+2|1-(\theta+2\xi_0)\xi_0|}{\sqrt{2}q_1(1+\theta^2)}\\ &\ge \frac{|\theta+2\xi_0|+1-|(\theta+2\xi_0)\xi_0|}{\sqrt{2}q_1(1+\theta^2)} = \frac{1+(1-\xi_0)|\theta+2\xi_0|}{\sqrt{2}q_1(1+\theta^2)} \ge \frac{1-2(1-\xi_0)\xi_0+(1-\bar{\xi}_1)|\theta|}{\sqrt{2}q_1(1+\theta^2)} \ge \frac{D_0}{1+|\theta|},\end{aligned}$$

where we used Lemma 5.1 as well as the facts that $\sqrt{2}|z| \geq |\Re z| + |\Im z|$ and $1 - 2x(1-x) \geq 1/2$ for $0 \leq x \leq 1$.

For (30) we note that if r_k and p are any polynomials of degrees ℓ_k and ℓ, then

$$\frac{d}{d\theta}\left(\frac{r_k}{p^k}\right) = \frac{r_k' p - r_k p'}{p^{k+1}} =: \frac{r_{k+1}}{p^{k+1}},$$

where the degree of r_{k+1} is $\ell_{k+1} = \ell_k + \ell - 1$. By induction,

$$\frac{d^n}{d\theta^n}\left(\frac{r_1}{p}\right) = \frac{r_{n+1}}{p^{n+1}},$$

and r_{n+1} has degree $\ell_{n+1} = \ell_1 + n(\ell - 1)$. Since m_{11} is the quotient of the first order polynomial $r_1(s) = (2i - \xi_0\beta - \beta s)/2$, and the second order polynomial $q(s)$, its n-th derivative, is

$$m_{11}^{(n)}(\xi_0 + \theta) = \frac{r_{n+1}(\xi_0 + \theta)}{q(\xi_0 + \theta)^{n+1}}$$

and r_{n+1} is of degree $n+1$. Using Lemma 5.1 and (5) we then obtain the required estimate,

$$|m_{11}^{(n)}(\xi_0 + \theta)| \leq \frac{C\left(1 + |\xi_0 + \theta|^{n+1}\right)}{(q_0(1+\theta^2))^{n+1}} \leq \frac{C\left(1 + 2^n(|\xi_0|^{n+1} + |\theta|^{n+1})\right)}{(q_0(1+\theta^2))^{n+1}} \leq \frac{d_n}{1 + |\theta|^{n+1}},$$

where d_n is independent of ξ_0. □

We are now ready to estimate the full phases ϕ_a and ϕ_g.

Lemma 5.3. *Let ϕ be either ϕ_a or ϕ_g. Suppose $|\delta| \leq 1$. Then there are constants c_0 and C_n, independent of η, θ, ξ_0 and δ such that*

$$|\phi_\theta(\delta, \theta, \eta)| \geq \frac{\theta^2}{16}, \quad \textit{when } |\theta| \geq c_0\left(1 + |\eta|^{1/2}\right), \tag{32}$$

$$|\partial_\theta^n \phi(\delta, \theta, \eta)| \leq C_n \begin{cases} |\theta|^2 + \delta + |\eta|, & n = 1, \\ |\theta| + \delta, & n = 2, \\ \frac{1}{1 + |\theta|^{n-3}}, & n \geq 3. \end{cases} \tag{33}$$

Additionally,

$$\Im\phi(\delta, \theta, \eta) \geq 0. \tag{34}$$

For ϕ_a we have $C_n = 0$ when $n \geq 4$.

Proof. We first prove the statements for $\phi = \phi_a$. Suppose $|\theta| \geq c_0(1 + |\eta|)^{1/2}$. Then $|\eta| \leq \theta^2/c_0 - 1$ and by (5)

$$|\partial_\theta \phi_a(\delta, \theta, \eta)| = |\theta^2 + 2\eta\eta_0 - \delta| \geq \theta^2 - 2|\eta| - 1 \geq \theta^2 - 2(\theta^2/c_0^2 - 1) - 1 \geq (1 - 2/c_0^2)\theta^2,$$

which gives (32) when $c_0 \geq \sqrt{32/15}$. Similarly,

$$|\partial_\theta \phi_a(\delta, \theta, \eta)| \leq \theta^2 + 2|\eta| + \delta \leq 2(\theta^2 + |\eta| + \delta),$$

showing (33) for $n = 1$. The bounds for larger n follow easily from an explicit calculation, yielding $C_n = 2$ for $1 \leq n \leq 3$ and $C_n = 0$ for $n \geq 4$.

To prove the statements for ϕ_g we denote the correction term by $w(\delta, \theta) = \phi_g - \phi_a$. Lemma 5.2 gives

$$\lim_{\theta \to \pm\infty} \frac{w_\theta(\delta, \theta)}{\theta^2} = \lim_{\theta \to \pm\infty} \frac{1}{\theta^2}\left(\frac{1}{2} m_{11}'(\xi_0 + \theta)(\theta^2 - \delta)^2 + 2m_{11}(\xi_0 + \theta)(\theta^2 - \delta)\theta\right) = -\frac{1}{4} + 1 = \frac{3}{4}.$$

Consequently, there is a K such that $|w_\theta(\delta, \theta)| \leq 7\theta^2/8$ for all $|\theta| \geq K$, uniformly in ξ_0 and δ thanks to (5). We now take $c_0 = \max(K, \sqrt{32})$. Then for $|\theta| \geq c_0(1 + |\eta|^{1/2}) \geq K$, we have

$$|\partial_\theta \phi_g(\delta, \theta, \eta)| \geq |\partial_\theta \phi_a(\delta, \theta, \eta)| - |\partial_\theta w(\delta, \theta)| \geq (1 - 2/c_0^2)\theta^2 - \frac{7}{8}\theta^2 = (1/8 - 2/c_0^2)\theta^2 \geq \frac{1}{16}\theta^2,$$

and (32) is proved.

For (33) we use (30) in Lemma 5.2. When $n=1$ we have, as above,

$$
\begin{aligned}
|\partial_\theta \phi_g(\delta,\theta,\eta)| &\leq \theta^2 + |\delta - 2\eta_0\eta| + \frac{1}{2}|m'_{11}(\xi_0+\theta)|(\theta^2-\delta)^2 + 2|m_{11}(\xi_0+\theta)|\theta(\theta^2-\delta)\\
&\leq \theta^2 + |\delta - 2\eta_0\eta| + \frac{d_1}{2}\frac{|\theta|^4+\delta^2}{1+\theta^2} + 2d_0\frac{|\theta|^3+\delta|\theta|}{1+|\theta|} \leq \left(1+\frac{1}{2}d_1+2d_0\right)(\theta^2+\delta) + 2\eta_0|\eta|,
\end{aligned}
$$

which shows the result for $n=1$ with $C_1 = \max(1+d_1/2+2d_0, 2)$. For $n=2$ we get

$$
\begin{aligned}
|\partial_{\theta\theta}\phi_g(\delta,\theta,\eta)| &= \left|-2\theta + \frac{1}{2}m_{11}^{(2)}(\xi_0+\theta)(\theta^2-\delta)^2 + 4m_{11}^{(1)}(\xi_0+\theta)(\theta^2-\delta)\theta + 2m_{11}(\xi_0+\theta)(3\theta^2-\delta)\right|\\
&\leq 2|\theta| + \frac{d_2}{2}\frac{|\theta|^4+\delta^2}{1+|\theta|^3} + 4d_1\frac{|\theta|^3+\delta|\theta|}{1+\theta^2} + 2d_0\frac{3\theta^2+\delta}{1+|\theta|}\\
&\leq \left(2+\frac{1}{2}d_2+4d_1+6d_0\right)(|\theta|+\delta) =: C_2(|\theta|+\delta).
\end{aligned}
$$

For $n \geq 3$,

$$
\begin{aligned}
|\partial_\theta^n w(\delta,\theta)| &= \frac{1}{2}\left|\sum_{\ell=0}^{n}\binom{n}{\ell} m_{11}^{(n-\ell)}(\xi_0+\theta)\frac{d^\ell}{d\theta^\ell}(\theta^2-\delta)^2\right| = \frac{1}{2}\left|\sum_{\ell=0}^{\max(4,n)}\binom{n}{\ell} m_{11}^{(n-\ell)}(\xi_0+\theta)\frac{d^\ell}{d\theta^\ell}(\theta^2-\delta)^2\right|\\
&\leq C\sum_{\ell=0}^{\max(4,n)}\binom{n}{\ell}\frac{1}{1+|\theta|^{n-\ell+1}}(1+|\theta|^{4-\ell}) \leq \frac{C_n}{1+|\theta|^{n-3}},
\end{aligned}
$$

which shows the result for $n \geq 3$ as $\partial_\theta^3\phi_g = -1 + \partial_\theta^3 w$ and $\partial_\theta^n\phi_g = \partial_\theta^n w$ for $n \geq 4$.

Finally, statement (34) for ϕ_a is trivial, as $\Im\phi_a = 0$, and for ϕ_g it follows from (29) in Lemma 5.2 and (5), since

$$
\Im\phi_g(\delta,\theta,\eta) = \frac{1}{2}\Im m_{11}(\xi_0+\theta)(\theta^2-\delta)^2 = \frac{(\theta^2-\delta)^2\eta_0^2}{2|q(\xi_0+\theta)|^2} \geq 0.
$$

This concludes the proof. □

In the final part of this section we consider a space of function that is used in Lemma B.1. For a fixed phase function ϕ and order p we first introduce the basis functions

$$
\mathcal{W}_p(\phi) = \left\{\prod_k \frac{\phi^{(\alpha_k+1)}}{\phi'} \;:\; \sum_k \alpha_k = p,\ \alpha_k \geq 1,\right\}, \tag{35}
$$

when $p \geq 1$ and let $\mathcal{W}_0(\phi)$ be the constant function equal to one. Second, we denote by $\mathcal{U}_p(\phi)$ the linear span of these functions over the complex numbers,

$$
\mathcal{U}_p(\phi) = \operatorname{span}_{\mathbb{C}} \mathcal{W}_p(\phi). \tag{36}
$$

Functions in $\mathcal{U}_p(\phi)$ appear in Lemma B.1. Here we show that when ϕ is either ϕ_a or ϕ_g, these functions are bounded on subsets where the phase gradient grows at least quadratically.

Lemma 5.4. *Let $K = \{\theta \in \mathbb{R} \mid r_0 \leq |\theta| \leq r_1\}$, with $0 < R_0 \leq r_0 < r_1$ and, for some $c > 0$,*

$$
|\phi_\theta(\delta,\theta,\eta)| \geq c\theta^2, \qquad \text{for all } \theta \in K \text{ and } |\delta| \leq 1,
$$

where c and R_0 are independent of δ and η. Then for each $u \in \mathcal{U}_p(\phi(\delta,\cdot,\eta))$, where ϕ is either ϕ_a or ϕ_g, there is a uniform bound

$$
|u(\theta)| \leq C, \qquad \forall\theta \in K.
$$

The constant C depends on c and R_0 but is independent of r_0, r_1, δ and η.

Proof. We get from Lemma 5.3,

$$
|\phi_{\theta\theta}(\delta,\theta,\eta)| \leq C_2(|\theta|+\delta) \leq C_2(|\theta|+1) \leq C_2(1+1/R_0)|\theta|, \qquad \forall\theta \in K,
$$

while for $n \geq 3$ we have

$$
|\partial_\theta^n\phi(\delta,\theta,\eta)| \leq \frac{C_n}{1+|\theta|^{n-3}} \leq C_n, \qquad \forall\theta \in \mathbb{R}.
$$

The constants C_2 and C_n are independent of δ and η. Consider next $w \in \mathcal{W}_p(\phi(\delta,\cdot,\eta))$ and assume it has $M \geq 1$ factors and, without loss of generality, that the first $M' \leq M$ factors have $\alpha_k = 1$. Then, for all $\theta \in K$,

$$\begin{aligned}
|w(\theta)| &= \prod_{k=1}^{M'} \frac{|\phi_{\theta\theta}(\delta,\theta,\eta)|}{|\phi_\theta(\delta,\theta,\eta)|} \prod_{k=M'+1}^{M} \frac{|\partial_\theta^{\alpha_k+1}\phi(\delta,\theta,\eta)|}{|\phi_\theta(\delta,\theta,\eta)|} \leq \prod_{k=1}^{M'} \frac{C_2(1+1/R_0)|\theta|}{c\theta^2} \prod_{k=M'+1}^{M} \frac{C_{\alpha_k+1}}{|c\theta^2|} \\
&\leq \left(\max_{2\leq j\leq p+1} C_j\right)^M \frac{(1+1/R_0)^{M'}}{c^M|\theta|^{2M-M'}} \leq C\frac{(1+1/R_0)^{M'}}{R_0^{2M-M'}} =: \tilde{C},
\end{aligned}$$

where $\tilde{C}$ is independent of δ, η, r_0 and r_1, but depends on c and R_0. Since $u \in \mathcal{U}_p(\phi(\delta,\cdot,\eta))$ is a linear combination of functions in $\mathcal{W}_p(\phi(\delta,\cdot,\eta))$ the same bound holds true for u on K. □

6. Estimates of oscillatory integrals

We consider integrals of the type

$$k^{1/3}\int \theta^p r(\theta) e^{ik\phi(\delta,\theta,\eta)}d\theta,$$

where ϕ is either ϕ_a or ϕ_g and $r \in W^{n,\infty}(\mathbb{R})$, whose norm is defined by

$$||r||_{W^{n,\infty}(\mathbb{R})} = \sum_{j=0}^{n} \left\|\frac{d^j r}{d\theta^j}\right\|_{L^\infty(\mathbb{R})}$$

In general the integrand is then not in $L^1(\mathbb{R})$ and the integral must be defined in a generalized sense as an oscillatory integral. In this section, however, we only estimate the integral over bounded intervals that are defined using a smooth cutoff function $\psi \in C_c^\infty(\mathbb{R})$, which takes values in $[0,1]$, is equal to one on $[-1,1]$ and has $\mathrm{supp}(\psi) \subset [-2,2]$. This leaves us with integrals over compact domains with smooth integrands. Our main tool for estimating them are the identities stated in Lemma B.1. They are used to rewrite the integral on the domain where $\phi_\theta \neq 0$. Lemma 5.3 in the previous section tells us when this is true. Lemma B.1 uses the space of functions $\mathcal{U}_p$ defined in (36). Functions in $\mathcal{U}_p$ are bounded on the domains we consider here, which is proved in Lemma 5.4. Together, Lemma B.1 and Lemma 5.4 constitute a precise version of the non-stationary phase lemma.

We will also make use of the simple inequalities

$$(37)\qquad ||uv||_{W^{n,\infty}(\mathbb{R})} \leq C_n \sum_{j=0}^{n}\sum_{k=0}^{j} \left\|\frac{d^k u}{d\theta^k}\right\|_{L^\infty(\mathbb{R})} \left\|\frac{d^{j-k} v}{d\theta^{j-k}}\right\|_{L^\infty(\mathbb{R})} \leq C_n ||u||_{W^{n,\infty}(\mathbb{R})}||v||_{W^{n,\infty}(\mathbb{R})},$$

for all $u, v \in W^{n,\infty}(\mathbb{R})$, and

$$(38)\quad ||u(\cdot/\sigma)||_{W^{n,\infty}(\mathbb{R})} = \sum_{j=0}^{n} \sigma^{-j} \left\|\frac{d^j u}{d\theta^j}\right\|_{L^\infty(\mathbb{R})} \leq \max(1,\sigma^{-n})||u||_{W^{n,\infty}(\mathbb{R})}, \qquad \forall u \in W^{n,\infty}(\mathbb{R}),\ \sigma \neq 0.$$

We start with an estimate of the integral between R and $2t$ where t is arbitrarily large. For this we consider a smooth cutoff around $\theta = R$ and $\theta = t > R$ and obtain bounds that are independent of t.

Lemma 6.1. *Let ϕ be either ϕ_a or ϕ_g and set*

$$I_t = k^{1/3}\int (1-\psi(\theta/R))\psi(\theta/t)\theta^p r(\theta) e^{ik\phi(\delta,\theta,\eta)}d\theta,$$

where $c_0 \leq R < t$ with c_0 as in Lemma 5.3, $|\delta| \leq 1$ and $r \in W^{n,\infty}(\mathbb{R})$. If $c_0(1+|\eta|^{1/2}) \leq R$ and $n \geq 1 + p/2$, there is a constant C_n independent of k, R, δ, η and t such that

$$|I_t| \leq C_n k^{1/3-n}||r||_{W^{n,\infty}(\mathbb{R})}.$$

Proof. On this domain the results in Section 5 show that the phase gradient does not vanish, and $|\phi_\theta| \geq c\theta^2$. Since the integrand is smooth and compactly supported we can therefore use the non-stationary phase lemma to estimate the integral. For sufficiently regular r the repeated partial integrations in this lemma enables us to offset the growing θ^p factor and obtain a bound that is independent of t.

To be precise, let

$$b(\theta) = (1-\psi(\theta/R))\psi(\theta/t)r(\theta),$$

which is supported in the compact set $K = \{\theta \in \mathbb{R} \mid R \le |\theta| \le 2t\}$. Then by Lemma 5.3 we have $|\phi_\theta(\delta,\theta,\eta)| \ge \theta^2/16$ on K, independent of δ, η and t, since $|\theta| \ge R > c_0(1+|\eta|^{1/2})$. We apply Lemma B.1 with $a(\theta) = b(\theta)\theta^p$ and let D be any bounded open set containing K. This gives

$$\begin{aligned} I_t &= k^{1/3}\int_D b(\theta)\theta^p e^{ik\phi(\delta,\theta,\eta)}d\theta \\ &= k^{1/3}(ik)^{-n}\sum_{\ell=0}^{n}\int_K \left(\frac{d^\ell}{d\theta^\ell}b(\theta)\theta^p\right)\frac{u_{\ell,n}(\theta)}{\phi_\theta(\delta,\theta,\eta)^n}e^{ik\phi(\delta,\theta,\eta)}d\theta, \qquad u_{\ell,n}\in\mathcal{U}_{n-\ell}(\phi(\delta,\cdot,\eta)). \end{aligned} \tag{39}$$

where the space $\mathcal{U}_p$ is defined in (36). Since K satisifies all conditions in Lemma 5.4 and $R \ge c_0 > 0$ we obtain a uniform bound,

$$|u_{\ell,n}(\theta)| \le C_{\ell,n}, \qquad \forall\theta\in K,$$

where $C_{\ell,n}$ depends on c_0, but is independent of δ, η, R and t. This allows us to estimate I_t as

$$\begin{aligned} |I_t| &\le Ck^{1/3-n}\sum_{\ell=0}^{n}\sum_{j=0}^{\min(\ell,p)}\int_K |b^{(\ell-j)}(\theta)|\frac{|\theta|^{p-j}}{|\theta|^{2n}}d\theta \le Ck^{1/3-n}\sum_{\ell=0}^{n}\sum_{j=0}^{\min(\ell,p)}||b^{(\ell-j)}||_{L^\infty(\mathbb{R})}\int_R^\infty \frac{d\theta}{|\theta|^{2n-p+j}} \\ &\le Ck^{1/3-n}\sum_{\ell=0}^{n}||b^{(\ell)}||_{L^\infty(\mathbb{R})}\sum_{j=0}^{p}\frac{1}{R^{2n-p+j+1}} \le C\frac{k^{1/3-n}}{c_0^{2n-p+1}}||b||_{W^{n,\infty}(\mathbb{R})}, \end{aligned}$$

where we also used the fact that $2(n-1)\ge p$ and $c_0 \le R \le t$. Moreover, by (37) and (38),

$$||b||_{W^{n,\infty}(\mathbb{R})} \le C_n^2\max(1,c_0^{-n})^2||1-\psi||_{W^{n,\infty}(\mathbb{R})}||\psi||_{W^{n,\infty}(\mathbb{R})}||r||_{W^{n,\infty}(\mathbb{R})} \le C||r||_{W^{n,\infty}(\mathbb{R})}$$

The result in the lemma follows. ☐

Next we consider the main part of the integral for small η and δ with the Airy phase. The estimate involves the norm of r with an argument scaled by $k^{1/3}$.

Lemma 6.2. *Let*

$$I = k^{1/3}\int \psi(\theta/R)\theta^p r(\theta)e^{ik\phi_a(\delta,\theta,\eta)}d\theta,$$

where $0 < R_0 \le R \le R_1$ *and* $r \in W^{n,\infty}(\mathbb{R})$ *with* $n = \lceil (p+1)/5\rceil$. *Moreover, suppose*

$$2|\eta| + |\delta| \le \frac{\xi_0^2}{2}k^{-2/3}, \qquad k \ge 1. \tag{40}$$

Then there is a constant C*, depending on* R_0 *and* R_1*, but independent of* k*,* R*,* η *and* δ *such that*

$$|I| \le Ck^{-p/3}||\tilde r_k||_{W^{n,\infty}(\mathbb{R})}, \qquad \tilde r_k(\zeta) := r\left(k^{-1/3}\zeta\right),$$

Proof. For $\delta - 2\eta\eta_0 = 0$ the phase ϕ_a has a degenerate stationary point at the origin. We will therefore treat the integral in the vicinity of the origin separately. Away from the origin we have the same type of lower bound $\partial_\theta\phi_a \ge c\theta^2$ as in Lemma 6.1 and we can therefore once again use the non-stationary phase lemma to estimate the integral.

For the proof we use the rescaled variables $\zeta = k^{1/3}\theta$, $\tilde\delta = k^{2/3}\delta$, $\tilde\eta = k^{2/3}\eta$. Since

$$k\phi_a(\tilde\delta/k^{2/3},\zeta/k^{1/3},\tilde\eta/k^{2/3}) = -\frac{1}{3}\zeta^3 + \zeta(\tilde\delta - 2\eta_0\tilde\eta) = \phi_a(\tilde\delta,\zeta,\tilde\eta),$$

we can rewrite the integral as

$$I = k^{-p/3}\int \psi\left(\frac{\zeta}{k^{1/3}R}\right)\zeta^p\tilde r_k(\zeta)e^{i\phi_a(\tilde\delta,\zeta,\tilde\eta)}d\zeta.$$

We then divide the integral into two pieces,

$$I = k^{-p/3}\int \psi(\zeta)\,\psi\left(\frac{\zeta}{k^{1/3}R}\right)\zeta^p\tilde{r}_k(\zeta)e^{i\phi_a(\tilde{\delta},\zeta,\tilde{\eta})}d\zeta$$
$$+ k^{-p/3}\int (1-\psi(\zeta))\psi\left(\frac{\zeta}{k^{1/3}R}\right)\zeta^p\tilde{r}_k(\zeta)e^{i\phi_a(\tilde{\delta},\zeta,\tilde{\eta})}d\zeta = k^{-p/3}(I_1+I_2),$$

where I_1 is the part close to the origin containing the stationary point, and I_2 is the remaining part. Note that I_2 matches the general form of the integral in Lemma 6.1 if we take $t = k^{1/3}R^2$.

For I_1 we simply have

$$|I_1| \le \int_{-2}^{2} |\zeta|^p\,|\tilde{r}_k(\zeta)|\,d\zeta \le C||\tilde{r}_k||_{L^\infty(\mathbb{R})},$$

with C independent of k. If $2k^{1/3}R \le 1$ we have $I_2 = 0$ and the proof is complete. We assume henceforth that $2k^{1/3}R > 1$ and let

$$b(\zeta) = (1-\psi(\zeta))\psi\left(\frac{\zeta}{k^{1/3}R}\right)\tilde{r}_k(\zeta),$$

the support of which lies in the compact set $K = \{\zeta \in \mathbb{R} \mid 1 \le |\zeta| \le 2k^{1/3}R\}$. Then for $\zeta \in K$, by (40),

$$|\partial_\zeta\phi_a(\tilde{\delta},\zeta,\tilde{\eta})| = |\zeta^2 + k^{2/3}(2\eta_0\eta-\delta)| \ge |\zeta^2| - k^{2/3}(2|\eta|+|\delta|) \ge |\zeta^2| - \frac{\xi_0^2}{2} \ge \frac{1}{2}|\zeta^2|.$$

Thus, since ϕ_a has no stationary points on K we can use Lemma B.1 with $a(\zeta) = b(\zeta)\zeta^p$ and D an open bounded set containing K. This gives

$$I_2 = (ik)^{-n}\sum_{\ell=0}^{n}\int_K \left(\frac{d^\ell}{d\zeta^\ell}b(\zeta)\zeta^p\right)\frac{u_{\ell,n}(\zeta)}{\tilde{\phi}_\zeta(\tilde{\delta},\zeta,\tilde{\eta})^n}e^{ik\phi_a(\tilde{\delta},\zeta,\tilde{\eta})}d\zeta,$$

where $u_{\ell,n} \in \mathcal{U}_{n-\ell}$, with $\mathcal{U}_p$ defined in (36). This expression is now estimated in the same way as (39) above. Since $|\tilde{\delta}| \le \xi_0^2/2 \le 1/2$ and K satsfies the assumptions of Lemma 5.4 we obtain a uniform bound for $u_{\ell,n}$ on K. Then

$$\begin{aligned}|I_2| &\le Ck^{-n}\sum_{\ell=0}^{n}\sum_{j=0}^{\min(\ell,p)}||b^{(\ell-j)}||_{L^\infty(\mathbb{R})}\int_1^{2k^{1/3}R}\frac{d\zeta}{|\zeta|^{2n-p+j}} \le Ck^{-n}||b||_{W^{n,\infty}(\mathbb{R})}\int_1^{2k^{1/3}R}\frac{d\zeta}{|\zeta|^{2n-p}}\\
&\le Ck^{-n}||b||_{W^{n,\infty}(\mathbb{R})}\max(1,(2k^{1/3}R)^{p-2n})2k^{1/3}R \le C||b||_{W^{n,\infty}(\mathbb{R})}\max(k^{1/3-n},k^{(p-5n+1)/3}R^{p-2n})R_1\\
&\le C\max(1,R_0^{p-2n},R_1^{p-2n})R_1||b||_{W^{n,\infty}(\mathbb{R})} =: \tilde{C}||b||_{W^{n,\infty}(\mathbb{R})},\end{aligned}$$

since $p+1 \le 5n$ and $k \ge 1$. Moreover, as in the proof of Lemma 6.1, by (37) and (38), since $k^{1/3}R > 1/2$,

$$|I_2| \le \tilde{C}||b||_{W^{n,\infty}(\mathbb{R})} \le \tilde{C}C_n^2 2^n||1-\psi||_{W^{n,\infty}(\mathbb{R})}||\psi||_{W^{n,\infty}(\mathbb{R})}||\tilde{r}_k||_{W^{n,\infty}(\mathbb{R})} \le C||\tilde{r}_k||_{W^{n,\infty}(\mathbb{R})}.$$

Together the estimates of I_1 and I_2 then prove the lemma. We finally note that since $r \in W^{n,\infty}(\mathbb{R})$ the norm $||\tilde{r}_k||_{W^{n,\infty}(\mathbb{R})}$ is bounded because of (38) and (40). □

Finally, we show that the derivatives of the Airy function are well approximated by an oscillatory integral with a monomial factor and the Airy phase.

Lemma 6.3. *Let*

$$I = k^{1/3}\int \psi(\theta/R)\theta^p e^{ik\phi_a(\delta,\theta,\eta)}d\theta,$$

where $c_0 \le R$ with c_0 as in Lemma 5.3 and $|\delta| \le 1$. If $c_0(1+|\eta|^{1/2}) \le R$ and $n \ge 1+p/2$, there is a constant C_n, independent of k, R, δ and η, such that

$$\left|I - \frac{2\pi i^p}{k^{p/3}}\mathrm{Ai}^{(p)}(k^{2/3}(2\eta_0\eta-\delta))\right| \le C_n k^{1/3-n}.$$

Proof. The Fourier transform of $\mathrm{Ai}^{(p)}$ in $\mathcal{S}'(\mathbb{R})$ is

$$\mathcal{F}(\mathrm{Ai}^{(p)})(\zeta) = \frac{1}{\sqrt{2\pi}}(i\zeta)^p e^{i\frac{\zeta^3}{3}}.$$

Therefore, using Lemma 2.1, and noting that $\mathcal{F}^{-1}(\cdot)(\rho) = \mathcal{F}(\cdot)(-\rho)$,

$$\mathrm{Ai}^{(p)}(\rho) = \mathcal{F}^{-1}\left(\frac{1}{\sqrt{2\pi}}(i\zeta)^p e^{i\frac{\zeta^3}{3}}\right)(\rho) = \lim_{t\to\infty}\frac{1}{2\pi}\int \psi(\zeta/t)(i\zeta)^p e^{i\left(\frac{\zeta^3}{3}+\zeta\rho\right)}d\zeta.$$

After rescaling $\theta = k^{-1/3}\zeta$ we get

$$\int \psi(\zeta/t)(i\zeta)^p e^{i\left(\frac{\zeta^3}{3}+\zeta\rho\right)}d\zeta = i^p k^{\frac{p+1}{3}}\int \psi(\theta k^{1/3}/t)\theta^p e^{ik\left(\frac{\theta^3}{3}+k^{-2/3}\theta\rho\right)}d\theta.$$

It follows that if

$$\rho = k^{2/3}(2\eta_0\eta - \delta), \tag{41}$$

then

$$\begin{aligned}\frac{2\pi}{(ik^{1/3})^p}\mathrm{Ai}^{(p)}(k^{2/3}(2\eta_0\eta-\delta)) &= k^{1/3}\lim_{t\to\infty}\int \psi(\theta k^{1/3}/t)\theta^p e^{ik\left(\frac{\theta^3}{3}+k^{-2/3}\theta\rho\right)}d\theta \\ &= k^{1/3}\lim_{t\to\infty}\int \psi(\theta/t)\theta^p e^{-ik\phi_a(\delta,\theta,\eta)}d\theta \\ &= k^{1/3}(-1)^p\lim_{t\to\infty}\int \psi(-\theta/t)\theta^p e^{-ik\phi_a(\delta,-\theta,\eta)}d\theta.\end{aligned}$$

Since ϕ_a is odd and ψ is even in θ we obtain

$$\frac{2\pi i^p}{k^{p/3}}\mathrm{Ai}^{(p)}(k^{2/3}(2\eta_0\eta-\delta)) = I + k^{1/3}\lim_{t\to\infty}\int(1-\psi(\theta/R))\psi(\theta/t)\theta^p e^{ik\phi_a(\delta,\theta,\eta)}d\theta,$$

The result now follows from Lemma 6.1, with $r \equiv 1$. □

7. Solution estimates

In Sections 3 and 4 it was shown that the partial Fourier transform in y for both the exact solution and the Gaussian beam approximation can be written on the form

$$\hat{u}(x,\eta) = \hat{v}(x,\eta;k)\hat{A}(\eta),$$

where A is the amplitude function and $\hat{v}$ for the two cases are given in (10) and (25). In this section we prove bounds of those $\hat{v}$ in terms of k and η, which are valid for all $\eta \in \mathbb{R}$, $x \in [0, x_c]$ and $k \geq 1$. We start with the Gaussian beam superposition case and estimate $\hat{v}_{GB}$ as follows.

Lemma 7.1. *For $\hat{v}_{GB}$ defined in* (25)*, there is a constant M such that*

$$|\hat{v}_{GB}(\eta,x,k)| \leq M\left(1+\log\left(1+|\eta|^{1/2}\right)\right)k^{1/2}, \tag{42}$$

for all $\eta \in \mathbb{R}$, $x \in [0,x_c]$ and $k \geq 1$.

Proof. From (24) in Section 4.2 we have

$$I_t = k^{1/3}\int \psi(\theta/t)r(\theta)e^{ik\phi_g(\delta,\theta,\eta)}d\theta, \qquad r(\theta) = \frac{1}{2\pi}\sqrt{\frac{q(\xi_0)}{q(\xi_0+\theta)}}.$$

We note that $r \in W^{n,\infty}(\mathbb{R})$ for all n by Lemma 5.1 and that $|\delta| = |x_c - x| \leq |x_c| = \xi_0^2 \leq 1$. We divide the integral into two parts. Let $R = R(\eta) = c_0\left(1+|\eta|^{1/2}\right)$ with c_0 as in Lemma 5.3 and define

$$I_t = k^{1/3}\int \psi(\theta/R)\psi(\theta/t)r(\theta)e^{ik\phi_g(\delta,\theta,\eta)}d\theta + k^{1/3}\int(1-\psi(\theta/R))\psi(\theta/t)r(\theta)e^{ik\phi_g(\delta,\theta,\eta)}d\theta =: I_{1,t}+I_{2,t}.$$

For $I_{1,t}$ we have, again using Lemma 5.1, and that fact that $\Im\phi_g \geq 0$ by Lemma 5.3,

$$|I_{1,t}| \leq k^{1/3}\sqrt{\frac{q_1}{q_0}}\int_{-2R(\eta)}^{2R(\eta)} \frac{e^{-k\Im\phi_g(x,\theta,\eta)}}{\sqrt{1+\theta^2}}d\theta \leq k^{1/3}\sqrt{\frac{q_1}{q_0}}\int_{-2R(\eta)}^{2R(\eta)} \frac{d\theta}{\sqrt{1+\theta^2}}$$
$$\leq Ck^{1/3}\log(R(\eta)) \leq C'k^{1/3}\log(1+|\eta|^{1/2}).$$

For $I_{2,t}$ we use Lemma 6.1 with $p=0$, which says that for any $n \geq 1$ and $t > R(\eta)$,

$$|I_{2,t}| \leq C_n k^{1/3-n}||r||_{W^{n,\infty}(\mathbb{R})} \leq C_n' k^{1/3-n},$$

where C_n' is independent of k, $R(\eta)$, δ, η and t. By (23) we have $|P_{GB}| = 2(\pi\xi_0)^{1/2} \leq 2\sqrt{\pi}$ for all η and k. Then we get

$$|\hat{v}_{GB}(\eta,x,k)| \leq \lim_{t\to\infty} k^{1/6}|P_{GB}(x,\eta)|(|I_{1,t}|+|I_{2,t}|) \leq 2\sqrt{\pi}k^{1/6}\Big[C'k^{1/3}\log(1+|\eta|^{1/2}) + C_n'k^{1/3-n}\Big],$$

and upon taking $n=1$ the Lemma follows with $M = 2\sqrt{\pi}\max(C', C_1')$. □

Next, for the exact solution, we estimate $\hat{v}$.

Lemma 7.2. *For $\hat{v}$ defined in* (10) *there is a constant M such that*

$$|\hat{v}(\eta,x,k)| \leq Mk^{1/6},$$

for all $\eta \in \mathbb{R}$, $x \in [0,x_c]$ and $k \geq 1$.

Proof. From (10) we have

$$\hat{v}(\eta,x,k) = k^{1/6}P(k,\eta)\mathrm{Ai}(k^{\frac{2}{3}}(x-X)), \qquad P(k,\eta) = \frac{\bar{\alpha}k^{-1/6}}{\mathrm{Ai}(\alpha k^{2/3}X)}, \qquad X(\eta) = 1-(\eta_0+\eta)^2.$$

Then, using (68) and (70) in Lemma 9.1,

$$|\hat{v}(\eta,x,k)| = \frac{|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))|}{|\mathrm{Ai}(\alpha k^{\frac{2}{3}}X)|} \leq C\frac{(1+k^{\frac{2}{3}}|x-X|)^{-1/4}}{(1+k^{\frac{2}{3}}|X|)^{-1/4}} \leq C\left(\frac{1+k^{\frac{2}{3}}(|x|+|x-X|)}{1+k^{\frac{2}{3}}|x-X|}\right)^{1/4}$$
$$= C\left(1+\frac{k^{\frac{2}{3}}|x|}{1+k^{\frac{2}{3}}|x-X|}\right)^{1/4} \leq C\left(1+|x_c|k^{\frac{2}{3}}\right)^{1/4} \leq C(1+k^{1/6}).$$

The result follows as $k \geq 1$. (The estimate is sharp for $x = X$.) □

8. Proof of the Main Result

In this section we prove the main result Theorem 1.1 estimating the L^∞ error between the exact solution and the Gaussian beam solution. To estimate the difference between $u_{GB}(x,y)$ and the exact solution $u(x,y)$, it is enough to control the L^1 norm of the difference between their scaled Fourier transforms since

$$||u_{GB}(x,\cdot) - u(x,\cdot)||_{L^\infty} \leq \sqrt{\frac{k}{2\pi}}||\hat{u}_{GB}(x,\cdot) - \hat{u}(x,\cdot)||_{L^1}. \tag{43}$$

We will use this strategy. From (25) and (10) we get

$$\hat{u}_{GB}(x,\eta_0+\eta) - \hat{u}(x,\eta_0+\eta) = \big(\hat{v}_{GB}(\eta,x,k) - \hat{v}(\eta,x,k)\big)\hat{A}(\eta).$$

We divide the expression into two parts, one for $|\eta|$ smaller than $O(k^{-2/3})$ and one for $|\eta|$ larger than $O(k^{-2/3})$. Thus, for c to be determined below, we let

$$\sqrt{\frac{k}{2\pi}}||\hat{u}_{GB}(x,\cdot)-\hat{u}(x,\cdot)||_{L^1} \leq \sqrt{\frac{k}{2\pi}}\int_{|\eta|\leq ck^{-2/3}} |\hat{v}_{GB}(x,\eta)-\hat{v}(x,\eta)||\hat{A}(\eta)|d\eta$$
$$+\sqrt{\frac{k}{2\pi}}\int_{|\eta|\geq ck^{-2/3}} |\hat{v}_{GB}(x,\eta)-\hat{v}(x,\eta)||\hat{A}(\eta)|d\eta =: E_1+E_2. \tag{44}$$

For the large values of $|\eta|$ we can immediately get a bound of $O(k^{-1})$ by using the fact that $\hat{A}$ has very rapid decay, being the Fourier transform of $A \in \mathcal{S}$. This is used in the following lemma.

Lemma 8.1. *Suppose $f(\cdot,k)\in L^1_{\rm loc}(\mathbb{R})$ for each $k\geq 1$ and $A\in\mathcal{S}(\mathbb{R})$. Let $c,\beta>0$ be given. If there exist $r\in\mathbb{R}$ and $q,M\in\mathbb{R}_+$ such that,*

$$|f(\eta,k)|\leq M(1+|k\eta|^q)k^r,\qquad \forall k\geq 1,\quad |k\eta|\geq ck^\beta, \tag{45}$$

then, for each $p\geq 0$, there exists a constant C_p, independent of k, but dependent on A,M,q,r,c, such that

$$k^{1/2}\int_{|k\eta|\geq ck^\beta}\left|f(\eta,k)\hat{A}(\eta)\right|d\eta\leq C_pk^{-p}.$$

Proof. Since $A\in\mathcal{S}(\mathbb{R})$ for all $\ell\geq 0$ there exists c_ℓ such that $|\mathcal{F}(A)(\eta)|\leq c_\ell/(1+|\eta|^\ell)$. Hence,

$$|\hat{A}(\eta)|=|\sqrt{k}\mathcal{F}(A)(k\eta)|\leq\frac{c_\ell\sqrt{k}}{1+|k\eta|^\ell},\qquad\forall\eta.$$

Then for $\ell>q+1$, with $\xi=k\eta$,

$$\begin{aligned}k^{1/2}\int_{|k\eta|\geq ck^\beta}\left|f(\eta,k)\hat{A}(\eta)\right|d\eta&\leq c_\ell k\int_{|k\eta|\geq ck^\beta}\frac{|f(\eta,k)|}{1+|k\eta|^\ell}d\eta\leq c_\ell Mk^{r+1}\int_{|k\eta|\geq ck^\beta}\frac{1+|k\eta|^q}{1+|k\eta|^\ell}d\eta\\&=c_\ell Mk^r\int_{|\xi|\geq ck^\beta}\frac{1+|\xi|^q}{1+|\xi|^\ell}d\xi\leq c_\ell Mk^r\int_{|\xi|\geq ck^\beta}\frac{1+|\xi|^q}{|\xi|^\ell}d\xi\\&=2Mc_\ell\left(\frac{c^{1-\ell}}{\ell-1}k^{r+\beta(1-\ell)}+\frac{c^{1-\ell+q}}{\ell-1-q}k^{r+\beta(1-\ell+q)}\right).\end{aligned}$$

For the given p we now take $\ell=\ell_p:=\max((r+p)/\beta+q+1,q+2)$. Then $r+\beta(1-\ell+q)\leq -p$ and

$$\begin{aligned}k^{1/2}\int_{|k\eta|\geq ck^\beta}\left|f(\eta,k)\hat{A}(\eta)\right|d\eta&\leq 2Mc_{\ell_p}\left(\frac{c^{1-\ell_p}}{\ell_p-1}k^{-p-\beta q}+\frac{c^{1-\ell_p+q}}{\ell_p-1-q}k^{-p}\right)\\&\leq\underbrace{2Mc_{\ell_p}\left(\frac{c^{1-\ell_p}}{\ell_p-1}+\frac{c^{1-\ell_p+q}}{\ell_p-1-q}\right)}_{=:C_p}k^{-p},\end{aligned}$$

which is the desired estimate. ∎

In our case we let $f=(\hat{v}_{GB}-\hat{v})/\sqrt{2\pi}$ for fixed x. Then it follows from Lemma 7.1 and Lemma 7.2 that f satisfies (45), as for $k\geq 1$,

$$|\hat{v}_{GB}(\eta,x,k)-\hat{v}(\eta,x,k)|\leq Mk^{1/6}+M(1+\log(1+|\eta|^{1/2}))k^{1/2}\leq M'(1+|\eta|^{1/2})k^{1/2}\leq M'(1+|k\eta|^{1/2})k^{1/2},$$

where M' is in fact independent also of $x\in[0,x_c]$. Lemma 8.1 with $\beta=1/3$, $q=1/2$, $r=1/2$, $c=\xi_0^2/4$ and $p=1$ now shows that

$$|E_2|\leq Ck^{-1}. \tag{46}$$

The choice of c will be motivated below in the next step.

To estimate E_1 we will use more precise estimates of $|\hat{v}_{GB}(\eta,x,k)-\hat{v}(\eta,x,k)|$ for small $|\eta|$ and the following lemma.

Lemma 8.2. *Suppose $f(\cdot,k)\in L^1_{\rm loc}(\mathbb{R})$ for each $k\geq 1$ and $A\in\mathcal{S}(\mathbb{R})$. Let $c,\beta>0$ be given. If there exist $r\in\mathbb{R}$ and $q,M\in\mathbb{R}_+$ such that,*

$$|f(\eta,k)|\leq M(1+|k\eta|^q)k^r,\qquad\forall k\geq 1,\quad |k\eta|\leq ck^\beta, \tag{47}$$

then there exists a constant C, independent of k, but dependent on A,M,q,r,c, such that

$$k^{1/2}\int_{|k\eta|\leq ck^\beta}\left|f(\eta,k)\hat{A}(\eta)\right|d\eta\leq Ck^r.$$

Proof. As in the proof of Lemma 8.1, when $\ell>q+1$ we get

$$k^{1/2}\int_{|k\eta|\leq ck^\beta}\left|f(\eta,k)\hat{A}(\eta)\right|d\eta\leq c_\ell Mk^r\int_{|\xi|\leq ck^\beta}\frac{1+|\xi|^q}{1+|\xi|^\ell}d\xi\leq\underbrace{c_\ell M\int\frac{1+|\xi|^q}{1+|\xi|^\ell}d\xi}_{=:C}\,k^r.$$

This proves the lemma. ∎

As for E_2 we apply this lemma with $f=(\hat{v}_{GB}-\hat{v})/\sqrt{2\pi}$ and to get the bound (47) we need to estimate the difference between $\hat{v}_{GB}$ and $\hat{v}$ when $|k\eta|\le ck^{1/3}$. This estimate is the main part of the proof.

To examine $\hat{v}_{GB}-\hat{v}$ more carefully we first recall the expressions:

$$\begin{aligned}\hat{v}_{GB}(\eta,x,k)&=k^{1/6}P_{GB}(k,\eta)I(\eta,x,k),\\ \hat{v}(\eta,x,k)&=k^{1/6}P(k,\eta)\mathrm{Ai}(k^{\frac{2}{3}}(x-X(\eta))),\end{aligned}$$

where

$$I(\eta,x,k)=\lim_{t\to\infty}\frac{k^{1/3}\sqrt{q(\xi_0)}}{2\pi}\int\psi(\theta/t)\frac{e^{ik\phi_g(x,\theta,\eta)}}{\sqrt{q(\xi_0+\theta)}}d\theta\qquad P_{GB}(k,\eta)=2(\pi\xi_0)^{1/2}e^{-i\pi/4}e^{\frac{2}{3}ik\xi_0^3-2ik\eta_0\xi_0\eta},$$

and

$$X(\eta):=1-(\eta+\eta_0)^2,\qquad P(k,\eta)=\frac{\bar{\alpha}k^{-1/6}}{\mathrm{Ai}(\alpha k^{2/3}X)}. \tag{48}$$

We divide the difference $\hat{v}_{GB}(\eta,x,k)-\hat{v}(\eta,x,k)$ into three parts

$$\begin{aligned}\hat{v}(\eta,x,k)-\hat{v}_{GB}(\eta,x,k)&=k^{1/6}P_{GB}(k,\eta)[\mathrm{Ai}(k^{\frac{2}{3}}(x-X))-\mathrm{Ai}(k^{\frac{2}{3}}(x-X-\eta^2))]\\&\quad+k^{1/6}[P(k,\eta)-P_{GB}(k,\eta)]\mathrm{Ai}(k^{\frac{2}{3}}(x-X))\\&\quad+k^{1/6}P_{GB}(k,\eta)[\mathrm{Ai}(k^{\frac{2}{3}}(x-X-\eta^2))-I(\eta,x,k)]\\&=:R_1+R_2+R_3.\end{aligned}$$

In three Lemmas below we show that when $x=x_c$ and $\eta\le\xi_0^2k^{-2/3}/4$, there is a constant M such that

$$|R_1|\le M|k\eta|^2k^{-1},\qquad |R_2|\le M(1+|k\eta|^2)k^{-5/6},\qquad |R_3|\le M(1+|k\eta|^2)k^{-5/6}.$$

It follows that

$$|\hat{v}(\eta,x_c,k)-\hat{v}_{GB}(\eta,x_c,k)|\le|R_1+R_2+R_3|\le 3M(1+|k\eta|^2)k^{-5/6},\qquad\text{when } |k\eta|\le\frac{\xi_0^2}{4}k^{1/3}\text{ and } k\ge1,$$

Then applying Lemma 8.2 with $\beta=1/3$, $q=2$, $r=-5/6$ and $c=\xi_0^2/4$ shows that

$$|E_1|\le Ck^{-5/6},$$

when $x=x_c$. Together with (43), (44) and (46) this proves Theorem 1.1.

Note that the estimates of R_1 and R_2 above are shown to be valid for all $x\in[0,x_c]$, while the R_3 estimate is considered, in this paper, only for $x=x_c$. Furtheremore, note that R_2 and R_3 exhibit the same loss of decay through the term $k^{-5/6}$. In R_2 this comes from the estimate (52) and R_3 has $k^{1/6}$ embedded in (54).

We now turn to proving the lemmas about R_j.

8.1. **Estimate of R_1.**

Lemma 8.3. *There is a constant M independent of η and $k\ge1$, such that*

$$|R_1|\le Mk\eta^2,\qquad \textit{when } |\eta|\le1.$$

Proof. Since $|P_{GB}|\le2(\pi\xi_0)^{1/2}$ we have

$$|R_1|\le2\sqrt{\pi}k^{1/6}|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))-\mathrm{Ai}(k^{\frac{2}{3}}(x-X-\eta^2))|.$$

Moreover, from (69) in Lemma 9.1 we get

$$\begin{aligned}|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))-\mathrm{Ai}(k^{\frac{2}{3}}(x-X-\eta^2))|&=k^{\frac{2}{3}}\left|\int_0^{\eta^2}\mathrm{Ai}'(k^{\frac{2}{3}}(x-X-s))ds\right|\\&\le C_3k^{\frac{2}{3}}\eta^2\max_{0\le s\le\eta^2}(1+|k^{\frac{2}{3}}(x-X-s)|)^{1/4}.\end{aligned}$$

Then, since

$$|X|\le(|\eta|+|\eta_0|)^2-1\le3,\qquad |x|\le1,\qquad s\le\eta^2\le1,$$

we obtain

$$|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))-\mathrm{Ai}(k^{\frac{2}{3}}(x-X-\eta^2))|\leq C_3k^{\frac{2}{3}}\eta^2(1+5k^{\frac{2}{3}})^{1/4}\leq C_3 6^{1/4}k^{5/6}\eta^2.$$

It follows that $|R_1|\leq Mk\eta^2$ where $M=2\sqrt{\pi}6^{1/4}C_3$ with C_3 being the constant in (69). □

8.2. **Estimate of R_2.**

Lemma 8.4. *There is a constant M dependent on x, but independent of η and $k\geq 1$, such that*

$$|R_2|\leq M(1+k^2\eta^2)k^r,\qquad \text{when } |\eta|\leq \frac{\xi_0^2}{4}k^{-2/3},\qquad r=\begin{cases}-5/6, & x=x_c,\\ -1, & 0\leq x<x_c.\end{cases}$$

Proof. We start by proving two estimates of $X(\eta)$. We use the inequalities $1-x\leq\sqrt{1-x}\leq 1-x/2$ which hold for $x\in[0,1]$. The definition (48) together with the assumption on η and the fact that $k\geq 1$, then gives

$$X(\eta)=1-(\eta+\eta_0)^2\geq 1-\left(\frac{\xi_0^2}{4}+\sqrt{1-\xi_0^2}\right)^2\geq 1-\left(1-\frac{\xi_0^2}{4}\right)^2\geq\frac{\xi_0^2}{4}=:X_0>0.$$

Moreover, since $x_c=\xi_0^2$,

$$|X(\eta)-x_c|=|\eta||2\eta_0-\eta|\leq|\eta|\left|2\sqrt{1-\xi_0^2}+\frac{\xi_0^2}{4}\right|\leq|\eta|\left|2-\xi_0^2+\frac{\xi_0^2}{4}\right|\leq 2|\eta|.$$

Clearly, we also have $X(\eta)\leq 1$, and therefore, in summary,

$$0<X_0\leq X(\eta)\leq 1,\qquad |X(\eta)-x_c|\leq 2|\eta|. \tag{49}$$

Next, we rewrite P_{GB}, adopting the the defintion

$$\widetilde{\mathrm{Ai}}(z)=\frac{1}{2\sqrt{\pi}}z^{-\frac{1}{4}}e^{-\frac{2}{3}z^{\frac{3}{2}}}, \tag{50}$$

from Lemma 9.1. Then for $x\in\mathbb{R}$,

$$\widetilde{\mathrm{Ai}}(\alpha k^{2/3}x)=\frac{k^{-1/6}}{2\sqrt{\pi}}e^{-i\pi/12}x^{-\frac{1}{4}}e^{-\frac{2}{3}ikx^{3/2}}\quad\Rightarrow\quad\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)=\widetilde{\mathrm{Ai}}(\alpha k^{2/3}x_c)\left(\frac{x_c}{X}\right)^{1/4}e^{-\frac{2}{3}ik(X^{3/2}-x_c^{3/2})},$$

and

$$P_{GB}=\frac{\bar\alpha k^{-1/6}}{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}x_c)}e^{-2ik\eta\eta_0\sqrt{x_c}}=\frac{\bar\alpha k^{-1/6}}{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)}\left(\frac{x_c}{X}\right)^{1/4}e^{-\frac{2}{3}ik(X^{3/2}-x_c^{3/2})-2ik\eta\eta_0\sqrt{x_c}}=:\frac{\bar\alpha k^{-1/6}g(\eta)}{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)}.$$

We get

$$\begin{aligned}k^{1/6}|P-P_{GB}|&=\left|\frac{1}{\mathrm{Ai}(\alpha k^{2/3}X)}-\frac{g(\eta)}{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)}\right|\leq\left|\frac{1}{\mathrm{Ai}(\alpha k^{2/3}X)}-\frac{1}{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)}\right|+\left|\frac{1-g(\eta)}{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)}\right|\\&=\frac{1}{|\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)|}\left(\left|\frac{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)-\mathrm{Ai}(\alpha k^{2/3}X)}{\mathrm{Ai}(\alpha k^{2/3}X)}\right|+|1-g(\eta)|\right).\end{aligned}$$

We can then estimate R_2 as

$$|R_2|\leq\frac{|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))|}{|\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)|}\left(\left|\frac{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)-\mathrm{Ai}(\alpha k^{2/3}X)}{\mathrm{Ai}(\alpha k^{2/3}X)}\right|+|1-g(\eta)|\right). \tag{51}$$

We will now study the different parts of this expression separately.

- **Estimate of $\left|\frac{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)-\mathrm{Ai}(\alpha k^{2/3}X)}{\mathrm{Ai}(\alpha k^{2/3}X)}\right|$.**
  This is given directly by (67) in Lemma 9.1 with $s_0=X_0$, as then $k^{2/3}X(\eta)\geq k^{2/3}X_0\geq s_0$. We get

  $$\left|\frac{\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)-\mathrm{Ai}(\alpha k^{2/3}X)}{\mathrm{Ai}(\alpha k^{2/3}X)}\right|\leq C_1|k^{2/3}X|^{-3/2}\leq C_1X_0^{-3/2}k^{-1}=:D_1k^{-1},$$

  where C_1 is the constant in (67).

- **Estimate of $|1-g(\eta)|$.**
Using Taylor's formula for $x \mapsto x^{3/2}$ around $x = X(0) = 1-\eta_0^2 = \xi_0^2 = x_c$, we compute

$$X(\eta)^{3/2} = X(0)^{3/2} + \frac{3}{2}X(0)^{1/2}(X(\eta)-X(0)) + R(X(\eta)-X(0))^2,$$
$$= x_c^{3/2} + \frac{3}{2}\sqrt{x_c}(2\eta\eta_0 - \eta^2) + R(X(\eta)-X(0))^2, \qquad |R| \le \sup_{\xi \ge X_0} \frac{3}{8}\xi^{-1/2} = \frac{3}{8\sqrt{X_0}}.$$

Therefore,

$$\frac{2}{3}(X^{3/2} - x_c^{3/2}) + 2\eta\eta_0\sqrt{x_c} = -\sqrt{x_c}\eta^2 + \frac{2}{3}R(X(\eta)-x_c)^2,$$

and consequently, by (49),

$$\left|\frac{2}{3}(X^{3/2} - x_c^{3/2}) + 2\eta\eta_0\sqrt{x_c}\right| \le \left(|\eta|^2 + \frac{|X(\eta)-x_c|^2}{4\sqrt{X_0}}\right) \le \left(|\eta|^2 + \frac{|\eta|^2}{\sqrt{X_0}}\right) =: D_2\eta^2.$$

This gives us

$$\begin{aligned}|1-g(\eta)| &\le \left|1 - e^{-ik\left(\frac{2}{3}(X^{3/2}-x_c^{3/2})+2\eta\eta_0\sqrt{x_c}\right)}\right| + \left|\left(\frac{x_c}{X(\eta)}\right)^{1/4} - 1\right| \\ &\le k\left|\frac{2}{3}(X^{3/2} - x_c^{3/2}) + 2\eta\eta_0\sqrt{x_c}\right| + \left|\left(1+\frac{2\eta}{X_0}\right)^{1/4} - 1\right| \\ &\le D_2 k\eta^2 + \frac{1}{2X_0}\eta \le D_3(1+k^2\eta^2)k^{-1},\end{aligned}$$

for $D_3 = \max(D_2, (2X_0)^{-1})$.
- **Estimate of $\frac{|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))|}{|\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)|}$.**
We divide this into three subcases. Suppose first that

$$|x-x_c| > |X(\eta)-x_c|.$$

Then by (49),

$$\begin{aligned}k^{2/3}|x-X(\eta)| &\ge k^{2/3}|x-x_c| - k^{2/3}|X(\eta)-x_c| \ge k^{2/3}|x-x_c| - 2k^{2/3}|\eta| \\ &\ge k^{2/3}|x-x_c| - \frac{\xi_0^2}{2}.\end{aligned}$$

By (68) in Lemma 9.1 and (50),

$$\begin{aligned}\frac{|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))|}{|\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)|} &\le 2\sqrt{\pi}k^{1/6}|X|^{1/4}C_2(1+|k^{\frac{2}{3}}(x-X)|)^{-1/4} \\ &\le 2\sqrt{\pi}k^{1/6}C_2(1-\xi_0^2/2+k^{\frac{2}{3}}|x-x_c|)^{-1/4} \le 2\sqrt{\pi}C_2|x-x_c|^{-1/4},\end{aligned}$$

where C_2 is the constant in (68). On the other hand, if

$$|X(\eta)-x_c| \ge |x-x_c| > 0,$$

then by (49),

$$k^{2/3} \le \frac{\xi_0^2}{4|\eta|} \le \frac{\xi_0^2}{2|X(\eta)-x_c|} \le \frac{\xi_0^2}{2|x-x_c|},$$

and we obtain the same estimate as above, via

$$\begin{aligned}\frac{|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))|}{|\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)|} &\le 2\sqrt{\pi}k^{1/6}|X|^{1/4}C_2(1+|k^{\frac{2}{3}}(x-X)|)^{-1/4} \le 2\sqrt{\pi}k^{1/6}C_2 \\ &\le 2\sqrt{\pi}C_2\left(\frac{\xi_0^2}{2|x-x_c|}\right)^{1/4} \le 2\sqrt{\pi}C_2|x-x_c|^{-1/4}.\end{aligned}$$

Finally, when $x = x_c$ (caustic case) we can not get better than

$$\frac{|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))|}{|\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)|} \leq 2\sqrt{\pi}k^{1/6}|X|^{1/4}C_2(1+|k^{\frac{2}{3}}(x-X)|)^{-1/4} \leq 2\sqrt{\pi}C_2k^{1/6}.$$

In summary, we have with $D_4 = 2\sqrt{\pi}C_2$,

$$\frac{|\mathrm{Ai}(k^{\frac{2}{3}}(x-X))|}{|\widetilde{\mathrm{Ai}}(\alpha k^{2/3}X)|} \leq D_4 \begin{cases} k^{1/6}, & x = x_c, \\ |x-x_c|^{-1/4}, & x < x_c. \end{cases} \tag{52}$$

We can now put the estimates together and apply them to R_2 in (51). We get

$$|R_2| \leq \left(D_1k^{-1} + D_3(1+k^2\eta^2)k^{-1}\right) D_4 \begin{cases} k^{1/6}, & x = x_c, \\ |x-x_c|^{-1/4}, & x < x_c, \end{cases} \leq M(1+k^2\eta^2)k^r,$$

where

$$M = \max(D_1, D_3)D_4 \begin{cases} 1, & x = x_c, \\ |x-x_c|^{-1/4}, & x < x_c. \end{cases}$$

This proves the lemma. □

8.3. **Estimate of R_3 at the caustic.** This is the main estimate. Here we assume that $x = x_c$.

Lemma 8.5. *For $x = x_c$ there is a constant M independent of η and $k \geq 1$, such that*

$$|R_3| \leq M(1+k^2\eta^2)k^{-5/6}, \qquad \text{when } |\eta| \leq \frac{\xi_0^2}{4}k^{-2/3}.$$

Proof. We consider $\rho = 2k^{2/3}\eta_0\eta$, which amounts to taking $\delta = 0$ in (41). By the assumption on η and k it is bounded as

$$|\rho| \leq 2k^{2/3}|\eta| \leq \xi_0^2/2 \leq \frac{1}{2}. \tag{53}$$

Moreover, since $k^{2/3}(x_c - X - \eta^2) = \rho$ and as before, $|P_{GB}| \leq 2(\pi\xi_0)^{1/2}$, we get

$$|R_3| = k^{1/6}|P_{GB}(k,\eta)| \cdot |\mathrm{Ai}(k^{\frac{2}{3}}(x_c - X - \eta^2)) - I(\eta, x_c, k)| \leq Ck^{1/6}|\mathrm{Ai}(\rho) - I(\eta, x_c, k)|. \tag{54}$$

Hence, we need to estimate $|\mathrm{Ai}(\rho) - I|$.

Let

$$r(\theta) = \frac{1}{2\pi}\sqrt{\frac{q(\xi_0)}{q(\xi_0+\theta)}}. \tag{55}$$

As in the proof of Lemma 7.1 we then use the fact that $I = \lim_{t\to\infty} I_t$ where I_t is defined and divided as

$$I_t = k^{1/3}\int \psi(\theta/t)r(\theta)e^{ik\phi_g(0,\theta,\eta)}d\theta = k^{1/3}\int \psi(\theta/R)\psi(\theta/t)r(\theta)e^{ik\phi_g(0,\theta,\eta)}d\theta$$
$$+ k^{1/3}\int (1-\psi(\theta/R))\psi(\theta/t)r(\theta)e^{ik\phi_g(0,\theta,\eta)}d\theta =: I_{\mathrm{main}} + I_{t,\mathrm{tail}}.$$

With c_0 as in Lemma 5.3 we choose here $R = 3c_0/2$, independent of η, which implies that for all η which we consider,

$$c_0(1+|\eta|^{1/2}) \leq c_0(1+\xi_0/2) \leq \frac{3}{2}c_0 = R. \tag{56}$$

Moreover, we take $t \geq 2R = 3c_0$ such that $\psi(\theta/R)\psi(\theta/t) = \psi(\theta/R)$. To analyze I_{main} we then first note that it can be written as

$$I_{\mathrm{main}} = k^{1/3}\int \psi(\theta/R)\tilde{r}(\theta)e^{ik\phi_a(0,\theta,\eta)}d\theta, \qquad \tilde{r}(\theta) = r(\theta)e^{ik\frac{1}{2}m_{11}(\xi_0+\theta)\theta^4}.$$

We next expand $\tilde{r}$ in terms of θ, first using the Taylor expansion of $\exp(iz)$,

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2} + (iz)^3 Z(z), \qquad Z(z) = \frac{1}{2i}\int_0^1 e^{isz}(1-s)^2ds.$$

This gives

$$\tilde{r}(\theta) = r(\theta)\left(1 + \frac{ik}{2}m_{11}(\xi_0+\theta)\theta^4 + \frac{(ik)^2}{8}m_{11}(\xi_0+\theta)^2\theta^8\right) + \frac{(ik)^3}{8}r(\theta)Z\left(\frac{k}{2}m_{11}(\xi_0+\theta)\theta^4\right)m_{11}(\xi_0+\theta)^3\theta^{12}.$$

Furthermore, let

$$v_\ell(\theta) := r(\theta)m_{11}(\xi_0+\theta)^\ell,$$

and Taylor expand these functions as

$$v_\ell(\theta) = \sum_{j=0}^{p}\frac{v_\ell^{(j)}(0)\theta^j}{j!} + V_{p,\ell}(\theta)\theta^{p+1}, \qquad V_{p,\ell}(\theta) = \frac{1}{p!}\int_0^1 v_\ell^{(p+1)}(t\theta)(1-t)^p dt.$$

Then

$$\begin{aligned}\tilde{r}(\theta) &= v_0(0) + v_0'\theta + \frac{1}{2}v_0''(0)\theta^2 + V_{2,0}(\theta)\theta^3\\ &+ \frac{ik}{2}v_1(0)\theta^4 + \frac{ik}{2}v_1'(0)\theta^5 + \frac{ik}{2}V_{1,1}(\theta)\theta^6\\ &+ \frac{(ik)^2}{8}v_2(0)\theta^8 + \frac{(ik)^2}{8}V_{0,2}(\theta)\theta^9 + \frac{(ik)^3}{8}v_3(\theta)Z\left(k\frac{1}{2}m_{11}(\xi_0+\theta)\theta^4\right)\theta^{12}.\end{aligned}$$

From this expansion of $\tilde{r}$ we now get a corresponding expansion of I_{main},

$$I_{\text{main}} = I_S + I_{V_{2,0}} + \frac{1}{2}I_{V_{1,1}} + \frac{1}{8}I_{V_{0,2}} + I_Z, \tag{57}$$

$$I_S = v_0(0)I_0 + v_0'(0)I_1 + \frac{1}{2}v_0''(0)I_2 + v_1(0)\frac{ik}{2}I_4 + \frac{ik}{2}v_1'(0)I_5 + \frac{(ik)^2}{8}v_2(0)I_8,$$

where

$$I_p = k^{1/3}\int \psi(\theta/R)\theta^p e^{ik\phi_a(0,\theta,\eta)}d\theta, \qquad I_{V_{p,\ell}} = (ik)^\ell k^{1/3}\int \psi(\theta/R)V_{p,\ell}(\theta)\theta^{3\ell+3}e^{ik\phi_a(0,\theta,\eta)}d\theta,$$

and

$$I_Z = \frac{(ik)^3}{8}k^{1/3}\int \psi(\theta/R)w(\theta,k)\theta^{12}e^{ik\phi_a(0,\theta,\eta)}d\theta,$$

with

$$w(\theta,k) = v_3(\theta)z(\theta,k), \qquad z(\theta,k) = Z\left(\frac{k}{2}m_{11}(\xi_0+\theta)\theta^4\right).$$

We will next show that the last four terms in (57) are at most of size $O(1/k)$. To see this, we note that by Lemma 5.1 and Lemma 5.2, both r and $m_{11}(\xi_0+\cdot)$ belong to $W^{n,\infty}(\mathbb{R})$ for all n, ξ_0, and their $W^{n,\infty}$-norms are bounded independent of ξ_0. By (37) the same is true for v_ℓ, for all ℓ. Therefore, by (38),

$$\begin{aligned}||V_{p,\ell}||_{W^{n,\infty}(\mathbb{R})} &\leq \frac{1}{p!}\int_0^1 ||v_\ell^{(p+1)}(\cdot\, t)||_{W^{n,\infty}(\mathbb{R})}(1-t)^p dt \leq \frac{1}{p!}\int_0^1 \max(1,t^n)||v_\ell^{(p+1)}||_{W^{n,\infty}(\mathbb{R})}(1-t)^p dt\\ &\leq \frac{1}{p!}||v_\ell||_{W^{n+p+1,\infty}(\mathbb{R})},\end{aligned}$$

showing that also $V_{p,\ell} \in W^{n,\infty}(\mathbb{R})$ for all n,p,ℓ. Since (40) is satisfied under the assumptions on η, δ and k, we can use Lemma 6.2 with $n = \lceil(3\ell+4)/5\rceil$ to estimate

$$|I_{V_{p,\ell}}| \leq Ck^{\ell-(3\ell+3)/3}||V_{p,\ell}(\cdot/k^{\frac{1}{3}})||_{W^{n,\infty}(\mathbb{R})} \leq Ck^{-1}\max(1,k^{-\frac{n}{3}})||V_{p,\ell}||_{W^{n,\infty}(\mathbb{R})} \leq Ck^{-1}.$$

For I_Z we first observe that

$$z(\theta/k^{\frac{1}{3}},k) = Z\left(\frac{k^{-\frac{1}{3}}}{2}m_{11}(\xi_0+k^{-\frac{1}{3}}\theta)\theta^4\right).$$

By appealing to Lemma C.1 with $\varepsilon = k^{-1/3}$ we conclude that $||z(\cdot/k^{\frac{1}{3}}, k)||_{W^{3,\infty}(\mathbb{R})}$ is bounded uniformly for $k \geq 1$. Consequently, we can use Lemma 6.2 with $n = \lceil (12+1)/5 \rceil = 3$ together with (37) and (38) to show that

$$|I_Z| \leq Ck^{3-12/3}||w(\cdot/k^{\frac{1}{3}}, k)||_{W^{3,\infty}(\mathbb{R})} \leq Ck^{-1}\max(1, k^{-1})||v_3||_{W^{3,\infty}(\mathbb{R})}||z(\cdot/k^{\frac{1}{3}}, k)||_{W^{3,\infty}(\mathbb{R})} \leq Ck^{-1}.$$

We have thus proved that

$$|I_{\text{main}} - I_S| \leq Ck^{-1}. \tag{58}$$

From Lemma 6.3 we know that $I_p \approx 2\pi \mathrm{Ai}^{(p)}(\rho) i^p / k^{p/3}$ and we therefore introduce the approximation $\tilde{I}_S$ of I_S obtained by replacing I_p with the corresponding Airy function,

$$\begin{aligned}\tilde{I}_S &= v_0(0) 2\pi \mathrm{Ai}(\rho) + v_0'(0)\frac{2\pi i}{k^{1/3}}\mathrm{Ai}'(\rho) + \frac{1}{2}v_0''(0)\frac{2\pi i^2}{k^{2/3}}\mathrm{Ai}^{(2)}(\rho) \\ &\quad + v_1(0)\frac{ik}{2}\frac{2\pi i^4}{k^{4/3}}\mathrm{Ai}^{(4)}(\rho) + \frac{ik}{2}v_1'(0)\frac{2\pi i^5}{k^{5/3}}\mathrm{Ai}^{(5)}(\rho) + \frac{(ik)^2}{8}v_2(0)\frac{2\pi i^8}{k^{8/3}}\mathrm{Ai}^{(8)}(\rho).\end{aligned}$$

By (56) we can use Lemma 6.3 with large enough n to obtain

$$\left| I_p - \frac{2\pi i^p}{k^{p/3}}\mathrm{Ai}^{(p)}(\rho) \right| \leq Ck^{-3},$$

where C is uniform in ρ. Then

$$|I_S - \tilde{I}_S| \leq C(|v_0(0)| + |v_0'(0)| + |v_0''(0)| + k|v_1(0)| + k|v_1'(0)| + k^2|v_2(0)|)k^{-3} \leq C'k^{-1}, \tag{59}$$

as $||v_\ell||_{W^{n,\infty}(\mathbb{R})} \leq C$ uniformly in ξ_0.

The next step is to show that $\tilde{I}_S$ is close to $\mathrm{Ai}(\rho)$. Upon using the identities for $\mathrm{Ai}^{(m)}$ with $m = 2, 4, 5, 8$ given in Remark 9.1 we can simplify the expression for $\tilde{I}_S$ as follows

$$\begin{aligned}\frac{\tilde{I}_S}{2\pi} &= v_0(0)\mathrm{Ai}(\rho) + iv_0'(0)k^{-\frac{1}{3}}\mathrm{Ai}'(\rho) - \frac{1}{2}v_0''(0)k^{-\frac{2}{3}}\mathrm{Ai}^{(2)}(\rho) \\ &\quad + \frac{i}{2}v_1(0)k^{-\frac{1}{3}}\mathrm{Ai}^{(4)}(\rho) - \frac{1}{2}v_1'(0)k^{-\frac{2}{3}}\mathrm{Ai}^{(5)}(\rho) - \frac{1}{8}v_2(0)k^{-\frac{2}{3}}\mathrm{Ai}^{(8)}(\rho) \\ &= v_0(0)\mathrm{Ai}(\rho) + iv_0'(0)k^{-\frac{1}{3}}\mathrm{Ai}'(\rho) - \frac{1}{2}v_0''(0)k^{-\frac{2}{3}}\rho\mathrm{Ai}(\rho) + \frac{i}{2}v_1(0)k^{-\frac{1}{3}}(\rho^2\mathrm{Ai}(\rho) + 2\mathrm{Ai}'(\rho)) \\ &\quad - \frac{1}{2}v_1'(0)k^{-\frac{2}{3}}(4\rho\mathrm{Ai}(\rho) + \rho^2\mathrm{Ai}'(\rho)) - \frac{1}{8}v_2(0)k^{-\frac{2}{3}}((\rho^4 + 28\rho)\mathrm{Ai}(\rho) + 12\rho^2\mathrm{Ai}'(\rho)) \\ &= \left(v_0(0) - \frac{1}{2}v_0''(0)k^{-\frac{2}{3}}\rho + \frac{i}{2}v_1(0)k^{-\frac{1}{3}}\rho^2 - 2v_1'(0)k^{-\frac{2}{3}}\rho - \frac{1}{8}v_2(0)k^{-\frac{2}{3}}(\rho^4 + 28\rho)\right)\mathrm{Ai}(\rho) \\ &\quad + \left(iv_0'(0)k^{-\frac{1}{3}} + iv_1(0)k^{-\frac{1}{3}} - \frac{1}{2}v_1'(0)k^{-\frac{2}{3}}\rho^2 - \frac{3}{2}v_2(0)k^{-\frac{2}{3}}\rho^2\right)\mathrm{Ai}'(\rho).\end{aligned}$$

From (55), (7) and (16), we obtain

$$v_0(0) = \frac{1}{2\pi}, \qquad v_0'(0) = -\frac{q'(\xi_0)}{4\pi q(\xi_0)}, \qquad v_1(0) = \frac{2i - 2\xi_0\beta}{4\pi q(\xi_0)} = \frac{q'(\xi_0)}{4\pi q(\xi_0)}.$$

Hence, $v_0'(0) + v_1(0) = 0$ and

$$\begin{aligned}\tilde{I}_S &= \left(1 + i\pi k^{-\frac{1}{3}}\rho^2 v_1(0) - \pi k^{-\frac{2}{3}}\rho\left(v_0''(0) + 4v_1'(0) + \frac{1}{4}v_2(0)(\rho^3 + 28)\right)\right)\mathrm{Ai}(\rho) \\ &\quad - \pi k^{-\frac{2}{3}}\rho^2\left(v_1'(0) + 3v_2(0)\right)\mathrm{Ai}'(\rho).\end{aligned}$$

Since ρ is bounded by (53) and Ai is smooth around $\rho = 0$, this shows that

$$|\tilde{I}_S - \mathrm{Ai}(\rho)| \leq C\left[k^{-\frac{1}{3}}\rho^2 + k^{-\frac{2}{3}}|\rho|\right] \leq C'\left[k|\eta|^2 + |\eta|\right] = C'\left[|k\eta|^2 + |k\eta|\right]k^{-1}. \tag{60}$$

Note that the dependence on $k^2\eta^2$ which appears here, also appears in the estimate of R_2 in Lemma 8.4.

It remains to estimate $I_{t,\text{tail}}$. By (56) we get from Lemma 6.1 with $p = 0$ and $n = 2$, for all $t > R$, that

$$|I_{t,\text{tail}}| \leq C_2 k^{1/3-2} ||r||_{W^{2,\infty}(\mathbb{R})} \leq C k^{-5/3}, \tag{61}$$

where the constant is independent of t. In conclusion, using (58), (59), (60) and (61) we have shown that

$$\begin{aligned}|I(\eta, x_c, k) - \text{Ai}(\rho)| &\leq |I_{\text{main}} - I_S| + |I_S - \tilde{I}_S| + |\tilde{I}_S - \text{Ai}(\rho)| + \lim_{t\to\infty} |I_{t,\text{tail}}| \\ &\leq C' \left[1 + |k\eta|^2 + |k\eta| + k^{-2/3}\right] k^{-1} \leq C'' \left[1 + |k\eta|^2\right] k^{-1}.\end{aligned}$$

Together with (54) this concludes the proof of Lemma 8.5.

Finally note that, away from the caustic point, i.e. $x < x_c$, the method used here to estimate R_3 will not give sharp results; if the stationary phase method is applied directly to ϕ_g extra decay in k follows. ☐

9. Properties of the Airy function

Here we show some known properties of the Airy function and we derive a few consequences in two lemmas. A more complete source for information about Airy functions is [7], which we frequently cite below. We consider the Airy function of the first kind Ai and second kind Bi.

(P1) The Airy functions are linearly independent solutions of the Airy differential equation

$$\text{Ai}''(z) = z\text{Ai}(z), \qquad \text{Bi}''(z) = z\text{Bi}(z). \tag{62}$$

(P2) Ai and Ai$'$ only have zeros on the negative real line. The zeros do not coincide. Ai(s) is positive and decreasing for $s \geq 0$.

(P3) Bi and Bi$'$ also only have zeros on the negative real line. The zeros do not coincide. Bi(s) is positive and increasing for $s \geq 0$.

(P4) Let

$$\widetilde{\text{Ai}}(z) := \frac{1}{2}\pi^{-\frac{1}{2}} z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}}. \tag{63}$$

Then, for real $s > 0$,

$$\widetilde{\text{Ai}}(-s) = \frac{1}{2}\pi^{-\frac{1}{2}} s^{-\frac{1}{4}} \left(\cos\left(\frac{2}{3}s^{\frac{3}{2}} - \frac{\pi}{4}\right) + i\sin\left(\frac{2}{3}s^{\frac{3}{2}} - \frac{\pi}{4}\right)\right),$$

and it follows easily from [7, Section 9.7 (ii,iii)] that

$$\left|\text{Ai}(s) - \widetilde{\text{Ai}}(s)\right| \leq C s^{-3/2} \left|\widetilde{\text{Ai}}(s)\right|, \qquad \left|\text{Ai}(-s) - 2\Re\widetilde{\text{Ai}}(-s)\right| \leq C s^{-3/2} \tag{64}$$

$$\left|\text{Ai}'(s) + \sqrt{s}\widetilde{\text{Ai}}(s)\right| \leq C s^{-1} \left|\widetilde{\text{Ai}}(s)\right|, \qquad \left|\text{Ai}'(-s) - 2\sqrt{s}\Im\widetilde{\text{Ai}}(-s)\right| \leq C s^{-1}. \tag{65}$$

We can now prove the following lemmas.

Lemma 9.1. *Let $\beta = e^{i\theta}$ with $|\theta| \leq \pi/3$. There is a constant C such that*

$$\left|\text{Ai}(\beta s) - \widetilde{\text{Ai}}(\beta s)\right| \leq C s^{-3/2} \left|\widetilde{\text{Ai}}(\beta s)\right|, \qquad s > 0. \tag{66}$$

Moreover, for each $s_0 > 0$ there is a constant $C(s_0)$ such that

$$\left|\text{Ai}(\beta s) - \widetilde{\text{Ai}}(\beta s)\right| \leq C(s_0) s^{-3/2} \left|\text{Ai}(\beta s)\right|, \qquad s \geq s_0. \tag{67}$$

Moreover, for $s \in \mathbb{R}$ and $\alpha = \exp(i\pi/3)$ there is a constant C such that

$$|\text{Ai}(s)| \leq C(1 + |s|)^{-1/4}, \tag{68}$$

$$|\text{Ai}'(s)| \leq C(1 + |s|)^{1/4}, \tag{69}$$

$$|\text{Ai}(\alpha s)| \geq C(1 + |s|)^{-1/4}, \tag{70}$$

$$|\text{Ai}'(s) + i\sqrt{-s}\text{Ai}(s)| \geq C \begin{cases} (1 + |s|)^{1/2} |\text{Ai}(s)|, & s \geq 0, \\ (1 + |s|)^{1/4}, & s < 0. \end{cases} \tag{71}$$

Proof. We define Φ_0 by the relation $\mathrm{Ai}(z) = \widetilde{\mathrm{Ai}}(z)\Phi_0(z)$. It has an asymptotic expansion,

$$\Phi_0(z) \simeq \sum_{n=0}^{\infty} c_n z^{-3n/2}, \qquad c_n = \frac{(-1)^n\Gamma(n+5/6)\Gamma(n+1/6)\left(\frac{3}{4}\right)^n}{2\pi n!}, \qquad c_0 = 1, \qquad c_1 \approx -0.104.$$

From the estimates on P and Q of [1, Appendix A, Lemma 7] one obtains the uniform estimate

$$|\Phi_0(z) - 1| \leq \frac{|c_1|}{|z|^{3/2}},$$

valid for $|\arg(z)| \leq \frac{\pi}{3}$. Then (66) follows directly with $C = |c_1|$,

$$\left|\mathrm{Ai}(\beta s) - \widetilde{\mathrm{Ai}}(\beta s)\right| = \left|\widetilde{\mathrm{Ai}}(\beta s)\right| |\Phi_0(\beta s) - 1| \leq |c_1| \frac{\left|\widetilde{\mathrm{Ai}}(\beta s)\right|}{s^{3/2}}.$$

Now suppose $s \geq s_0 > 0$. We first note that for all $s > 0$,

$$|\Phi_0(\beta s)| \geq 1 - |\Phi_0(\beta s) - 1| \geq 1 - \frac{|c_1|}{s^{3/2}}.$$

Hence, for $s \geq s_1 := (2|c_1|)^{2/3}$ we have $|\Phi_0(s\beta)| \geq 1/2$, and since $\mathrm{Ai} = \widetilde{\mathrm{Ai}}\Phi_0$ only has zeros on the negative real line, there is a positive infinum,

$$d(s_0) := \inf_{s \geq s_0} |\Phi_0(\beta s)| \geq \min\left(\frac{1}{2}, \min_{s_0 \leq s \leq s_1} |\Phi_0(\beta s)|\right) > 0.$$

Therefore, as above,

$$\left|\mathrm{Ai}(\beta s) - \widetilde{\mathrm{Ai}}(\beta s)\right| \leq |c_1| \frac{\left|\widetilde{\mathrm{Ai}}(\beta s)\right|}{s^{3/2}}. \leq \frac{|c_1|}{d(s_0)} \frac{\left|\widetilde{\mathrm{Ai}}(\beta s)\Phi_0(\beta s)\right|}{s^{3/2}} = \frac{|c_1|}{d(s_0)} \frac{|\mathrm{Ai}(\beta s)|}{s^{3/2}},$$

when $s \geq s_0$, proving (67) with $C(s_0) = |c_1|/d(s_0)$.

Next, to show (68) and (69) we note first that, for real $s > 0$,

$$|\widetilde{\mathrm{Ai}}(s)| = \left|\frac{1}{2}\pi^{-\frac{1}{2}} s^{-\frac{1}{4}} e^{-\frac{2}{3}s^{\frac{3}{2}}}\right| \leq C s^{-\frac{1}{4}}, \qquad |\widetilde{\mathrm{Ai}}(-s)| = \left|\frac{1}{2}\pi^{-\frac{1}{2}} (-s)^{-\frac{1}{4}} e^{\frac{2}{3}is^{\frac{3}{2}}}\right| \leq C s^{-\frac{1}{4}}.$$

Then (64, 65) readily give

$$|\mathrm{Ai}(s)| \leq \left|\widetilde{\mathrm{Ai}}(s)\right| + C\left(1 + \left|\widetilde{\mathrm{Ai}}(s)\right|\right)|s|^{-3/2} \leq C|s|^{-1/4}$$

and

$$\left|\mathrm{Ai}'(s)\right| \leq \sqrt{|s|}\left|\widetilde{\mathrm{Ai}}(s)\right| + C\left(1 + \left|\widetilde{\mathrm{Ai}}(s)\right|\right)|s|^{-1} \leq C|s|^{1/4}.$$

which extend to (68) and (69) as Ai(0) is bounded.

The lower bound (70) follows for $s \geq s_0$ from the previous estimates,

$$\left|\widetilde{\mathrm{Ai}}(\alpha s)\right| \leq \left|\mathrm{Ai}(\alpha s) - \widetilde{\mathrm{Ai}}(\alpha s)\right| + |\mathrm{Ai}(\alpha s)| \leq |c_1| \frac{\left|\widetilde{\mathrm{Ai}}(\alpha s)\right|}{s^{3/2}} + |\mathrm{Ai}(\alpha s)| = \frac{1}{2}\frac{s_0^{3/2}}{s^{3/2}}\left|\widetilde{\mathrm{Ai}}(\alpha s)\right| + |\mathrm{Ai}(\alpha s)|,$$

as then

$$|\mathrm{Ai}(\alpha s)| \geq \frac{1}{2}\left|\widetilde{\mathrm{Ai}}(\alpha s)\right| = \frac{1}{4\sqrt{\pi}}\left|\alpha^{-\frac{1}{4}} s^{-\frac{1}{4}} e^{-\frac{2}{3}is^{\frac{3}{2}}}\right| = \frac{1}{4\sqrt{\pi}} s^{-\frac{1}{4}}.$$

By (P2) we also have $|\mathrm{Ai}(\alpha s)| \geq c$ for $0 \leq s \leq s_0$ and some $c > 0$. Moreover, the identity [7, Eq. 9.2.11]

$$\mathrm{Ai}(\alpha s) = \frac{\bar{\alpha}}{2}[\mathrm{Ai}(-s) + i\mathrm{Bi}(-s)], \tag{72}$$

and (P3) implies that $|\mathrm{Ai}(\alpha s)| \geq |\mathrm{Bi}(-s)|/2 = \mathrm{Bi}(-s)/2 \geq \mathrm{Bi}(0)/2 > 0$ for $s \leq 0$. This gives the bound (70) also for $s \leq s_0$.

For (71) we consider $s \geq 0$ and use (64, 65)

$$\begin{aligned}2\sqrt{s}|\widetilde{\mathrm{Ai}}(-s)| &\leq |2\sqrt{s}\widetilde{\mathrm{Ai}}(-s) - i\mathrm{Ai}'(-s) - \sqrt{s}\mathrm{Ai}(-s)| + |i\mathrm{Ai}'(-s) + \sqrt{s}\mathrm{Ai}(-s)| \\ &\leq |2\sqrt{s}\Im\widetilde{\mathrm{Ai}}(-s) - \mathrm{Ai}'(-s)| + \sqrt{s}|2\Re\widetilde{\mathrm{Ai}}(-s) - \mathrm{Ai}(-s)| + |i\mathrm{Ai}'(-s) + \sqrt{s}\mathrm{Ai}(-s)| \\ &\leq Cs^{-1} + |i\mathrm{Ai}'(-s) + \sqrt{s}\mathrm{Ai}(-s)| = Cs^{-1} + |\mathrm{Ai}'(-s) + i\sqrt{s}\mathrm{Ai}(-s)|.\end{aligned}$$

Hence,

$$|\mathrm{Ai}'(-s) + i\sqrt{s}\mathrm{Ai}(-s)| \geq 2\sqrt{s}|\widetilde{\mathrm{Ai}}(-s)| - Cs^{-1} = Cs^{1/4} - Cs^{-1}.$$

Since the zeros of Ai' and Ai do not coincide (no double roots) we get

$$|\mathrm{Ai}'(-s) + i\sqrt{s}\mathrm{Ai}(-s)| = |\mathrm{Ai}'(-s)| + \sqrt{s}|\mathrm{Ai}(-s)| \neq 0, \qquad s > 0,$$

and the estimate (71) for $s < 0$ follows. Moreover, by (P2), when $s \geq 0$,

$$\begin{aligned}|\mathrm{Ai}'(s) + i\sqrt{-s}\mathrm{Ai}(s)| &= |\mathrm{Ai}'(s) - \sqrt{s}\mathrm{Ai}(s)| = \sqrt{s}\mathrm{Ai}(s) - \mathrm{Ai}'(s) = \sqrt{s}|\mathrm{Ai}(s)| + |\mathrm{Ai}'(s)| \\ &\geq \max\left(\sqrt{s}|\mathrm{Ai}(s)|,\ |\mathrm{Ai}(s)| \min_{0\leq t\leq s} \frac{|\mathrm{Ai}'(t)|}{|\mathrm{Ai}(t)|}\right) \geq \frac{1}{2}(\sqrt{s} + C)|\mathrm{Ai}(s)|,\end{aligned}$$

which gives (71) for $s \geq 0$. Here we also used the fact that $\lim_{s\to+\infty} \frac{|\mathrm{Ai}'(s)|}{|\mathrm{Ai}(s)|} = \lim_{s\to+\infty} \frac{|\sqrt{s}\widetilde{\mathrm{Ai}}(s)|}{|\widetilde{\mathrm{Ai}}(s)|} = \infty$ by (64) and (65). ∎

Lemma 9.2. *For the Airy function we have*

$$\mathrm{Ai}^{(m)}(x) = p_m(x)\mathrm{Ai}(x) + q_m(x)\mathrm{Ai}'(x),$$

where p_m and q_m are polynomials given by the recursions

$$p_{m+1} = p_m' + xq_m, \qquad q_{m+1} = p_m + q_m', \qquad p_0 = 1, \quad q_0 = 0. \tag{73}$$

The degree of their sum satisfies $deg(p_m + q_m) = \lfloor m/2 \rfloor$ and, for $|x| < 1$,

$$|p_m(x)| + |q_m(x)| \leq \frac{(d_{m+1}!)^2}{1 - |x|}, \qquad d_m = \left\lfloor \frac{m}{2} \right\rfloor.$$

Furthermore,

$$\mathrm{Ai}^{(3p+2)}(0) = 0, \qquad p = 0, 1, \ldots \tag{74}$$

Proof. Using the form of $\mathrm{Ai}^{(m)}(x)$ given and using (62) we note that

$$\mathrm{Ai}^{(m+1)} = p_m'\mathrm{Ai} + q_m'\mathrm{Ai}'(x) + p_m\mathrm{Ai}' + q_m\mathrm{Ai}'' = p_m'\mathrm{Ai} + q_m'\mathrm{Ai}'(x) + p_m\mathrm{Ai}' + xq_m\mathrm{Ai},$$

where we used the Airy differential equation $\mathrm{Ai}'' = x\mathrm{Ai}$. This gives the recursion (73). The statement about the degree is easily checked for $m = 0, 1$. Suppose it holds upto a general $m \geq 2$. Then

$$\begin{aligned}\deg(p_{m+1} + q_{m+1}) &= \deg(p_m' + q_m' + p_m + xq_m) = \deg(p_m + xq_m) \\ &= \deg(p_{m-1}' + xq_{m-1} + xp_{m-1} + xq_{m-1}') = \deg(x(p_{m-1} + q_{m-1})) \\ &= \deg(p_{m-1} + q_{m-1}) + 1.\end{aligned}$$

That $\deg(p_m + q_m) = d_m$ follows by induction. Since the polynomials all have positive coefficients, it also follows that $d_m = \max(\deg(p_m), \deg(q_m))$. For a polynomial p, let $|p|_\infty$ denote its largest coefficient in magnitude. Then $|xp|_\infty = |p|_\infty$ and $|p'|_\infty \leq \deg(p)|p|_\infty$. Consequently,

$$\begin{aligned}|p_{m+1}|_\infty + |q_{m+1}|_\infty &= |p_m' + xq_m|_\infty + |p_m + q_m'|_\infty \leq |p_m'|_\infty + |xq_m|_\infty + |p_m|_\infty + |q_m'|_\infty \\ &\leq \deg(p_m)|p_m|_\infty + |q_m|_\infty + |p_m|_\infty + \deg(q_m)|q_m|_\infty \\ &\leq (\max(\deg(p_m), \deg(q_m)) + 1)(|p_m|_\infty + |q_m|_\infty) = (d_m + 1)(|p_m|_\infty + |q_m|_\infty).\end{aligned}$$

Therefore, since $|p_0|_\infty + |q_0|_\infty = 1$,

$$|p_m|_\infty + |q_m|_\infty \leq \Pi_{\ell=0}^{m-1}(d_\ell + 1) \leq \Pi_{\ell=2}^{m+1} d_\ell \leq (d_{m+1}!)^2.$$

Finally, we have for a polynomial p of degree d, and $|x| < 1$,

$$\frac{|p(x)|}{|p|_\infty} \leq 1 + |x| + \cdots + |x|^d \leq \frac{1}{1-|x|}.$$

The last statement of the lemma is known for $p = 0$. Suppose it holds for p and use the Airy differential equation $\mathrm{Ai}'' = x\mathrm{Ai}$. That gives $\mathrm{Ai}^{(3p+2)} = x\mathrm{Ai}^{(3p)} + (3p-1)\mathrm{Ai}^{(3(p-1)+2)}$, which shows the claim. □

Remark 9.1. The first few polynomials p_m and q_m in the theorem are given by

$$\begin{aligned}
\mathrm{Ai}^{(2)}(x) &= x\mathrm{Ai}(x),\\
\mathrm{Ai}^{(3)}(x) &= \mathrm{Ai}(x) + x\mathrm{Ai}'(x),\\
\mathrm{Ai}^{(4)}(x) &= x^2\mathrm{Ai}(x) + 2\mathrm{Ai}'(x),\\
\mathrm{Ai}^{(5)}(x) &= 4x\mathrm{Ai}(x) + x^2\mathrm{Ai}'(x),\\
\mathrm{Ai}^{(6)}(x) &= (x^3+4)\mathrm{Ai}(x) + 6x\mathrm{Ai}'(x),\\
\mathrm{Ai}^{(7)}(x) &= 9x^2\mathrm{Ai}(x) + (x^3+10)\mathrm{Ai}'(x),\\
\mathrm{Ai}^{(8)}(x) &= (x^4+28x)\mathrm{Ai}(x) + 12x^2\mathrm{Ai}'(x).
\end{aligned}$$

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Appendix A. Proof of Lemma 2.1

We show this for $k = 1$ so that $\mathcal{F}_k = \mathcal{F}$. The case with general k follows from a simple rescaling. Let $\phi \in \mathcal{S}$ be a test function and $\langle \cdot, \cdot \rangle$ the duality pairing between $\mathcal{S}$ and $\mathcal{S}'$. Then since $f\psi_t \in L^\infty \subset \mathcal{S}'$ and, by dominated convergence,

$$\begin{aligned}\langle \mathcal{F}(f\psi_t), \phi\rangle = \langle f\psi_t, \mathcal{F}(\phi)\rangle &= \int f(x)\psi_t(x)\mathcal{F}(\phi)(x)dx \\ &\to \int f(x)\mathcal{F}(\phi)(x)dx = \langle f, \mathcal{F}(\phi)\rangle = \langle \mathcal{F}(f), \phi\rangle,\end{aligned}$$

where we used the facts that $\mathcal{F}(\phi) \in \mathcal{S} \subset L^1$, $|\psi_t| \leq 1$ for all t and $f\psi_t \to f$ pointwise. This is true for all $\phi \in \mathcal{S}$ and therefore $\mathcal{F}(f\psi_t) \to \mathcal{F}(f)$ in $\mathcal{S}'$, proving the first statement.

That $\mathcal{F}((f * g)\psi_t) \to \mathcal{F}(f)\mathcal{F}(g)$ follows from the first statement since $f * g \in L^\infty$ when $g \in \mathcal{S}$ and $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$. The last part of the second statement is true, since $\mathcal{F}(g)\phi \in \mathcal{S}$ and therefore the first part gives

$$\langle \mathcal{F}(f\psi_t)\mathcal{F}(g), \phi\rangle = \langle \mathcal{F}(f\psi_t), \mathcal{F}(g)\phi\rangle \to \langle \mathcal{F}(f), \mathcal{F}(g)\phi\rangle = \langle \mathcal{F}(f)\mathcal{F}(g), \phi\rangle.$$

This shows the lemma.

Appendix B. Proof of the non-stationary phase identities

Below is a proof of identities used in the non-stationary phase lemma. The identities show how the rewritten integral depends on the derivatives of the phase function. In order to do that we use the spaces of functions defined in (35) and (36).

Lemma B.1. *Suppose $D \subset \mathbb{R}$ is a bounded open set and $K \subset D$ is compact. Let $a \in W^{n,1}(D)$ and $\phi \in C^{n+1}(D)$. If $\phi' \neq 0$ on K and $\operatorname{supp} a \subset K$, then there exist functions $u_{\ell,n}$ such that*

$$\int_D a(y)e^{i\phi(y)/\varepsilon}dy = (ik)^{-n}\sum_{\ell=0}^{n}\int_K a^{(\ell)}(y)\frac{u_{\ell,n}(y)}{\phi'(y)^n}e^{ik\phi(y)}dy, \qquad u_{\ell,n} \in \mathcal{U}_{n-\ell}(\phi).$$

Proof. Define the differential operator,

$$L[a] := \left(\frac{a}{\phi'}\right)'.$$

Then, since $\operatorname{supp} u \subset K \subset D$ and $|\phi'| > 0$ on K integration by parts gives

$$\int_D a(y)e^{i\phi(y)/\varepsilon}dy = \frac{1}{ik}\int_D L[a](y)e^{ik\phi(y)}dy. \tag{75}$$

Since and u and ϕ are sufficiently regular, this can be repeated n times, giving

$$\int_D u(y)e^{ik\phi(y)}dy = (ik)^{-n}\int_D L^n[a](y)e^{ik\phi(y)}dy.$$

We thus need to show that there exist $u_{\ell,n}$ such that

$$L^n[a] = \sum_{\ell=0}^{n} a^{(\ell)}\frac{u_{\ell,n}}{\phi'^n}, \qquad u_{\ell,n} \in \mathcal{U}_{n-\ell}(\phi).$$

When $n = 0$ this simply says that $u_{0,0} = 1 \in \mathcal{U}_0(\phi)$. Suppose the claim holds for n and consider

$$\begin{aligned}L^{n+1}[a] = L\sum_{\ell=0}^{n} a^{(\ell)}\frac{u_{\ell,n}}{\phi'^n} &= \sum_{\ell=0}^{n}\frac{d}{dy}\left(a^{(\ell)}\frac{u_{\ell,n}}{\phi'^{n+1}}\right) \\ &= \sum_{\ell=0}^{n} a^{(\ell+1)}\frac{u_{\ell,n}}{\phi'^{n+1}} + a^{(\ell)}\frac{u'_{\ell,n}}{\phi'^{n+1}} - (n+1)a^{(\ell)}\frac{u_{\ell,n}\frac{\phi''}{\phi'}}{\phi'^{n+1}}.\end{aligned}$$

For the first term we have

$$u_{\ell,n} \in \mathcal{U}_{n-\ell}(\phi) = \mathcal{U}_{n+1-(\ell+1)}(\phi).$$

For the third term
$$u_{\ell,n}\frac{\phi''}{\phi'} \in \mathcal{U}_{n+1-\ell}(\phi).$$
For the second term, consider one basis function $w \in \mathcal{W}_p(\phi)$,
$$w = \prod_{k=0}^{M} \frac{\phi^{(\alpha_k+1)}}{\phi'}, \qquad \sum_{k=0}^{M} \alpha_k = p.$$
for some M. Then
$$w' = \sum_{\ell=0}^{M} \prod_{k=0}^{M} \frac{\phi^{(\alpha_k+1+\delta_{\ell,k})}}{\phi'} - \sum_{\ell=0}^{M} \frac{\phi''}{\phi'} \prod_{k=0}^{M} \frac{\phi^{(\alpha_k+1)}}{\phi'} \in \mathcal{U}_{p+1}(\phi).$$
Hence, $u'_{\ell,n} \in \mathcal{U}_{n+1-\ell}(\phi)$, as it is a linear combination of derivatives of functions in $\mathcal{U}_{n-\ell}(\phi)$. This shows that $L^{n+1}[a]$ is of the correct form and the lemma is proved. □

Appendix C. Boundedness of Z

Here we consider the scaled remainder term in the Taylor expansion of $\exp(iz)$,

$$Z(z) = \frac{1}{z^3}\left(e^{iz} - (1 + iz + (iz)^2/2)\right) = \frac{1}{2i}\int_0^1 e^{isz}(1-s)^2 ds. \tag{76}$$

We have the following lemma.

Lemma C.1. *Let*
$$\sigma^\varepsilon(\theta) = Z\left(\frac{\varepsilon}{2} m_{11}(\xi_0 + \varepsilon\theta)\theta^4\right).$$
Then there is a constant C such that
$$||\sigma^\varepsilon||_{W^{3,\infty}(\mathbb{R})} \leq C,$$
for all $0 < \varepsilon \leq 1$.

Proof. We begin by estimating Z and its first three derivatives for z with non-negative imaginary part. Then all derivatives of Z are bounded, since
$$\left|\frac{d^p Z(z)}{dz^p}\right| = \frac{1}{2}\left|\int_0^1 e^{isz}(1-s)^2(is)^p ds\right| \leq \frac{1}{2}\int_0^1 e^{-s\Im z}(1-s)^2 s^p ds < \frac{1}{2}.$$
Furthermore, from the first part of the definition (76) we get, for $p = 0, \dots, 3$ and $|z| \geq 1$,
$$\left|\frac{d^p Z(z)}{dz^p}\right| \leq C\left(\sum_{j=0}^{p} |z|^{-3-p+j} + |z|^{-3-p} + |z|^{-2-p} + |z|^{-1-p}\right) \leq \frac{C'}{|z|^{\min(3,p+1)}}.$$
It follows that there is a constant C such that
$$\left|\frac{d^p Z(z)}{dt^p}\right| \leq \frac{C}{1 + |z|^p}, \qquad p = 0, \dots, 3, \qquad \Im z \geq 0. \tag{77}$$
Next, let
$$G^\varepsilon(\theta) = \frac{\varepsilon}{2} m_{11}(\xi_0 + \varepsilon\theta)\theta^4,$$
so that $\sigma^\varepsilon(\theta) = Z(G^\varepsilon(\theta))$. Then, by Lemma 5.2 for $0 \leq p \leq 4$,
$$\left|\frac{d^p G^\varepsilon(\theta)}{d\theta^p}\right| \leq C\varepsilon \sum_{j=0}^{p} \varepsilon^j |m_{11}^{(j)}(\xi_0 + \varepsilon\theta)||\theta|^{4-p+j} \leq C\varepsilon|\theta|^{4-p} \sum_{j=0}^{p} \frac{d_j|\varepsilon\theta|^j}{1 + |\varepsilon\theta|^{j+1}} \leq C' \frac{\varepsilon|\theta|^{4-p}}{1 + |\varepsilon\theta|},$$
and since $\Im m_{11} > 0$ by Lemma 5.2, we get from (77) and (31) that
$$|Z^{(p)}(G^\varepsilon(\theta))| \leq \frac{C}{1 + |G^\varepsilon(\theta)|^p} \leq \frac{C}{1 + \left|\frac{D_0\varepsilon\theta^4}{2(1+|\varepsilon\theta|)}\right|^p} \leq C'' \frac{(1 + |\varepsilon\theta|)^p}{1 + \varepsilon^p|\theta|^{4p}}, \qquad p = 0, \dots, 3.$$

Therefore, if $p = 0, \dots, 3$ and $j_1 + \cdots + j_p = p' \le 4p$,

$$\left| Z^{(p)}(G^\varepsilon(\theta)) \frac{d^{j_1} G^\varepsilon(\theta)}{d\theta^{j_1}} \cdots \frac{d^{j_p} G^\varepsilon(\theta)}{d\theta^{j_p}} \right| \le \frac{C''(1+|\varepsilon\theta|)^p}{1+\varepsilon^p|\theta|^{4p}} \frac{(C'\varepsilon)^{j_1+\cdots+j_p}|\theta|^{4-j_1+4-j_2+\cdots+4-j_p}}{(1+|\varepsilon\theta|)^{j_1+\cdots+j_p}}$$

$$= C''' \frac{\varepsilon^p |\theta|^{4p-p'}}{1+\varepsilon^p\,|\theta|^{4p}} = C''' \frac{\varepsilon^{p'/4}(\varepsilon^{1/4}|\theta|)^{4p-p'}}{1+(\varepsilon^{1/4}|\theta|)^{4p}} \le C''' \varepsilon^{p'/4}.$$

From these estimates we get, with $p = p' = 0, \dots, 3$,

$$|\sigma^\varepsilon(\theta)| = |Z(G^\varepsilon(\theta))| \le C,$$

$$\left| \frac{d}{d\theta} \sigma^\varepsilon(\theta) \right| = \left| Z^{(1)}(G^\varepsilon(\theta)) \frac{dG^\varepsilon(\theta)}{d\theta} \right| \le C\varepsilon^{1/4},$$

$$\left| \frac{d^2}{d\theta^2} \sigma^\varepsilon(\theta) \right| = \left| Z^{(2)}(G^\varepsilon(\theta)) \left(\frac{dG^\varepsilon(\theta)}{d\theta} \right)^2 + Z^{(1)}(G^\varepsilon(\theta)) \frac{d^2 G^\varepsilon(\theta)}{d\theta^2} \right| \le C\varepsilon^{1/2},$$

$$\left| \frac{d^3}{d\theta^3} \sigma^\varepsilon(\theta) \right| = \left| Z^{(3)}(G^\varepsilon(\theta)) \left(\frac{dG^\varepsilon(\theta)}{d\theta} \right)^3 + 3Z^{(2)}(G^\varepsilon(\theta)) \frac{d^2 G^\varepsilon(\theta)}{d\theta^2} \frac{dG^\varepsilon(\theta)}{d\theta} + 3Z^{(1)}(G^\varepsilon(\theta)) \frac{d^3 G^\varepsilon(\theta)}{d\theta^3} \right| \le C\varepsilon^{3/4}.$$

This shows the lemma. □

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