HIGH FREQUENCY LIMIT OF THE HELMHOLTZ EQUATION II: SOURCE ON A GENERAL SMOOTH MANIFOLD

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1 Introduction

In this paper, we consider the following high frequency Helmholtz equation,

$$+ i\frac{\alpha_\varepsilon}{\varepsilon}u^\varepsilon + \Delta u^\varepsilon + \frac{n^2(x)}{\varepsilon^2}u^\varepsilon = \mathcal{S}_\varepsilon(x)\ ,\ x \in \mathbb{R}^d\ , \tag{1}$$

where the source term $\mathcal{S}_\varepsilon$ in (1) concentrates on a p-dimensional manifold $\Gamma \subset \mathbb{R}^d$, i.e. it is chosen of the form,

$$\mathcal{S}_\varepsilon(x) := -\frac{1}{\varepsilon^q}\int_\Gamma A(y)e^{i\phi(y)/\varepsilon}S\left(\frac{x-y}{\varepsilon}\right)d\sigma(y)\ . \tag{2}$$

The factor $n(x)/\varepsilon$ in (1) is the scaled refraction index, and the scaling parameter ε prescribes the typical wavelength of solutions u^ε to the given Helmholtz equation. The prefactor $\alpha_\varepsilon > 0$ is a regularizing parameter, with,

$$\alpha_\varepsilon \to \alpha \geq 0 \quad \text{as} \quad \varepsilon \to 0 \ , \tag{3}$$

so that α_ε may vanish asymptotically, but the positivity of α_ε for any given $\varepsilon > 0$ uniquely determines u^ε in (1). The right hand side of (1) models a source concentrating on a surface Γ *via* the concentration profile $\varepsilon^{-q} S((x-y)/\varepsilon)$. Note that the exact value of the scaling exponent $q \geq 0$ in terms of the dimensions p and d is made precise below, and it depends on the behavior of the refraction index $n(x)$ together with the phase $\phi(x)$. The source emits waves with amplitude $A \in \mathbb{R}$ and phase ϕ, oscillating with the same wavelength ε as u^ε, *via* the term $A(y)\exp(i\phi(y)/\varepsilon)$, so as to create resonance effects between the highly oscillating function u^ε and the source itself. Finally, Γ is a smooth p-dimensional manifold ($0 \leq p \leq d$) with euclidian surface measure $d\sigma$. We actually assume that Γ is given as the range of the embedding (i.e. the proper injective immersion) $\gamma : \mathbb{R}^p \mapsto \mathbb{R}^d$. We also assume that for fixed t the vectors $\{\frac{\partial \gamma}{\partial t_j}(t)\}$ form an orthonormal basis for the p-dimensional tangent space to Γ, $T_{\gamma(t)}\Gamma$. These assumptions can always be realized locally, and we simply assume for convenience here that they are satisfied globally.

Using the global parametrization γ we can thus write (1) as,

$$\begin{aligned} +i\frac{\alpha_\varepsilon}{\varepsilon}u^\varepsilon + \Delta u^\varepsilon + \frac{n^2(x)}{\varepsilon^2}u^\varepsilon = \\ -\frac{1}{\varepsilon^q}\int_{\mathbb{R}^p} A(\gamma(t))e^{i\phi(\gamma(t))/\varepsilon} S\left(\frac{x-\gamma(t)}{\varepsilon}\right) dt. \end{aligned} \tag{4}$$

The goal of this paper is to present a description of the high frequency limit $\varepsilon \to 0$ in (1).

Before going further, we mention that the model under consideration (1) may seem fairly general, since the refraction index $n(x)$ is allowed to vary with x, and the source Γ may have non-vanishing curvature. We are indeed able to present a *formal* description of the asymptotics $\varepsilon \to 0$ in (1) in the case of a variable refraction index $n(x)$, as well as for a general smooth manifold Γ. However, the *rigorous* results presented in this paper are merely concerned with the case of a *constant* refraction index $n(x) \equiv n_0$,

together with a p-dimensional *affine subspace* Γ. Even the formal analysis sometimes requires the assumption that $n(x)$ is constant, depending on the exact regime under consideration (see below). These two restrictions on n and Γ deserve some comments. First, the restriction on the refraction index is not just a technicality, and it is linked to the difficulty of finding a reasonable "radiation condition at infinity" in this case, a difficulty that we are not able to overcome in the present framework. In other words, the regularizing parameter $+i(\alpha_\varepsilon/\varepsilon)u^\varepsilon$ uniquely prescribes u^ε in (1), and the difficulty lies in defining the limiting value of u^ε as the regularizing parameter $\alpha_\varepsilon/\varepsilon$ is set to 0^+. On the other hand, the restriction on Γ (namely that Γ should have vanishing curvature) stems from more technical reasons, and a future work intends to state rigorous results for *general* sources Γ (and in all regimes), see [3].

Now, we wish to describe the propagation of quadratic quantities, like the local energy density $|u^\varepsilon(x)|^2$, as $\varepsilon \to 0$. As it is classical since the works of Tartar [10], Lions and Paul [7], Gérard, Mauser, Markowich and Poupaud, [4, 6], this can be done via the Wigner measure f associated with the family u^ε. The outcome of our study is that we can identify three different regimes.

The first possible regime is a *propagative* regime, obtained under the typical assumption $|\nabla\phi(x)| < n(x)$ for any x (or more precisely $|\nabla^\tau\phi(x)| < n(x)$, $\forall x \in \Gamma$, see below). In this case, the scaling exponent q in (2) has to be taken as,

$$q = \frac{3+d+p}{2}\,, \tag{5}$$

and the limiting Wigner measure is described by a transport equation,

$$+\,\alpha f(x,\xi) + \xi\cdot\nabla_x f + \frac{1}{2}\nabla_x n^2(x)\cdot\nabla_\xi f = Q(x,\xi). \tag{6}$$

This equation describes the propagation of energy in the medium having refraction index $n(x)$ as in geometrical optics (propagation along the rays of geometrical optics), and in this picture $f(x,\xi)$ can be seen as the energy density carried by rays located at the position x with velocity ξ. The source term Q in the right-hand-side of (6) is the remainder of the source $\mathcal{S}_\varepsilon$ in the Helmholtz equation (1) as it is defined in (2), and it cannot be obtained *via* geometrical optics analysis. It keeps track of the interaction between the high oscillations of the source term $\mathcal{S}_\varepsilon$ and the ones generated by the high

frequency Helmholtz operator $\Delta_x + n^2(x)/\varepsilon^2$. For this reason, the limiting energy source $Q(x,\xi)$ is naturally concentrated on the curve Γ in the space variable x. Also, it is concentrated on velocities ξ such that $\xi^2 = n^2(x)$ and ξ should have a specific direction in terms of $\nabla\phi(x)$. For instance, when, $0 < p < d$, we are able to get the explicit formula,

$$Q(x,\xi) = 2^p \pi^{p+1} |\widehat{S}(\xi)|^2 \int_\Gamma \delta(x-y)\, \delta(\xi_y^\tau - \nabla^\tau \phi(y))\, \delta\left(n^2(y) - |\xi|^2\right) |A(y)|^2 d\sigma(y)\,, \tag{7}$$

where ξ_y^τ is the component of ξ tangential to the curve at y, ξ_y^ν its normal component, and $\nabla^\tau\phi(y)$ denotes in the same vein the part of $\nabla\phi(y)$ which is tangential to Γ at y, a convention that we shall keep throughout the paper. Hence, the source emits rays in the direction of $\nabla^\tau\phi(y)$. This role of $\nabla^\tau\phi(y)$ is well-known and, for instance, the basis for the working of so-called phase array antennae. We actually have to supplement (6) with a radiation condition at infinity in the case $\alpha \equiv 0$, analogous to the Sommerfeld radiation condition for the Helmholtz equation, and this creates the key difficulty in passing to the limit from (1) to (6). We refer to the sequel for details on this point.

The second possible limit is obtained in a *resonant* regime, where $|\nabla^\tau\phi(x)| > n(x)$, $\forall x \in \Gamma$. Our formal analysis is now restricted to the case of a constant refraction index $n(x) \equiv n_0$. In this regime the waves created by the source term $\mathcal{S}_\varepsilon$ in (1) and the ones created by the high frequency Helmholtz operator, interact in a *weaker* way than in the propagative regime. For this reason, the source $\mathcal{S}_\varepsilon$ in (1) has to be amplified in a stronger way than in the propagative regime above, so that the scaling exponant q in (2) has the greater value,

$$q = \frac{4+d+p}{2}\,, \tag{8}$$

in this case (compare with (5)). The assumption $|\nabla^\tau\phi(x)| > n(x)$ implies indeed that no ray vector ξ satisfies at the same time the two conditions $\xi_y^\tau = \nabla^\tau\phi(y)$ *and* $\xi^2 = n^2(y)$ needed in the propagative regime, see (7). For this reason, the interaction between the two waves is entirely located on the curve Γ, hence the energy density f itself is located on Γ, and it does not propagate outside Γ. We find that the Wigner measure is directly given by,

$$f(x,\xi) = (2\pi)^p \int_\Gamma \delta(x-y)\, \delta(\xi_y^\tau - \nabla^\tau\phi(y))\, \frac{|\widehat{S}(\xi)^2 |A(y)|^2}{(n^2(y) - |\nabla\phi(y)|^2)^2} d\sigma(y)\,, \tag{9}$$

under the assumption that $|\nabla\phi|(x) > n(x)$ $(\equiv n_0)$.

The third possible regime, called the *characteristic* regime, is the borderline between the resonant and the propagative regimes. It corresponds to the typical situation where $|\nabla^\tau\phi|(x) \equiv n(x)$ on Γ, and our formal study is again restricted to the case of a constant refraction index $n(x) \equiv n_0$. In this regime, the two typical sets $\xi^2 = n^2(y)$ and $\xi^\tau_y = \nabla^\tau\phi(y)$ intersect tangentially. The interaction between waves created by the source term and waves created by the Helmholtz operator is stronger than in the resonant regime where these two sets do not intersect. Also, this interaction may be either weaker or stronger than in the propagative case where these two sets intersect transversely, depending on the value of $d-p$ (compare (10), (8) and (5)). We perform the complete analysis of the characteristic regime under the simplifying assumption that the phase ϕ depends *linearly* on its argument. It turns out that the energy density is again located on Γ in this case (the interaction is still too weak to propagate outside Γ). Though the energy does not propagate *outside* Γ, we show that it is propagated *inside* Γ, according to the flow of $\nabla^\tau\phi$. Also, the following choice of the scaling exponent q is prescribed,

$$q = \frac{4+3d+p}{4}\ . \tag{10}$$

However, we do not know whether these conclusions still hold true in more general situations, where the curve Γ has non-vanishing curvature, or the phase ϕ depends non-linearly upon its argument.

We wish to mention here that the distinction between these three regimes (propagative/resonant/characteristic) naturally arising in the present context is reminiscent of the standard distinction between, respectively, hyperbolic, elliptic, and glancing covectors arising in the study of semi-classical measures for boundary value problems. See e.g. [5] for the asymptotic analysis of the eigenfunctions of the Laplacian on a bounded domain, or [8] for the asymptotic analysis of a Schrödinger equation with interface.

We wish to end this introduction by mentioning that the kind of limit under consideration here is already studied in [2], where the simple case of a point source ($p = 0$) is treated, but the refraction index $n(x)$ is allowed to depend on x. As in [2], one important difficulty in passing to the limit in (1) lies in the fact that the solution $u^\varepsilon(x)$ typically decays too slowly with x, in the sense that it belongs to weighted spaces of the form $L^2(\langle x\rangle^{-\beta}dx)$,

for some (large enough) $\beta > 0$. This is due to the fact that the Helmholtz operator is not elliptic and generates roughly speaking a singularity of the form $\delta(\xi^2 - n^2(x))$. in the Fourier space (here ξ is the Fourier variable corresponding to x). The situation actually gets more difficult to control in the present context, due to the highly oscillating source terms (2) that we consider in (1). They generate a new singularity of the form $\delta(\xi_y^\tau - \nabla^\tau \phi(y))$, $y \in \Gamma$, in the Fourier space. (Like above, ξ_y^τ is the component of ξ tangent to the manifold Γ at y). As a consequence, we have to control, say, the singularity created by the product of these two Dirac masses in order to obtain uniform weighted L^2 estimates. In the case treated in [2], the specific shape of the source term makes it possible to use new uniform (in ε) estimates for the Helmholtz operator established by B. Perthame and L. Vega in [9], also valid for a variable refraction index $n(x)$, and the problem is reduced to controlling the Wigner transform of functions merely lying in weighted L^2 spaces, instead of the natural L^2 framework. In the present paper, the estimates established in [9] cannot be applied due to the lack of decay of the source in the directions of Γ, and the new difficulty lies in obtaining the desired a priori weighted-L^2 estimates. We are only able to get these bounds in the case of a constant refraction index ($n(x) \equiv n_0$), where reasonably explicit formulae are at hand, and this is the reason of our restriction on n when proving rigorous convergence results.

Notation

For a given p-dimensional surface Γ, a point $y \in \Gamma$, and a given vector ξ, we use the notation $\xi = \xi_y^\tau + \xi_y^\nu \in T_y\Gamma + T_y^\perp\Gamma$. According to the context, ξ_y^τ is considered either as a vector in $\mathbb{R}^p$, or a vector in (a p-dimensional subspace of) $\mathbb{R}^d$, without notational distinction.

If Γ is a linear subspace, we drop the index y and simply write ξ^τ and ξ^ν instead of ξ_y^τ and ξ_y^ν. For instance, if $\Gamma = \mathbb{R}^p \times \{0\} \subset \mathbb{R}^d$, each vector $x \in \mathbb{R}^d$ is decomposed as $x = x^\tau + x^\nu \in \mathbb{R}^p + \mathbb{R}^{d-p}$.

Finally, we shall make repeated use of the standard notation, valid for any $x \in \mathbb{R}^d$,

$$\langle x \rangle := (1 + x^2)^{1/2} .$$

2 Formal derivation

The aim of this section is to present the formal computations leading from the high frequency Helmholtz equation (1) to equations of the form (6) or (9). In particular, we wish to point out the distinction between the different regimes (propagative/resonant/characteristic). In this section, all profiles (ϕ, S, A, γ) are assumed smooth (say C^∞). Also, the profiles S and A are assumed to decay fast enough at infinity if needed, and we shall never make this point more precise in the sequel.

We already mentioned in the introduction that the high frequency limit $\varepsilon \to 0$ in (1) combines the emergence of a first Dirac mass $\delta(\xi^2 - n^2(x))$ created by the singular support of the Helmholtz operator in Fourier space, and a second one $\delta(\xi_y^\tau - \nabla^\tau \phi(y))$ ($y \in \Gamma$) due to the oscillations of the source in (1). In particular, for a given $x \in \Gamma$, the Helmholtz operator selects those frequencies with given modulus $\xi^2 = n^2(x)$, while the source term concentrates on frequencies such that $\xi_x^\tau = \nabla\phi(x)$, and a resonance may occur, depending on whether the sphere $\xi^2 = n^2(x)$ and the affine subspace $\xi_x^\tau = \nabla\phi(x)$ intersect or not. The fact that such resonance can occur in some cases, but cannot occur in other cases, is the very reason for the different regimes we point out, as we show below.

We begin this section by briefly introducing the main tool of our analysis, namely the Wigner transform. With an exception in Section 2.4, this formal derivation is concerned with the study of (1) in the most singular case where the asymptotic regularizing parameter α vanishes,

$$\alpha \equiv 0 \ .$$

2.1 The Wigner transform

We begin by defining the Fourier transform of $u(x)$,

$$\hat{u}(\xi) = (\mathcal{F}u)(\xi) = \int_{\mathbb{R}^d} \exp(-ix\cdot\xi) u(x) dx \tag{11}$$

and its inverse

$$(\mathcal{F}^{-1}\hat{u})(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(ix\cdot\xi)\hat{u}(\xi) d\xi. \tag{12}$$

Next, letting $u^\varepsilon(x)$ be the unique solution to (1) (in $L^2(\mathbb{R}^d)$, say), we introduce its Wigner transform $f^\varepsilon(x,\xi)$ as,

$$f^\varepsilon(x,\xi) := \mathcal{F}_{y\to\xi}\left(u^\varepsilon(x+\varepsilon\frac{y}{2})\,\overline{u^\varepsilon(x-\varepsilon\frac{y}{2})}\right). \tag{13}$$

As desired, f^ε is obviously a quadratic function of u^ε. We mention in passing the following easy equality, valid for any (smooth enough) function u^ε, namely,

$$\int_{\mathbb{R}^d} f^\varepsilon(x,\xi)d\xi = (2\pi)^d|u^\varepsilon|^2(x)\ ,$$

and the right-hand-size is precisely the energy density, a quantity whose asymptotic behaviour we wish to describe. As a second comment concerning the definition (13), we recall that, from (1), $u^\varepsilon(x)$ typically varies on scales of the order ε ($\sim$ wavelength), so that the decorrelations $x+\varepsilon y/2$ and $x-\varepsilon y/2$ in (13) allow to "read" the oscillations of u^ε on this scale, and to get much better informations than the mere weak convergence of u^ε in some weighted L^2 space. These two remarks briefly motivate the introduction of f^ε as in (13), but we refer to [11], [7], [6] among others for a more complete discussion of the Wigner transform.

We now turn to computing the equation satisfied by f^ε. Using the notation,

$$\widehat{f^\varepsilon}(x,y) := \mathcal{F}^{-1}_{\xi\to y} f^\varepsilon(x,\xi) = u^\varepsilon(x+\varepsilon\frac{y}{2})\,\overline{u^\varepsilon(x-\varepsilon\frac{y}{2})}\ ,$$

it is classical to observe the identity,

$$\mathrm{div}_y\nabla_x\widehat{f^\varepsilon}(x,y) = \frac{\varepsilon}{2}\left[\Delta_x u^\varepsilon(x+\varepsilon\frac{y}{2})\,\overline{u^\varepsilon(x-\varepsilon\frac{y}{2})} - u^\varepsilon(x+\varepsilon\frac{y}{2})\,\overline{\Delta_x u^\varepsilon(x-\varepsilon\frac{y}{2})}\right]\ .$$

From this, we get, using (1),

$$\begin{aligned}
&\xi\cdot\nabla_x f^\varepsilon(x,\xi) = -i\mathcal{F}_{y\to\xi}\left(\mathrm{div}_y\nabla_x\widehat{f^\varepsilon}(x,y)\right)\\
&= \varepsilon\ \mathrm{Im}\ \mathcal{F}_{y\to\xi}\left(\mathcal{S}_\varepsilon(x+\varepsilon\frac{y}{2})\overline{u^\varepsilon(x-\varepsilon\frac{y}{2})}\right)\\
&\qquad -\varepsilon\ \mathrm{Im}\ \mathcal{F}_{y\to\xi}\left(\left[\frac{n^2(x+\varepsilon\frac{y}{2})}{\varepsilon^2}+i\frac{\alpha_\varepsilon}{\varepsilon}\right]u^\varepsilon(x+\varepsilon\frac{y}{2})\overline{u^\varepsilon(x-\varepsilon\frac{y}{2})}\right)\ .
\end{aligned}$$

Here we used the identity $\overline{(\mathcal{F}g)(\xi)} = (\mathcal{F}\overline{g})(-\xi)$. Hence, as it is well known (See e.g. [7]), the Wigner transform turns the Helmholtz equation (1) into a "transport equation" with a source term of the form,

$$+\alpha_\varepsilon f^\epsilon(x,\xi) + \xi\cdot\nabla_x f^\epsilon - \frac{i}{2}\frac{1}{(2\pi)^d}\delta n^\varepsilon(x,\xi)\ast_\xi f^\varepsilon = Q_\varepsilon(x,\xi)\ . \tag{14}$$

We use here the notation,

$$\delta n^\varepsilon(x,\xi) := \mathcal{F}_{y\to\xi}\left(\frac{n^2(x+\varepsilon\frac{y}{2}) - n^2(x-\varepsilon\frac{y}{2})}{\varepsilon}\right) . \tag{15}$$

Also, the source term in (14) is given by,

$$Q_\varepsilon(x,\xi) = \varepsilon \,\mathrm{Im}\, \mathcal{F}_{y\to\xi}\left(\mathcal{S}_\varepsilon(x+\varepsilon\frac{y}{2})\,\overline{u^\varepsilon(x-\varepsilon\frac{y}{2})}\right) . \tag{16}$$

Obviously, the source term (16) is the remainder of the right-hand-side $\mathcal{S}_\varepsilon$ in (1), and it will be clear below that Q_ε actually contains most of the difficulties while passing to the limit $\varepsilon \to 0$ in (14).

The above computations give the desired equation for the Wigner transform f^ε of u^ε. We now turn to computing the limiting equation for f^ε. More precisely, assume for the moment that we are able to compute the weak limit $Q(x,\xi)$ of $Q_\varepsilon(x,\xi)$. It is clear from (15) that the following weak convergence holds,

$$\delta n^\varepsilon(x,\xi) \to_{\varepsilon\to 0} \mathcal{F}_{y\to\xi}\left(y\cdot\nabla_x n^2(x)\right) = i(2\pi)^d \nabla_\xi\delta(\xi)\cdot\nabla_x n^2(x) . \tag{17}$$

Hence, at least formally, we readily deduce that f^ε, solution to (14), converges weakly towards the solution $f(x,\xi)$ to the following transport equation,

$$+0f + \xi\cdot\nabla_x f(x,\xi) + \frac{1}{2}\nabla_x n^2(x)\cdot\nabla_\xi f = Q(x,\xi) , \tag{18}$$

where the limiting source Q is still unknown at present.

The term $+0f$ in (18) deserves detailed comments. It is the remainder of the term $+\alpha_\varepsilon f^\varepsilon$ in (14), and it means that we are considering the unique outgoing solution to $\xi\cdot\nabla_x f + \nabla_x n^2(x)\cdot\nabla_\xi f/2 = Q$. In other terms, the solution f to (18) can be defined as the limit when $\eta\to 0^+$ of the solution g_η to $+\eta g_\eta + \xi\cdot\nabla_x g_\eta + \nabla_x n^2(x)\cdot\nabla_\xi g_\eta/2 = Q$. To rephrase this definition, we may say that f is also given by,

$$f(x,\xi) = \int_0^{+\infty} Q(X_0^{-s}(x,\xi),\Xi_0^{-s}(x,\xi))\,ds , \tag{19}$$

where the trajectories $X_0^s(x,\xi)$ and $\Xi_0^s(x,\xi)$ are at the same time the bicharacteristics of the Helmholtz operator $-\Delta_x - n^2(x)$ (as predicted by geometrical

optics), as well as the characteristic curves of the transport equation (18), i.e. the solutions to the ordinary differential system,

$$\begin{aligned} \frac{d}{ds}X_0^s(x,\xi) &= \Xi_0^s(x,\xi)\ , & X_0^0(x,\xi) &= x\ , \\ \frac{d}{ds}\Xi_0^s(x,\xi) &= \left(\frac{1}{2}\nabla_x n^2\right)(X_0^s(x,\xi))\ , & \Xi_0^0(x,\xi) &= \xi\ . \end{aligned} \tag{20}$$

As a conclusion we have shown that, at least formally, the asymptotic behaviour of the Wigner transform f^ε of u^ε is described by the transport equation (18), where Q is the weak limit of Q_ε given by (16), which is assumed to exist for the moment. The subsequent subsections are concerned with the actual computation of Q, the limiting source of energy. To be more precise, the asymptotic process $\varepsilon \to 0$ is not always naturally described in terms of Q, and we shall sometimes consider f_ε itself and directly compute its limit f. Indeed, and as we shall demonstrate below, the propagative regime naturally leads to the propagation of energy along the bicharacteristic curves (20) so considering Q_ε is natural. The resonant regime, on the other hand, leads to a situation where the energy remains entirely localized on the source Γ without being propagated neither outside nor inside Γ, and considering f_ε itself is somehow more natural. Finally the characteristic regime leads to a situation where energy is actually propagated inside the curve Γ, and we chose to consider f_ε as well in this case.

2.2 The propagative regime

This regime is obtained under the assumption,

$$0 \leq p < d\ ,\ \ |\nabla^\tau \phi(y)| < n(y)\ ,\ \forall y \in \Gamma\ . \tag{21}$$

In this case, the manifold Γ on which the source in (1) concentrates does not fill the whole space (assumption $p < d$). Moreover, when x lies on Γ, the two singular sets $\{\xi^2 = n^2(x)\}$ and $\{\xi_x^\tau = \nabla^\tau \phi(x)\}$ do intersect. As a consequence, we are able to compute the limit of Q_ε in this case, and the Wigner measure is indeed described by a Liouville equation with a source term as in (6) and (18). This is shown in the computations below.

As mentioned in the introduction, the following choice for the scaling exponent q (see (2)) is prescribed,

$$q = \frac{3+d+p}{2}\ . \tag{22}$$

2.2.1 The case $p > 0$

In order to compute the limit of Q_ε and thus establish the exact form of the equation (18) obtained in the previous subsection, we will go through some intermediate steps.

Firstly, due to the concentration profile $S((x-y)/\varepsilon)$ in the definition (2) of $\mathcal{S}_\varepsilon$, it is natural to introduce, for any point $\gamma(z)$ lying on Γ ($z \in \mathbb{R}^p$), the rescaled function $w^\varepsilon_{\gamma(z)}(y)$ ($y \in \mathbb{R}^d$) defined as,

$$w^\varepsilon_{\gamma(z)}(y) \equiv \varepsilon^{(d-p-1)/2} u^\varepsilon(\gamma(z) + \varepsilon y) e^{-i\phi(\gamma(z))/\varepsilon}. \tag{23}$$

Obviously, $w^\varepsilon_{\gamma(z)}(\cdot)$ measures the concentration of $u^\varepsilon(x)$ on Γ close to the point $\gamma(z)$, and it also carries the relevant oscillations of u^ε at this point. These two facts allow us to compute the limit of $w^\varepsilon_{\gamma(z)}(\cdot)$ in the next step.

Secondly, $w^\varepsilon_{\gamma(z)}$ carries the right scaling in ε, and it turns out that the source term Q_ε and its asymptotic behaviour are easily expressed in terms of this scaled help function. Indeed, inverting formula (23), we obtain from (16) and the definition of $\mathcal{S}_\varepsilon$,

$$Q_\varepsilon(x,\xi) =$$

$$= -\mathrm{Im}\mathcal{F}_{y\to\xi} \frac{1}{\varepsilon^{\frac{1+d+p}{2}}} \int_{\mathbb{R}^p} A(\gamma(t)) \exp\left(i\frac{\phi(\gamma(t))}{\varepsilon}\right) S\left(\frac{x-\gamma(t)}{\varepsilon} + \frac{y}{2}\right) \overline{u^\varepsilon\left(x - \varepsilon\frac{y}{2}\right)} dt$$

$$= -\mathrm{Im}\mathcal{F}_{y\to\xi} \frac{1}{\varepsilon^d} \int_{\mathbb{R}^p} A(\gamma(t)) S\left(\frac{x-\gamma(t)}{\varepsilon} + \frac{y}{2}\right) \overline{w^\varepsilon_{\gamma(t)}\left(\frac{x-\gamma(t)}{\varepsilon} - \frac{y}{2}\right)} dt. \tag{24}$$

Now, letting $\psi(x,\xi)$ be a smooth test function, and testing (24) against ψ, the change of variables $X = (x-\gamma(t))/\varepsilon$ readily gives the obvious concentration effect on Γ,

$$\langle Q_\varepsilon, \psi\rangle = \int_{\mathbb{R}^{2d}} Q_\varepsilon(x,\xi)\ \psi(x,\xi)\ dx\ d\xi$$

$$= -\mathrm{Im} \int_{\mathbb{R}^{2d}} \mathcal{F}_{y\to\xi} \left(\int_{\mathbb{R}^p} A(\gamma(t)) S\left(X + \frac{y}{2}\right) \overline{w^\varepsilon_{\gamma(t)}\left(X - \frac{y}{2}\right)} dt \right) \psi(\gamma(t)+\varepsilon X, \xi) dX d\xi$$

$$\sim_{\varepsilon\to 0} -\mathrm{Im} \int_{\mathbb{R}^{2d}} \mathcal{F}_{y\to\xi} \left(\int_{\mathbb{R}^p} A(\gamma(t)) S\left(X + \frac{y}{2}\right) \overline{w_{\gamma(t)}\left(X - \frac{y}{2}\right)} dt \right) \psi(\gamma(t), \xi) dX d\xi$$

$$= -\mathrm{Im} \int_{\mathbb{R}^d} \int_{\mathbb{R}^p} A(\gamma(t)) \widehat{S}(\xi) \overline{\widehat{w_{\gamma(t)}}(\xi)} \psi(\gamma(t),\xi) d\xi dt, \tag{25}$$

so that the problem is reduced to determining the limit $w_{\gamma(t)}(y) := \lim_{\varepsilon\to 0} w^\varepsilon_{\gamma(t)}(y)$, or more precisely its Fourier transform $\widehat{w_{\gamma(t)}}(\xi)$. This is done in the next two steps.

Thirdly, we observe from (1) that, for any fixed z, the function $w^\varepsilon_{\gamma(z)}(y)$ satisfies the rescaled equation,

$$+ i\varepsilon\alpha_\varepsilon\, w^\varepsilon_{\gamma(z)}(y) + \Delta_y w^\varepsilon_{\gamma(z)} + n^2(\gamma(z)+\varepsilon y)\, w^\varepsilon_{\gamma(z)} = W^\varepsilon_{\gamma(z)}(y)\ , \tag{26}$$

where the rescaled source term $W^\varepsilon_{\gamma(z)}(y)$ is given by,

$$\begin{aligned} W^\varepsilon_{\gamma(z)}(y) &:= -\varepsilon^{(d-p+3)/2}\mathcal{S}_\varepsilon(\gamma(z)+\varepsilon y)\exp\left(-i\phi(\gamma(z))/\varepsilon\right) \qquad (27)\\ &= -\frac{1}{\varepsilon^p}\int_{\mathbb{R}^p} A(\gamma(t))\, \exp\left(i\frac{\phi(\gamma(t))-\phi(\gamma(z))}{\varepsilon}\right)\, S\left(y+\frac{\gamma(z)-\gamma(t)}{\varepsilon}\right)\, dt\ . \end{aligned}$$

As in the previous subsection, we can now formally pass to the limit in equation (26) up to computing the actual limit of the source term $W^\varepsilon_{\gamma(z)}$. We have from (27),

$$\begin{aligned} W^\varepsilon_{\gamma(z)} &= -\int_{\mathbb{R}^p} A(\gamma(z+\varepsilon t))\exp(i\frac{\phi(\gamma(z+\varepsilon t))-\phi(\gamma(z))}{\varepsilon})\\ &\qquad S(y+\frac{\gamma(z)-\gamma(z+\varepsilon t)}{\varepsilon})dt\\ &\to_{\varepsilon\to 0} -\int_{\mathbb{R}^p} A(\gamma(z))\exp\left(i\langle\nabla\phi(\gamma(z)), D\gamma(z)\cdot t\rangle\right)\, S\left(y - D\gamma(z)\cdot t\right)\, dt \qquad (28)\\ &=: W_{\gamma(z)}(y)\ , \end{aligned}$$

where $D\gamma(z)$ denotes the Jacobian matrix of γ at z and $\langle .,.\rangle$ denotes the scalar product in $\mathbb{R}^d$. Hence, we deduce that $w^\varepsilon_{\gamma(z)}$ formally converges towards the solution $w_{\gamma(z)}$ to,

$$+ i0 w_{\gamma(z)}(y) + \Delta_y w_{\gamma(z)} + n^2(\gamma(z))\, w_{\gamma(z)} = W_{\gamma(z)}(y)\ . \tag{29}$$

By this we mean that $w_{\gamma(z)}$ should be the limit at $\eta\to 0^+$ of the unique solution to $+i\eta w(y) + \Delta_y w + n^2(\gamma(z))w = W_{\gamma(z)}(y)$. In other words, the term $+i0w_{\gamma(z)}$ in (29) specifies the radiation condition at infinity for $w_{\gamma(z)}$, and this means in view of (26) that the radiation condition for $w^\varepsilon_{\gamma(z)}$ given by the term $+i\varepsilon\alpha_\varepsilon w^\varepsilon_{\gamma(z)}$ is somehow preserved along the limiting process.

Before actually computing $w_{\gamma(z)}(y)$ as it is given by (29), we emphasize here that the assumed preservation of the radiation condition as $\varepsilon\to 0$

is far from obvious. More precisely, it is easy to prove that $w_{\gamma(z)}$ satisfies $\Delta_y w_{\gamma(z)} + n^2(\gamma(z)) w_{\gamma(z)} = W_{\gamma(z)}$, even for a variable refraction index $n(x)$. However, one should note that both the regularizing parameter $\varepsilon\alpha_\varepsilon$ *and* the refraction index $n(\gamma(z) + \varepsilon y)$ vary with ε in (26). (The fact that the source term also depends on ε turns out to be less difficult to handle). In particular, while for $\varepsilon > 0$ the refraction index $n(\gamma(z) + \varepsilon y)$ behaves like, say, $n(\infty)$ for large values of y, the limiting refraction index $n(\gamma(z))$ in (29) has constant value $n(\gamma(z)) \neq n(y)$ at infinity in y. As in [2], this turns out to create considerable difficulties in keeping track of the radiation condition at infinity while $\varepsilon \to 0$. Even for a given smooth right-hand-side in (26) (independent of ε), it is a conjecture that (26) actually goes to (29), with the correct radiation condition. In this perspective, the only case where we are able to rigorously keep track of this condition is obtained when the refraction index does not depend on y (see the next sections). In this case indeed, one can explicitely invert the Helmholtz operator, and the problem becomes considerably easier to handle. We may say as a conclusion that, though all the preceeding formal limits may be turned into rigorous ones even for a variable n, the deep difficulty in passing to the limit from (26) to (29) is the reason why we restrict ourselves with a constant n in Section 3.

As a fourth step, when the formal limit (29) is correct, we readily deduce from (29) and (28) the actual value of $\widehat{w_{\gamma(z)}}(\xi)$. Indeed, we have from (29), that

$$\widehat{w_{\gamma(z)}} = \lim_{\eta \downarrow 0} \frac{\widehat{W}_{\gamma(z)}(\xi)}{n^2(\gamma(z)) - |\xi|^2 + i\eta} =: \frac{\widehat{W}_{\gamma(z)}(\xi)}{n^2(\gamma(z)) - |\xi|^2 + i0}\,, \tag{30}$$

where we implicitely use the fact that one can compose the distribution,

$$\frac{1}{x + i0} = \text{p.v.}\left(\frac{1}{x}\right) - i\pi\delta(x) \;\in \mathcal{D}'(\mathbb{R})\,, \tag{31}$$

with the map $\xi \longrightarrow n^2(\gamma(z)) - \xi^2$ from $\mathbb{R}^d$ to $\mathbb{R}$, at least when $n(\gamma(z)) \neq 0$, which is the case here. (We let p.v. denote the principal value in (31).) On the other hand, upon Fourier transforming (28), we obtain,

$$\begin{aligned}\widehat{W_{\gamma(z)}}(\xi) &= -\int_{\mathbb{R}^{p+d}} A(\gamma(z)) e^{i\langle \nabla\phi(\gamma(z)), D\gamma(z)\cdot t\rangle}\; e^{-iy\cdot\xi}\; S(y - D\gamma(z)\cdot t)\; dt\; dy \\ &= -\int_{\mathbb{R}^p} A(\gamma(z)) \exp\left(i\langle \nabla\phi(\gamma(z)) - \xi, D\gamma(z)\cdot t\rangle\right)\; \widehat{S}(\xi)\; dt\end{aligned}$$

$$= -(2\pi)^p A(\gamma(z))\ \widehat{S}(\xi)\ \delta(\xi^\tau_{\gamma(z)} - \nabla^\tau\phi(\gamma(z)))\ , \tag{32}$$

where we used the notation given in the introduction. (For fixed $z \in \mathbb{R}^p$ and $\xi \in \mathbb{R}^d$ we let $\xi = \xi^\tau_{\gamma(z)} + \xi^\nu_{\gamma(z)} \in T_{\gamma(z)}\Gamma + T^\perp_{\gamma(z)}\Gamma$, and the actual value of $\xi^\tau_{\gamma(z)}$ is $\xi^\tau_{\gamma(z)} = {}^\tau D\gamma(z)\cdot\xi$.) Finally, combining (30) and (32) we get

$$\widehat{w_{\gamma(z)}}(\xi) = -\frac{A(\gamma(z))\ \widehat{S}(\xi)\ \delta(\xi^\tau_{\gamma(z)} - \nabla^\tau\phi(\gamma(z)))}{n^2(\gamma(z)) - |\xi|^2 + i0}\ . \tag{33}$$

This, together with (25) and (31), then gives,

$$\langle Q_\varepsilon, \psi\rangle \sim_{\varepsilon\to 0}$$

$$\sim_{\varepsilon\to 0}\ \mathrm{Im}\int_{\mathbb{R}^{d+p}} \frac{(2\pi)^p |A|^2(\gamma(t))|\widehat{S}|^2(\xi)}{n^2(\gamma(t)) - |\xi|^2 - i0}\delta(\xi^\tau_{\gamma(t)} - \nabla^\tau\phi(\gamma(t)))\ \psi(\gamma(t),\xi)\ d\xi\ dt$$

$$= (2\pi)^p\pi \int_{\mathbb{R}^{d+p}} \delta(\xi^\tau_{\gamma(t)} - \nabla^\tau\phi(\gamma(t)))\ \delta(n^2(\gamma(t)) - |\xi|^2) \tag{34}$$
$$|A|^2(\gamma(t))\ |\widehat{S}|^2(\xi)\ \psi(\gamma(t),\xi)\ d\xi\ dt\ .$$

The fifth and last step is merely a summary of the preceeding steps, together with the analysis of the previous subsection. The points (34) and (14) show that, in the case under consideration here, the limiting Wigner distribution $f(x,\xi)$ associated with u^ε satisfies the transport equation with a source term,

$$+0 f(x,\xi) + \xi\cdot\nabla_x f + \frac{1}{2}\nabla_x n^2(x)\cdot\nabla_\xi f = 2^p\pi^{p+1}|\widehat{S}(\xi)|^2 \tag{35}$$

$$\int_\Gamma \delta(x-y)\ \delta(\xi^\tau_y - \nabla^\tau\phi(y))\ \delta\left(n^2(y) - |\xi|^2\right)\ |A(y)|^2 d\sigma(y)\ .$$

This ends the present study and justifies the statement of the introduction.

Remark 1. It is obvious from the definition of these Dirac masses that the product (of distributions in ξ),

$$\delta(\xi^\tau_y - \nabla^\tau\phi(y))\ \delta\left(n^2(y) - |\xi|^2\right) =$$
$$\delta(\xi^\tau_y - \nabla^\tau\phi(y))\ \delta\left(n^2(y) - |\nabla^\tau\phi(y)|^2 - |\xi^\nu_y|^2\right)\ ,$$

is ill-defined when $|\nabla^\tau \phi(y)|^2 = n^2(y)$. This is related to the homogeneity property,

$$\delta(\lambda^2 - \xi^2) = (2\lambda)^{-1}\delta(\lambda - |\xi|) \, ,$$

valid for any $\lambda > 0$, which shows that a singularity occurs as $\lambda \to 0$. This simple observation explains in part why the limit $\varepsilon \to 0$ is different in the propagative regime ($|\nabla^\tau \phi(y)| < n(y)$) and in the characteristic regime ($|\nabla^\tau \phi(y)| \equiv n(y)$).

2.2.2 The case p=0

In fact the same scaling as aboves is correct for $p = 0$, up to dropping A, ϕ, replacing Γ with $\{0\}$, and interpreting (1) as,

$$+ i\alpha_\varepsilon u^\varepsilon + \Delta u^\varepsilon + \frac{n^2(x)}{\varepsilon^2} u^\varepsilon = -\frac{1}{\varepsilon^{(3+d)/2}} S\left(\frac{x}{\varepsilon}\right) , \qquad x \in \mathbb{R}^d. \tag{36}$$

In this case we have $W_0(y) = S(y)$. Consequently, $\widehat{w}_0(\xi) = \widehat{S}(\xi)/(n^2(0) - \xi^2 + i0)$ and we obtain the limiting equation,

$$+ 0f(x,\xi) + \xi \cdot \nabla_x f + \frac{1}{2}\nabla_x n^2(x) \cdot \nabla_\xi f = \pi\delta(x)\, \delta(n^2(0) - \xi^2)\, |\widehat{S}(\xi)|^2 \, . \tag{37}$$

This case was studied in [2].

2.3 The resonant regime

This regime occurs under the assumptions,

$$0 < p \leq d \, , \;\; |\nabla^\tau \phi|(y) > n(y) \, , \; \forall y \in \Gamma \, . \tag{38}$$

(To be complete, we may add that the resonant regime also occurs in the case when the source Γ fills the whole space, $p = d$, and $|\nabla^\tau \phi(y)| < n(y)$, $\forall y \in \Gamma$). The following choice for the scaling exponent q in (2) is prescribed in this regime,

$$q = \frac{4 + d + p}{2} \, . \tag{39}$$

The second restriction in (38) forces the two singular sets (in Fourier space) $\{\xi^2 = n^2(y)\}$ and $\{\xi_y^\tau = \nabla^\tau \phi(y)\}$ ($y \in \Gamma$), to have void intersection. In view of formula (35) established in the previous subsection, this case can not be correctly described by the same transport equation (35) as in the

case $p < d$, $|\nabla^\tau \phi|(y) < n(y)$, since the right-hand-side of (35) then vanishes. Indeed, the product $\delta(\xi^\tau_{\gamma(z)} - \nabla_x \phi(x))\ \delta(n^2(\gamma(z)) - \xi^2)$ formally vanishes under these circumstances. This leads to thinking that the scaling $q = (3+d+p)/2$ used in the propagative regime is actually incorrect in the resonant regime under consideration, hence the need for the appropriate rescaling (39).

To be more precise, using results which are only proven later, we may show that the solution u^ε actually tends to zero as $\varepsilon \to 0$ in the present case when the scaling in (22) is used. Indeed, by Theorem 2 we have, with the notations of the previous subsection,

$$||u^\varepsilon||_{B^*_{\frac{p}{2}}} \leq \varepsilon^{1/2} C ||w^\varepsilon_{\gamma(z)}||_{B^*_{\frac{p}{2}}} \tag{40}$$

and a simple adaptation of Theorem 1 gives in this case,

$$||w^\varepsilon_{\gamma(z)}||_{B^*_{\frac{p}{2}}} \leq C ||S||_{B_{\frac{p}{2}}}. \tag{41}$$

Following the above remarks and indications, we now prove that, upon correctly rescaling (1, 2), it is possible to directly compute the Wigner measure f. As we prove below, the function f *itself* turns out to be supported only on the curve Γ.

Firstly, with the choice of q given in (39), we now wish to study the limit $\varepsilon \to 0$ in,

$$\begin{aligned} +i\frac{\alpha_\varepsilon}{\varepsilon} u^\varepsilon(x) + \Delta_x u^\varepsilon + \frac{n^2(x)}{\varepsilon^2} u^\varepsilon = \\ -\frac{1}{\varepsilon^{2+\frac{p+d}{2}}} \int_\Gamma A(y)\ \exp\left(i\frac{\phi(y)}{\varepsilon}\right)\ S\left(\frac{x-y}{\varepsilon}\right)\ d\sigma(y)\ . \end{aligned} \tag{42}$$

Also, we rescale and extend the definition of w^ε_x to any $x \in \mathbb{R}^d$ by simply setting,

$$w^\varepsilon_x(y) \equiv u^\varepsilon(x + \varepsilon y) \exp(-i\frac{\phi(x)}{\varepsilon}) \qquad (x \in \mathbb{R}^d)\ . \tag{43}$$

At last, we restrict the present analysis to the case where the refraction index is constant, that is,

$$n(x) \equiv n_0\ ,\ \ \forall x \in \mathbb{R}^d\ . \tag{44}$$

The formal analysis we propose below holds analogously for a more general refraction index, though it leads to a questionable limiting procedure in a

non-linear function of u^ε, see below. This is the reason for the present restriction.

Secondly, we now express the Wigner function f^ε itself in terms of the rescaled help function w_x^ε. This gives,

$$f^\epsilon(x,\xi) = \mathcal{F}_{y\to\xi} u^\varepsilon\left(x+\frac{\varepsilon}{2}y\right)\overline{u^\varepsilon\left(x-\frac{\varepsilon}{2}y\right)} = \mathcal{F}_{y\to\xi} w_x^\varepsilon\left(\frac{y}{2}\right)\overline{w_x^\varepsilon\left(-\frac{y}{2}\right)}$$
$$= \frac{2^{2d}}{(2\pi)^d}\left(\hat{w}_x^\varepsilon \star \overline{\hat{w}_x^\varepsilon}\right)(2\xi) = \frac{2^d}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{w_x^\varepsilon}(2\xi-\eta)\,\overline{\widehat{w_x^\varepsilon}(\eta)}\,d\eta\ . \tag{45}$$

Hence we have left to pass to the limit on the non-linear expression $\widehat{w_x^\varepsilon} * \overline{\widehat{w_x^\varepsilon}}$.

Thirdly, we now perform the desired limiting procedure by using the equation for w_x^ε. Indeed, for any fixed $x\in\mathbb{R}^d$ the function $w_x^\varepsilon(y)$ satisfies the equation,

$$+i\varepsilon\alpha_\varepsilon w_x^\varepsilon(y) + \Delta_y w_x^\varepsilon + n^2(x+\varepsilon y)w_x^\varepsilon = W_x^\varepsilon(y)\ ,$$

that is, under the assumption (44),

$$+\,i\varepsilon\alpha_\varepsilon w_x^\varepsilon(y) + \Delta_y w_x^\varepsilon + n_0^2\ w_x^\varepsilon = W_x^\varepsilon(y)\ , \tag{46}$$

where the rescaled source term W_x^ε is given by,

$$W_x^\varepsilon(y) = -\frac{1}{\varepsilon^{\frac{p+d}{2}}}\int_{\mathbb{R}^p} A(\gamma(t))\exp\left(i\frac{\phi(\gamma(t))-\phi(x)}{\varepsilon}\right) S\left(\frac{x-\gamma(t)}{\varepsilon}+y\right)dt. \tag{47}$$

Now, roughly speaking, we way summarize the computations below in the following way: the factor $\varepsilon^{-d/2}S(y+(x-\gamma(t)/\varepsilon)$ in (47) enforces the appearance of the Dirac mass $\delta(x-\gamma(t))$ in (45), while for $x\in\Gamma$, $x=\gamma(z)$, the factor $\varepsilon^{-p/2}\exp(i[\phi(\gamma(t))-\phi(\gamma(z))]/\varepsilon$ generates a term $\exp(i\langle\nabla\phi(\gamma(z)), D\gamma(z)\cdot(z-t)\rangle)$ upon rescaling. These two facts briefly justify the formula (50) below.

The above rough summary is made precise in a fourth step. We write, using (46) together with (47),

$$\widehat{w_x^\varepsilon}(\xi) = -\frac{\varepsilon^{-\frac{p+d}{2}}\widehat{S}(\xi)}{n_0^2-|\xi|^2+i\varepsilon\alpha_\varepsilon} \tag{48}$$
$$\int_{\mathbb{R}^p} A(\gamma(t))\exp\left(i\frac{\phi(\gamma(t))-\phi(x)+\langle x-\gamma(t),\xi\rangle}{\varepsilon}\right)\,dt.$$

This gives in (45),

$$f^\epsilon(x,\xi) = \frac{2^d}{(2\pi)^d \varepsilon^{p+d}} \int_{\mathbb{R}^{d+2p}} \frac{\widehat{S}(2\xi-\eta)\,\overline{\widehat{S}(\eta)}}{[n_0^2-(2\xi-\eta)^2+i\varepsilon\alpha_\varepsilon]\,[n_0^2-\eta^2-i\varepsilon\alpha_\varepsilon]} \quad (49)$$
$$A(\gamma(t))\,A(\gamma(t'))\exp\left(i\frac{\phi(\gamma(t))-\phi(\gamma(t'))+\langle\gamma(t')-\gamma(t),\eta\rangle}{\varepsilon}\right)$$
$$\exp\left(2i\frac{\langle x-\gamma(t),\xi-\eta\rangle}{\varepsilon}\right)\,dt\,dt'\,d\eta\ .$$

Now one has to take care of the highly oscillating terms $\exp(i\cdot/\varepsilon)$ in (49). This can be done by using the non-stationary phase lemma. Alternatively, the natural rescaling,

$$t' \mapsto t+\varepsilon t', \qquad \eta \mapsto \xi+\varepsilon\eta/2\ ,$$

gives the desired concentration phenomena in the variables t' and η,

$$f^\epsilon(x,\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d+2p}} \frac{\widehat{S}(\xi-\varepsilon\eta/2)\,\overline{\widehat{S}(\xi+\varepsilon\eta/2)}}{[n_0^2-(\xi-\varepsilon\eta/2)^2+i\varepsilon\alpha_\varepsilon]\,[n_0^2-(\xi+\varepsilon\eta/2)^2-i\varepsilon\alpha_\varepsilon]}$$
$$A(\gamma(t))\,A(\gamma(t+\varepsilon t'))\ \exp\left(-i\langle x-\gamma(t),\eta\rangle\right)$$
$$\exp\left(i\frac{\phi(\gamma(t))-\phi(\gamma(t+\varepsilon t'))+\langle\gamma(t+\varepsilon t')-\gamma(t),\xi+\varepsilon\eta/2\rangle}{\varepsilon}\right)\,dt\,dt'\,d\eta\ ,$$

so that,

$$f^\varepsilon(x,\xi) \rightharpoonup \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d+2p}} \frac{|\widehat{S}|^2(\xi)}{[n_0^2-\xi^2]^2}\ |A|^2(\gamma(t))\ \exp\left(-i\langle x-\gamma(t),\eta\rangle\right)$$
$$\exp\left(i\left[\langle D\phi(\gamma(t)),D\gamma(t)\cdot t'\rangle-\langle D\gamma(t)\cdot t',\xi\rangle\right]\right)\ dtdt'd\eta$$
$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{d+2p}} \frac{|\widehat{S}|^2(\xi)}{[n_0^2-\xi^2]^2}\ |A|^2(\gamma(t))\ \exp\left(-i\langle x-\gamma(t),\eta\rangle\right)$$
$$\exp\left(-i\langle\xi^\tau_{\gamma(t)}-\nabla^\tau\phi(\gamma(t)),t'\rangle\right)\ dtdt'd\eta\ .$$

Upon explicitely performing the $d\eta$ and dt' integrations we thus obtain the weak limit f of the Wigner function f^ε,

$$f(x,\xi) = (2\pi)^p \frac{|\widehat{S}|^2(\xi)}{[n_0^2-\xi^2]^2} \int_\Gamma \delta(x-y)\delta\left(\xi^\tau_y-\nabla^\tau\phi(y)\right)|A|^2(y)d\sigma(y)\ . \quad (50)$$

This ends the analysis of the resonant regime. As we see, the limiting value f is directly computed here, up to the restriction (44) at least. The semi-classical measure f is entirely supported on positions $x\in\Gamma$, as well as frequencies ξ having prescribed tangential part $\xi^\tau_{\gamma(t)}=\nabla^\tau\phi(\gamma(t))$, as it was already the case while computing the limiting source term Q in the previous subsection.

2.4 The characteristic regime

This case is the borderline between the propagative and the resonant regimes. It is obtained by assuming,

$$\nabla^\tau \phi(y) \equiv n(y) \ , \qquad \forall y \in \Gamma \ . \tag{51}$$

In this case indeed, for any point $y \in \Gamma$, the corresponding sets $\{\xi_y^\tau = \nabla^\tau \phi(y)\}$ and $\{\xi^2 = n^2(y)\}$, where the frequencies concentrate in Fourier space, intersect *tangentially*. Roughly speaking, and in analogy with (34), the product $\delta(\xi_y^\tau - \nabla^\tau \phi(y))\ \delta(\xi^2 - n^2(y))$ naturally arising along the high frequency limit is therefore more singular than in the case where the two singular sets intersect *transversely*. See the remark at the end of Section 2.2.1. The apriori estimates in Section 3 are actually no longer valid. This subsection is devoted to showing that, upon correctly rescaling $\mathcal{S}_\varepsilon$, one can directly compute the actual limiting value f of f_ε. Again, we restrict ourselves in this subsection to the case of a constant refraction index,

$$n(x) \equiv n_0 \ .$$

Also, it turns out that, for sake of homogeneity, we have to impose a non-vanishing regularising parameter α to get a meaningful limit in this case,

$$\alpha_\varepsilon \to \alpha > 0 \ .$$

Without this assumption, the Wigner function f^ε might diverge. Indeed, when $\alpha_\varepsilon \to 0$, one has to replace the prefactor ε in (58) by $\varepsilon\alpha_\varepsilon$, and easy computations lead to a (possibly) diverging f^ε, depending on the specific values of ϕ, γ. We refer also to Remark 3, where the constant $C(\alpha)$ is easily seen to blow-up when $\alpha \to 0$.

However, these prescriptions are not enough, and we have to make two last simplifying assumptions, namely that the phase ϕ and the parametrization γ of the manifold Γ are *linear*. In the general case indeed where ϕ and/or γ are true non-linear functions of their arguments, additional oscillations occur, leading to an intricate limiting procedure. Under these assumptions at least, the correct scaling exponent q in (2) for the characteristic regime turns out to be,

$$q = \frac{4 + 3d + p}{4} \ . \tag{52}$$

It is clear from the computations below that the present scaling is imposed by the singularity created by the product $\delta(\xi_y^\tau - \nabla^\tau \phi(y))\ \delta(\xi^2 - n^2(y))$ at the point where the two surfaces intersect tangentially, see (57). We now turn to the detailed analysis of the present case.

Without loss of generality, the simplifying assumption we need may be written,

$$\begin{aligned} &\phi(x) = \nabla\phi \cdot x\ ,\ \ \forall x \in \mathbb{R}^d\ , \\ &\gamma(t) = (t,0) \in \mathbb{R}^p \times \mathbb{R}^{d-p}\ ,\ \ \forall t \in \mathbb{R}^p\ , \end{aligned} \tag{53}$$

where $\nabla\phi \in \mathbb{R}^d$ is a *fixed* vector. It turns out that the complete treatment of the present case requires the additional constraint that the source Γ has large enough a dimension, namely,

$$p > d - 4\ , \tag{54}$$

and we refer to Remark 3 at the end of this section for this technical point. We are now interested in passing to the limit in the equation,

$$\begin{aligned} &+i\frac{\alpha_\varepsilon}{\varepsilon}u^\varepsilon(x) + \Delta_x u^\varepsilon + \frac{n^2(x)}{\varepsilon^2}u^\varepsilon = \\ &\qquad -\frac{1}{\varepsilon^{\frac{3d+p}{4}+1}}\ \int_\Gamma A(y)\ \exp\left(i\frac{\phi(y)}{\varepsilon}\right)\ S\left(\frac{x-y}{\varepsilon}\right)\ d\sigma(y)\ . \end{aligned} \tag{55}$$

As a second step, we compute the actual value of the Wigner measure f^ε (before any scaling limit), as in the resonant case treated in Section 2.3. The computations are the same, since only the scaling differs in the present case. Upon testing $f^\varepsilon(x,\xi)$ against some smooth test function $\psi(x,\xi)$, we readily obtain,

$$\begin{aligned} \langle f^\epsilon, \psi\rangle = &\frac{2^d}{(2\pi)^d \varepsilon^{\frac{3d+p}{2}-2}} \int_{\mathbb{R}^{3d+2p}} \psi(x,\xi)\ \exp\left(2i\frac{\langle x-\gamma(t), \xi-\eta\rangle}{\varepsilon}\right) \\ &A(\gamma(t))\ \overline{A}(\gamma(t'))\exp\left(i\frac{\phi(\gamma(t)) - \phi(\gamma(t')) + \langle \gamma(t') - \gamma(t), \eta\rangle}{\varepsilon}\right) \\ &\frac{\widehat{S}(2\xi-\eta)\ \overline{\widehat{S}(\eta)}}{[n_0^2 - (2\xi-\eta)^2 + i\varepsilon\alpha_\varepsilon]\ [n_0^2 - \eta^2 - i\varepsilon\alpha_\varepsilon]}\ dt\ dt'\ d\eta\ d\xi\ dx\ , \end{aligned} \tag{56}$$

where γ and ϕ are at present still general functions.

Before going further, we now comment as a third step on the formal asymptotic behaviour of f^ε as it is given in (56). As in the previous subsection, the highly oscillating phases in (56) are expected to cause, among other things, the concentration of the above integral on the set $\xi = \eta$, $\nabla^\tau \phi(x) = \xi_x^\tau$. What differs from the previous case is the fact that a new singularity develops in the denominator of (56). Indeed, due to the concentration on those sets and a characteristic phase $|\nabla^\tau \phi|^2 \equiv n_0^2$, the denominator should behave like,

$$[n_0^2 - (\xi_x^\tau)^2 - (\xi_x^\nu)^2 + i\varepsilon\alpha_\varepsilon]\,[n_0^2 - (\xi_x^\tau)^2 - (\xi_x^\nu)^2 - i\varepsilon\alpha_\varepsilon]$$

$$\sim_{\varepsilon\to 0} [-(\xi_x^\nu)^2 + i\varepsilon\alpha]\,[-(\xi_x^\nu)^2 - i\varepsilon\alpha] = [(\xi_x^\nu)^4 + \varepsilon^2\alpha^2]\,, \tag{57}$$

Hence, we are a priori led to manipulate the distribution,

$$\frac{1}{(\xi_x^\nu)^4 + \varepsilon^2\alpha^2} \sim_{\varepsilon\to 0} \ \mathrm{const}(\alpha)\ \varepsilon^{\frac{d-p}{2}-2}\ \delta(\xi_x^\nu)\ \in \mathcal{D}'(\mathbb{R}^{d-p})\,,$$

for some constant depending on $\alpha = \lim_{\varepsilon\to 0}\alpha_\varepsilon$. The new factor $\varepsilon^{(d-p)/2-2}$ created by the additional singularity of the denominator is the reason for our rescaling of the source term $\mathcal{S}_\varepsilon$ in the characteristic regime.

The fourth step is now devoted to the precise description of the above rough analysis. Following the intuition already described, we naturally consider the change of variables,

$$\xi \mapsto \eta + \varepsilon\frac{\xi}{2}, \qquad \eta^\tau_{\gamma(t)} \mapsto \nabla^\tau\phi(\gamma(t)) + \varepsilon\eta^\tau_{\gamma(t)}, \qquad \eta^\nu_{\gamma(t)} \mapsto \varepsilon^{1/2}\eta^\nu_{\gamma(t)}, \tag{58}$$

in (56). The first two mappings take care of the (expected) concentration towards $\xi = \eta$, $\eta^\tau_{\gamma(t)} = \nabla^\tau\phi(\gamma(t))$. Note also that the rescaling of the normal part of η by a factor $\varepsilon^{1/2}$ is the natural one. With this scaling indeed, the term $(\xi^\nu_{\gamma(t)})^4 + \varepsilon^2\alpha^2$ which, by (57), is expected to come up in the limit is nicely rescaled into $\varepsilon^2\,[(\xi^\nu_{\gamma(t)})^4 + \alpha^2]$. Now the change of variables (58) transforms

(56) into

$$\langle f^\epsilon, \psi\rangle = \frac{\varepsilon^{d+p+\frac{d-p}{2}}}{(2\pi)^d \varepsilon^{\frac{3d+p}{2}-2}\varepsilon^2} \int_{\mathbb{R}^{3d+2p}} \psi\left(x, \eta+\varepsilon\frac{\xi}{2}\right)\ A(\gamma(t))\ \overline{A}(\gamma(t')) \tag{59}$$
$$\exp\left(i\langle x-\gamma(t),\xi\rangle\right)\ \exp\left(i\langle\gamma(t')-\gamma(t),\eta^\tau_{\gamma(t)}\rangle\right)$$
$$\exp\left(i\frac{\phi(\gamma(t))-\phi(\gamma(t'))+\langle\gamma(t')-\gamma(t),\nabla^\tau\phi(\gamma(t))\rangle}{\varepsilon}+i\frac{\langle\gamma(t')-\gamma(t),\eta^\nu_{\gamma(t)}\rangle}{\sqrt{\varepsilon}}\right)$$
$$\frac{\widehat{S}(\eta+\varepsilon\xi)\ \overline{\widehat{S}(\eta)}}{[\frac{n_0^2-(\eta+\varepsilon\xi)^2}{\varepsilon}+i\alpha_\varepsilon][\frac{n_0^2-\eta^2}{\varepsilon}-i\alpha_\varepsilon]}\ dt\ dt'\ d\eta\ d\xi\ dx\ ,$$

where we implicitely use the notation,

$$\eta = \nabla^\tau\phi(\gamma(t)) + \varepsilon\eta^\tau_{\gamma(t)} + \varepsilon^{1/2}\eta^\nu_{\gamma(t)}\ . \tag{60}$$

Note that the denominators in (59) are of order one, since, by (60) and the assumption $|\nabla^\tau\phi| \equiv n_0$,

$$\frac{n_0^2-(\eta+\varepsilon\xi)^2}{\varepsilon} = -(\eta^\nu_{\gamma(t)})^2 - 2\langle\nabla^\tau\phi(\gamma(t)),\xi^\tau_{\gamma(t)}+\eta^\tau_{\gamma(t)}\rangle + O(\varepsilon^{1/2})\ ,$$
$$\frac{n_0^2-\eta^2}{\varepsilon} = -(\eta^\nu_{\gamma(t)})^2 - 2\langle\nabla^\tau\phi(\gamma(t)),\eta^\tau_{\gamma(t)}\rangle + O(\varepsilon)\ .$$

It remains therefore to take care of the last oscillating term $\exp(i\cdot/\varepsilon + i\cdot/\varepsilon^{1/2})$ in (59). In the particular case where ϕ and γ are linear (assumption (53)), this term cancels out, since its value is,

$$\frac{\langle\nabla^\tau\phi, t-t'\rangle + \langle t'-t, \nabla^\tau\phi\rangle}{\varepsilon} + \frac{\langle t'-t,\eta^\nu\rangle}{\sqrt{\varepsilon}} \equiv 0\ .$$

This is the reason for the assumption of a linear phase we made above. Hence we can easily pass to the limit in this particular case, and by (53) we obtain,

$$\langle f^\epsilon,\psi\rangle \to \frac{1}{(2\pi)^d}\int_{\mathbb{R}^{3d+2p}} dt\ dt'\ d\eta\ d\xi\ dx\ \psi(x,\nabla^\tau\phi)\ A(t)\ \overline{A}(t') \tag{61}$$
$$\exp\left(i\langle x^\tau-t,\xi^\tau\rangle\right)\ \exp\left(i\langle x^\nu,\xi^\nu\rangle\right)\ \exp\left(i\langle t'-t,\eta^\tau\rangle\right)$$
$$\frac{|\widehat{S}|^2(\nabla^\tau\phi)}{[-(\eta^\nu)^2-2\langle\nabla^\tau\phi,\xi^\tau+\eta^\tau\rangle+i\alpha][-(\eta^\nu)^2-2\langle\nabla^\tau\phi,\eta^\tau\rangle-i\alpha]}$$
$$= \frac{1}{(2\pi)^p}\int_{\mathbb{R}^{4p+d}} dt\ dt'\ d\xi^\tau\ d\eta\ dx^\tau\ \psi(x^\tau,\nabla^\tau\phi)\ A(t)\ \overline{A}(t')$$
$$\exp\left(i\langle x^\tau-t,\xi^\tau\rangle\right)\ \exp\left(i\langle t'-t,\eta^\tau\rangle\right)$$
$$\frac{|\widehat{S}|^2(\nabla^\tau\phi)}{[-(\eta^\nu)^2-2\langle\nabla^\tau\phi,\xi^\tau+\eta^\tau\rangle+i\alpha][-(\eta^\nu)^2-2\langle\nabla^\tau\phi,\eta^\tau\rangle-i\alpha]}\ ,$$

where the last equality is obtained by explicitely performing the $d\xi^\nu$ integration, which gives $(2\pi)^{d-p}\delta(x^\nu)$. Now the change of variables $\xi^\tau \mapsto \xi^\tau - \eta^\tau$ gives,

$$\begin{aligned}
\langle f, \psi \rangle &= (2\pi)^{-p} \int_{\mathbb{R}^{4p+d}} dt\ dt'\ d\xi^\tau\ d\eta\ dx^\tau\ \psi(x^\tau, \nabla^\tau\phi)\ A(t)\ \overline{A}(t') \\
&\qquad \exp\left(i\langle x^\tau - t, \xi^\tau\rangle\right)\ \exp\left(-i\langle x^\tau - t', \eta^\tau\rangle\right) \\
&\qquad \frac{|\widehat{S}|^2(\nabla^\tau\phi)}{[-(\eta^\nu)^2 - 2\langle\nabla^\tau\phi, \xi^\tau\rangle + i\alpha][-(\eta^\nu)^2 - 2\langle\nabla^\tau\phi, \eta^\tau\rangle - i\alpha]} \\
&= (2\pi)^{-p} \int_{\mathbb{R}^p} dx^\tau\ \psi(x^\tau, \nabla^\tau\phi) \\
&\qquad \int_{\mathbb{R}^{d-p}} d\eta^\nu\ \left| \widehat{S}(\nabla^\tau\phi) \int_{\mathbb{R}^{2p}} dt\ d\eta^\tau\ \frac{A(t)\ \exp\left(i\langle x^\tau - t, \eta^\tau\rangle\right)}{[-(\eta^\nu)^2 - 2\langle\nabla^\tau\phi, \eta^\tau\rangle + i\alpha]} \right|^2 .
\end{aligned}$$

As a fifth and last step, we conclude by saying that the final value of f is,

$$f(x,\xi) = (2\pi)^{-p}\delta(x^\nu)\ \delta(\xi^\tau - \nabla^\tau\phi)\ \delta(\xi^\nu)\ \mathcal{A}(x^\tau)\ , \tag{62}$$

where the energy density $\mathcal{A}(x^\tau)$ is,

$$\begin{aligned}
\mathcal{A}(x^\tau) &= \int_{\mathbb{R}^{d-p}} d\eta^\nu\ \left| \widehat{S}(\nabla^\tau\phi) \int_{\mathbb{R}^{2p}} dt\ d\eta^\tau\ \frac{A(t)\ \exp\left(i\langle x^\tau - t, \eta^\tau\rangle\right)}{[-(\eta^\nu)^2 - 2\langle\nabla^\tau\phi, \eta^\tau\rangle + i\alpha]} \right|^2 \\
&= \int_{\mathbb{R}^{d-p}} d\eta^\nu\ \left| \widehat{S}(\nabla^\tau\phi) \int_{\mathbb{R}^p} d\eta^\tau\ \frac{\widehat{A}(\eta^\tau)\ \exp\left(i\langle x^\tau, \eta^\tau\rangle\right)}{[-(\eta^\nu)^2 - 2\langle\nabla^\tau\phi, \eta^\tau\rangle + i\alpha]} \right|^2 ,
\end{aligned} \tag{63}$$

and $\widehat{A}$ denotes the Fourier transform of A. The characteristic regime thus leads to a situation where the energy is entirely concentrated on the curve Γ in the space variable x, and on rays with purely tangential velocity $\xi^\tau_x = \nabla^\tau\phi(x)$, $\xi^\nu = 0$. The constraint $\xi^\tau = \nabla^\tau\phi(x)$ is a common feature with the propagative and resonant regimes, whereas the second constraint $\xi^\nu = 0$ is specific to the characteristic regime. Nevertheless, this characteristic regime is of propagative type because $f(x,\xi)$ depends in a non-local way upon the data, though it cannot be described by the mere propagation along bicharacteristics as in the propagative case. We make this point more precise below.

Remark 2. Energy propagation inside the source Γ.
Though in the present regime, the energy does not propagate *outside* the

source Γ, we may observe that, in some sense the energy density f is given as the average of a complex amplitude propagating along the source Γ *itself*.

More precisely, we write the energy density $\mathcal{A}$ as the following average over the normal direction,

$$\mathcal{A}(x^\tau) := \int_{\mathbb{R}^{d-p}} d\eta^\nu \ |a(x^\tau, \eta^\nu)|^2 \ ,$$

with the obvious notation for $a(x^\tau, \eta^\nu)$,

$$a(x^\tau, \eta^\nu) = \widehat{S}(\nabla^\tau \phi) \int_{\mathbb{R}^p} d\eta^\tau \ \frac{\widehat{A}(\eta^\tau) \ \exp\left(i\langle x^\tau, \eta^\tau\rangle\right)}{[-(\eta^\nu)^2 - 2\langle \nabla^\tau \phi, \eta^\tau\rangle + i\alpha]} \ .$$

We prove below that the complex amplitude $a(x^\tau, \eta^\tau)$ is the solution of the transport equation posed on the source Γ,

$$[\alpha + i(\eta^\nu)^2] a(x^\tau, \eta^\nu) + 2\nabla^\tau \phi \cdot \nabla_{x^\tau} a = i(2\pi)^p \widehat{S}(\nabla^\tau \phi) A(x^\tau) \ . \tag{64}$$

(Recall that, from the assumption (53), Γ is the linear subspace parameterized by x^τ).) This proves that the amplitude A in the source term $\mathcal{S}_\varepsilon$ acts in (64) as a source on Γ (weighted according to the concentration profile S). In this way, a collection of complex amplitudes $a(x^\tau, \eta^\nu)$ indexed by normal rays η^ν is built up, each one being propagated along Γ in the direction of $\nabla^\tau \phi$. Each such complex amplitude carries the energy $|a(x^\tau, \eta^\nu)|^2$, and the energy density f at the point x^τ and in the direction $\xi^\tau = \nabla^\tau \phi$ is then given from (62) as the total contribution $\int d\eta^\nu \ |a(x^\tau, \eta^\nu)|^2$.

Let us now turn to the proof of (64), and for sake of simplicity, we restrict ourselves to the case $\nabla^\tau \phi \equiv \boldsymbol{e}_1$, the general case being easily deduced then. The proof entirely relies on the well known formula in one dimension,

$$\int_{\mathbb{R}} d\xi \ \frac{\exp(ix\xi)}{\xi - z} = (2i\pi) \ \exp(izx) \ \mathbf{1}(x > 0) \ , \ \ \forall \ \mathrm{Im}\, z > 0 \ . \tag{65}$$

Let $x_{2,p} \in \mathbb{R}^{p-1}$ denote the vector formed by components $x_2, \ldots, x_p$ of x. Using (65) in the explicit formula for a, we can then indeed write,

$$a(x^\tau, \eta^\nu) = (2\pi)^{p-1} \widehat{S}(\boldsymbol{e}_1) \int_{\mathbb{R}^2} dt_1 \ d\eta_1 \ \frac{A(t_1, x_{2,p}, 0) \ \exp\left(i(x_1 - t_1)\eta_1\right)}{[-(\eta^\nu)^2 - 2\eta_1 + i\alpha]}$$

$$= \frac{-i}{2} (2\pi)^p \widehat{S}(\boldsymbol{e}_1) \int_{\mathbb{R}} dt_1 \ A(x_1 - t_1, x_{2,p}, 0) \ \exp\left(\frac{-t_1(\alpha + i(\eta^\nu)^2)}{2}\right) \ \mathbf{1}(t_1 > 0)$$

$$= -i(2\pi)^p \widehat{S}(\boldsymbol{e}_1) \int_0^{+\infty} dt_1 \; A(x_1 - 2t_1, x_{2,p}, 0) \; \exp\left(-t_1(\alpha + i(\eta^\nu)^2)\right) \; ,$$

so that a obviously satisfies,

$$[\alpha + i(\eta^\nu)^2] a(x^\tau, \eta^\nu) + 2\boldsymbol{e}_1 \cdot \nabla_{x^\tau} a = i(2\pi)^p \widehat{S}(\boldsymbol{e}_1) A(x^\tau) \; ,$$

and (64) is easily deduced in the case of a general value of $\nabla^\tau \phi$ with $|\nabla^\tau \phi| = n_0$.

Remark 3. Convergence of the integral in (63).
Note that the integral defining the amplitude $\mathcal{A}$ converges. Indeed, upon rotating the axes, we may again assume $\nabla^\tau \phi \equiv \boldsymbol{e}_1$, the first vector of the basis, so that by virtue of the Fourier inversion formula equation (63) gives,

$$\mathcal{A}(x^\tau) =$$

$$(2\pi)^{2(p-1)} \int_{\mathbb{R}^{d-p}} d\eta^\nu \; \left| \widehat{S}(\nabla^\tau \phi) \int_{\mathbb{R}} d\eta_1 \; \frac{\left(\mathcal{F}_{x_1 \to \eta_1} A\right)(\eta_1, x_{2,p}, 0) \; \exp\left(i x_1 \eta_1\right)}{[-(\eta^\nu)^2 - 2\eta_1 + i\alpha]} \right|^2 .$$

Using the assumed smoothness and fast decay of the profile A under the form $\left|\left(\mathcal{F}_{x_1 \to \eta_1} A\right)(\eta_1, x_{2,p}, 0)\right| \leq C\langle \eta_1 \rangle^{-N}$ for some large enough exponent N, we can upper bound,

$$|\mathcal{A}(x^\tau)| \leq C(\alpha) \int_{\mathbb{R}^{d-p}} d\eta^\nu \; \left| \int_{\mathbb{R}} \frac{d\eta_1}{\langle \eta_1 \rangle^N \langle \eta^1 + (\eta^\nu)^2 \rangle} \right|^2$$

$$\leq C(\alpha) \int_{\mathbb{R}^{d-p}} d\eta^\nu \; \left| \frac{1}{\langle \eta^\nu \rangle^2} \int_{\mathbb{R}} \frac{d\eta_1}{\langle \eta_1 \rangle^{N-1}} \right|^2 < +\infty$$

at least for a manifold Γ with dimension p satisfying, $4 > d - p$. This is where the constraint (54) on the dimension p enters.

3 Precise results in the propagative regime

In this section we show precise convergence results justifying the formal analysis of Section 2, at least for a more restricted form of (1). In particular, we only consider the case where Γ is a linear manifold and the index of refraction is constant. Moreover, in order to simplify the proof, we also confine ourselves to the case when the amplitude A has compact support, corresponding to a compact source. Also, we restrict ourselves to rigorously

justifying the convergence of the Wigner measure f^ε in the mere propagative regime $|\nabla^\tau \phi(x)| < n(x)$ (actually we use the more stringent assumption $|\nabla^\tau \phi(x)| < n(x) - \delta$, see H3 below).

We wish to mention that the same kind of approach proposed here allows to treat the characteristic and resonant regimes as well, at least for sources Γ having vanishing curvature. Also, and as mentioned in the introduction, the assumption that Γ is a linear subspace, i.e. has vanishing curvature, may be removed following essentially the same approach, and we may treat the three regimes (propagative/resonant/characteristic) for true manifolds Γ. This last work is in progress [3].

3.1 Convergence of the Wigner measure f^ε

As mentioned above, the rigorous convergence of f^ε is restricted to various assumptions on the index of refraction n, the parametrization γ, the concentration profile S, the amplitude A, and the phase ϕ. Before stating these restrictions, we need to introduce the following weighted L^2 spaces in which uniform estimates on $w^\varepsilon_{\gamma(z)}$ and u^ε are at hand.

Definition 1. *Define the ball* $C_{-1} = \{x \in \mathbb{R}^d \mid |x| \leq 1/2\}$ *and, for any* $j \in \mathbb{N}$, *define the annulae* $C_j = \{x \in \mathbb{R}^d \mid 2^{j-1} \leq |x| \leq 2^j\}$. *Then, for any* $s \in \mathbb{R}$, *the space* $B_s(\mathbb{R}^d)$ *is defined as the space of functions* $u \in L^2_{\rm loc}(\mathbb{R}^d)$ *satisfying,*

$$\|u\|_{B_s} \equiv \sum_{j=-1}^{\infty} 2^{js} \left(\int_{C_j} |u|^2(x) dx \right)^{1/2} < \infty. \tag{66}$$

The dual B_s^* *of* B_s *is the space of functions* $u \in L^2_{\rm loc}(\mathbb{R}^d)$ *such that,*

$$\|u\|_{B_s^*} \equiv \sup_{j \geq -1} 2^{-js} \left(\int_{C_j} |u|^2 dx \right)^{1/2} < \infty. \tag{67}$$

This definition is reminiscent of the usual Besov spaces, where similar norms are given in the Fourier space. Also, up to defining the more usual weighted L^2 spaces $L^2_s(\mathbb{R}^d)$ as $L^2(\mathbb{R}^d, \langle x \rangle^{2s} dx)$ ($s \in \mathbb{R}$), we may observe that,

$$L^2_{s+\lambda} \subset B_s \subset L^2_s, \qquad L^2_{-s} \subset B_s^* \subset L^2_{-(s+\lambda)}, \qquad \forall s \in \mathbb{R}, \qquad \forall \lambda > 0, \tag{68}$$

with continuous embedding. We finally mention here that we may as well work with the homogeneous version of the spaces B_s, B_s^*, where the ball $|x| \leq 1/2$ is not treated separately. Instead we can define C_j as the annulus $\{2^{j-1} \leq |x| \leq 2^j\}$ for *any* $j \in \mathbb{Z}$, and give the homogeneous space B_s the norm $\sum_{j\in\mathbb{Z}} 2^{js}\|u\|_{L^2(C_j)}$, and analogously for the homogeneous version of B_s^*. The reason for our introduction of B_s mainly comes from the standard fact that the constant coefficient inverse Helmholtz operator $(-\Delta_x - n_0^2 \pm i0)^{-1}$ sends B_s into B_s^* for any $s \geq 1/2$, see [1, 9]. Further useful properties of these spaces are given in Section 3.2 below.

Now, the exact assumptions we need are the following:

(H1) The index of refraction is constant,

$$n(x) \equiv n_0, \qquad \forall x \in \mathbb{R}^d\,. \tag{69}$$

(H2) The source Γ is a linear subspace, and more precisely,

$$\gamma(x) = \gamma(x^\tau, x^\nu) \equiv (x^\tau, 0), \qquad \forall x \in \mathbb{R}^d\,. \tag{70}$$

(H3) The phase ϕ is uniformly non-characteristic, in the sense that there exists a (small) $\delta > 0$, such that,

$$|\nabla^\tau \phi(\,x)| \leq n_0 - \delta < n_0, \qquad \forall x \in \mathbb{R}^d\,. \tag{71}$$

The assumption (H3) ensures indeed that the propagative regime is obtained uniformly in space, as was formally shown in Section 2.2 above.

(H4) The profiles ϕ, A has the following regularity

$$A \in C_c^{p+2}(\mathbb{R}^p), \qquad \phi \in C^{p+2}(\mathbb{R}^p). \tag{72}$$

(H5) The source function S decays at infinity such that

$$S \in B_{\frac{p+1}{2}}(\mathbb{R}^d). \tag{73}$$

With these assumptions, the following apriori estimates hold true.

Theorem 1. *Let w_z^ε be the solution to (26) under the assumptions H1–H5. Alternatively, we could replace H4 by $A = \text{const}$ and $\nabla\phi = \text{const}$. Then there is a positive constant C, independent of z, ε and α_ε such that*

$$\|w_z^\varepsilon\|_{B^*_{\frac{p+1}{2}}} \leq C\|S\|_{B_{\frac{p+1}{2}}}. \tag{74}$$

The estimate (74) is the key estimate of the present paper. It is obtained by cutting the Fourier space $\xi \in \mathbb{R}^d$ into pieces, according to whether the ray vector ξ satisfies $\xi^2 = n_0^2$ and/or $\xi^\tau = \nabla^\tau\phi(x)$ when $x \in \Gamma$. In this picture, the uniform assumption (71) is a geometric one. Also, the decay assumption (73) is the natural one since we shall need to take restrictions of $\mathcal{F}(w_z^\varepsilon)(\xi)$ over these subspaces of codimension 1 and $d-p$ respectively, an operation which is precisely allowed in the spaces B_s for s "large enough". We refer to the proof given in Section 4.1, as well as property (84) for details.

It turns out that the bound (74) immediately gives the following bound on u^ε.

Theorem 2. *Let u^ε be the solution to (1) under the assumptions H1–H5 and $\lambda < (p+1)/2$ a real number. Then there is a positive constant C independent of ε and z such that,*

$$\|u^\varepsilon\|_{B^*_{\frac{p+1}{2}-\lambda}} \leq C\varepsilon^\lambda \langle\gamma(z)\rangle^{\frac{p+1}{2}-\lambda} \|w_z^\varepsilon\|_{B^*_{\frac{p+1}{2}-\lambda}}. \tag{75}$$

Finally, the convergence of the Wigner measure f^ε is an easy consequence of (75) as well. We have

Theorem 3. *Assume that H1–H5 hold and that f^ϵ is the Wigner transform of u^ε, the solution to (1). Define also, for any $t \in \mathbb{R}$, the space X_t of test functions ϕ as the completion of the Schwarz space $\mathcal{S}(\mathbb{R}^{2d})$ under the norm,*

$$\|\phi\|_{X_t} := \int_{\mathbb{R}^d} dy \sup_{x\in\mathbb{R}^d} \langle|x| + |y|\rangle^\tau \left|\left(\mathcal{F}_{\xi\to y}\phi\right)(x,y)\right|. \tag{76}$$

The space X_t is a Banach space with dual X_t^*. Under these circumstances, the following holds.

(i) For any $\lambda > 0$, the sequence f^ε is uniformly bounded in $X^*_{p+1+\lambda}$, i.e. there exists a constant C_λ, independent of ε, such that,

$$\|f^\varepsilon\|_{X^*_{p+1+\lambda}} \leq C_\lambda \, \|S\|^2_{B_{\frac{p+1}{2}}} \, . \tag{77}$$

(ii) There exists a subsequence of $\{f^\epsilon\}$ which converges in the weak- topology of $X^*_{p+1+\lambda}$, for any $\lambda > 0$. The limiting value f is a non-negative, locally bounded measure satisfying the more precise localisation property*

$$\int_{\mathbb{R}^{2d}} \frac{f(x,\xi)}{\langle x \rangle^{p+1+\lambda}} dx d\xi \leq C_\lambda \, \|S\|^2_{B_{\frac{p+1}{2}}}, \tag{78}$$

for some constant C_λ independent of ε.

Now, the assumptions (H1–H5) merely ensures the convergence of f^ε. In order to identify the limiting energy density f as the outgoing solution to the transport equation (6) we need the following additional assumptions.

(H6) The regularising parameter α_ε has polynomial decay, i.e. there exists a $\beta \geq 0$, such that,

$$\alpha_\varepsilon \to \alpha \geq 0, \qquad \alpha_\varepsilon \geq \varepsilon^\beta. \tag{79}$$

(H7) The source function S decays at infinity such that

$$\langle x \rangle^N S(x) \in L^2(\mathbb{R}^d) \, , \qquad \text{for some } N > \frac{p+1}{2} + \frac{\beta}{1+\beta} d. \tag{80}$$

Note that this requires more regularity of S than H5, since $L^2_N \subset B_{(p+1)/2}$. Also, β can be arbitrarily large, provided N satisfies (80).

Under these stronger assumptions, we have

Theorem 4. *Assume that H1–H4 together with H5–H6 hold. Assume also that f^ϵ is the Wigner transform of u^ε, solution to (1). Then,*

(i) The limiting measure f, given in Theorem 3, satisfies the transport equation (6), with Q given by (34),

$$\alpha f(x,\xi) + \xi \cdot \nabla_x f = Q(x,\xi) \, , \tag{81}$$

$$Q(x,\xi) = 2^p \pi^{p+1} \int_{\mathbb{R}^p} dt \; \delta(x-t) \; \delta(\xi^\tau - \nabla^\tau \phi(t)) \; \delta(n_0^2 - \xi^2) \; |A|^2(t) |\widehat{S}|^2(\xi).$$

(ii) The Sommerfeld radiation condition holds in the following weak form. For any function $R \in \mathcal{D}(\mathbb{R}^{2d} \setminus \{\xi = 0\})$, *let* $g(x,\xi)$ *solve the dual problem,*

$$-\alpha g + \xi \cdot \nabla_x g = R\,, \tag{82}$$

so that the function g *is given by* $g(x,\xi) = \int_0^\infty e^{-\alpha t} R(x+\xi t, \xi) dt$, *also for the case* $\alpha \equiv 0$ *(see Section 1). We have the duality property,*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} R(x,\xi) f(x,\xi) dx d\xi = -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} Q(x,\xi) g(x,\xi) dx d\xi\,. \tag{83}$$

In contrast to Theorem 2 and Theorem 3, Theorem 4 is not an easy consequence of Theorem 1, and its proof requires different ideas. The key difficulty lies in keeping track of the radiation condition at infinity along the limiting process $\varepsilon \to 0$ when the regularising parameter α_ε eventually vanishes, $\alpha \equiv 0$. Note in particular that g does note decay in the direction $x \cdot \xi > 0$, also for a very smooth R, so that even the mere convergence of the duality product $\langle Q, g\rangle$ in (83) is not obvious.

The proofs of the above theorems are now given in the most singular case where the regularising parameter α_ε vanishes asymptotically, $\alpha \equiv 0$.

3.2 Some properties of the spaces B_s

The use of the spaces B_s is standard in the study of the Helmholtz operator, as well as in the context of scattering theory in quantum mechanics. What makes these spaces interesting are their useful properties in the Fourier space. In particular, we shall need the following specific properties of the spaces B_s in the sequel, which we simply give here without proofs, refering to [1] for more details. These will be used in establishing the uniform estimate (74) on w_z^ε.

Property 1. For functions in $B_{p/2}$ we are allowed to take "traces" over p-dimensional planes, in the sense that, for any $u \in B_{p/2}(\mathbb{R}^d)$, we have,

$$\begin{aligned} \|u(x^\tau, x^\nu)\|_{L^1(\mathbb{R}^p, L^2(\mathbb{R}^{d-p}))} \;&:=\; \int_{\mathbb{R}^p} dx^\tau \left(\int_{\mathbb{R}^{d-p}} dx^\nu \; |u|^2(x^\tau, x^\nu) \right)^{1/2} \\ &\le C \|u\|_{B_{p/2}(\mathbb{R}^d)}. \end{aligned} \tag{84}$$

This follows from the definition (66) via an easy computation.

Property 2. The spaces B_s are ordered. If $s \geq t$, then $B_s \subset B_t$ with continuous embedding.

This is obvious from the definition (66).

Property 3. The spaces B_s are stable with respect to localization *in the Fourier space*. Indeed, for any localization $\chi \in C_c^N(\mathbb{R}^d)$, we have,

$$\|\mathcal{F}^{-1}\chi\mathcal{F}u\|_{B_s} \leq C_s\|\chi\|_{C_b^N}\|u\|_{B_s}, \qquad 0 < s < N, \tag{85}$$

where the norm in C_b^N is given as usual by,

$$||\psi||_{C_b^N} \equiv \sum_{|\alpha|\leq N} ||D^\alpha\psi||_{L^\infty} \ . \tag{86}$$

Property 4. More generally, the spaces B_s are stable with respect to local smooth changes of variables *in the Fourier space*. Let Ω_1 and Ω_2 be two open subsets of $\mathbb{R}^d$ and $\psi : \Omega_1 \mapsto \Omega_2$ a C^{N+1} diffeomorphism. For any localization $\chi \in C_c^N(\Omega_1)$, we have,

$$\|\mathcal{F}^{-1}\chi(\hat{u} \circ \psi)\|_{B_s} \leq C_s\|\chi\|_{C_b^N}\|\psi\|_{C_b^{N+1}}\|u\|_{B_s}, \qquad 0 < s < N. \tag{87}$$

The operators in (85) and (87) will be of repeated use in the sequel. A typical application of these result gives that the operator,

$$(Tu)(x) := \mathcal{F}^{-1}\left((\phi \circ \psi)|\det D\psi|(\hat{u} \circ \psi)\right)(x), \qquad \phi \in C_c^N(\Omega_2), \tag{88}$$

sends B_s into itself.

4 Proofs

4.1 Proof of Theorem 1

The proof relies on an appropriate decomposition of the Fourier space. The very first step consists in formulating the estimate (74) by duality. We take a test function $v \in B_{\frac{p+1}{2}}$, and consider the duality product,

$$\langle w_z^\varepsilon, v\rangle = \frac{1}{(2\pi)^d}\langle \widehat{w}_z^\varepsilon, \widehat{v}\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\widehat{R}_z^\varepsilon(\xi)}{n_0^2 - \xi^2 + i\varepsilon\alpha_\varepsilon}\widehat{S}(\xi)\overline{\widehat{v}(\xi)}d\xi \ , \tag{89}$$

where we used the fact that w_z^ε satisfies the rescaled Helmholtz equation (26) with a constant index $n \equiv n_0$. We also introduced

$$\widehat{R}_z^\varepsilon(\xi) = \int_{\mathbb{R}^p} e^{i\,\langle \xi, \gamma(z)-\gamma(z+\varepsilon t)\rangle/\varepsilon}\; A(\gamma(z+\varepsilon t))\; e^{-i\,(\phi(\gamma(z))-\phi(\gamma(z+\varepsilon t)))/\varepsilon}\; dt. \quad (90)$$

The result (74) follows once the bound,

$$|\langle w_z^\varepsilon, v\rangle| \leq C(A,\phi)\; \|S\|_{B_{p/2}}\; \|v\|_{B_{p/2}}\;, \quad (91)$$

is proved, which we do now.

We start by noting that the assumption of a linear source (70) simplifies (90) into,

$$\begin{aligned} \widehat{R}_z^\varepsilon(\xi) &\equiv \widehat{R}_z^\varepsilon(\xi^\tau) = \int_{\mathbb{R}^p} e^{-i\langle \xi^\tau, t\rangle} A(z+\varepsilon t) e^{-i\,(\phi(z)-\phi(z+\varepsilon t))/\varepsilon}\; dt \\ &= \mathcal{F}_{x\to\xi}\left(\delta(x^\nu) A(z+\varepsilon x^\tau) e^{-i\,(\phi(z)-\phi(z+\varepsilon x^\tau))/\varepsilon}\right), \end{aligned} \quad (92)$$

so that we may use in the next estimates the following trivial bound, valid for any test function ψ,

$$|\langle \widehat{R}_z^\varepsilon, \widehat{\psi}\rangle| \leq \|A\|_{L^\infty} \int_{\mathbb{R}^p} |\psi|(x^\tau, 0) dx^\tau\;. \quad (93)$$

Next, we partition the integral (89) into three different contributions which we treat separately. For each case we let the generic symbol $\chi \in C^\infty$ denote a smooth function which localizes on the corresponding set. Also, the variable δ used below refers to the assumption of a uniformly non-characteristic phase (71).

Case 1: Contribution of the set $|n_0 - |\xi|| \geq \delta/4$.

We begin with the set away from the circle $n_0^2 = \xi^2$, where the integrand denominator is non-singular. We introduce the operator T, defined as a smooth Fourier multiplier,

$$(Tu)(x) := \mathcal{F}^{-1}\left(\frac{\chi(\xi)}{n_0^2 - \xi^2 + i\varepsilon\alpha_\varepsilon}\widehat{u}(\xi)\right)(x). \quad (94)$$

By (85), this operator is bounded in B_s for any $0 < s < \infty$, independently of ε on this set. We now estimate

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \frac{\widehat{R}_z^\varepsilon(\xi^\tau)}{n_0^2 - \xi^2 + i\varepsilon\alpha_\varepsilon} \widehat{S}(\xi)\overline{\widehat{v}(\xi)}\chi(\xi) d\xi \right| \leq \|A\|_{L^\infty} \int_{\mathbb{R}^p} |(TS \star_x \overline{v})(-x^\tau, 0)|\, dx^\tau \\
&\leq C\|A\|_{L^\infty} \int_{\mathbb{R}^p} ||TS(x^\tau, \cdot)||_{L^2(\mathbb{R}^{d-p})} dx^\tau \int_{\mathbb{R}^p} ||v(x^\tau, \cdot)||_{L^2(\mathbb{R}^{d-p})} dx^\tau \\
&\leq C\|A\|_{L^\infty} \|TS\|_{B_{p/2}} \|v\|_{B_{p/2}} \\
&\leq C\|A\|_{L^\infty} \|S\|_{B_{p/2}} \|v\|_{B_{p/2}},
\end{aligned} \tag{95}$$

where we successively used (93), the Cauchy-Schwarz inequality in the variable x^ν, the property (84) of the space $B_{p/2}$ (traces over p-dimensional subspaces), and the uniform boundedness of the operator T.

Case 2: Contribution of the set $|n_0 - |\xi|| \leq \delta/4,\ |\xi^\nu| \geq \delta/4$.

On this set we explicitly use the fact that the quantities $\xi^2 - n_0^2$ and ξ^τ give $p+1$ independent variables. Indeed, the Jacobian matrix,

$$\frac{d(\xi^\tau, \xi^2 - n_0^2)}{d\xi} = \begin{pmatrix} I_p & 0 \\ 2\xi^\tau & 2\xi^\nu \end{pmatrix} \tag{96}$$

has maximal rank $p+1$ everywhere on the (compact) set under consideration. Hence, we are able to further subdivide this set and choose a finite open cover such that the change of variables $\xi \mapsto \Xi$ with,

$$\Xi^\tau = \xi^\tau, \qquad \Xi_{p+1} = \xi^2 - n_0^2\ , \tag{97}$$

defines a diffeomorphism on each subset.

We index the subdivision by θ and let $\xi^\theta(\Xi)$ denote the local inverse change of variables. Moreover, the corresponding localizing function χ^θ may always be written as a product of two localizing functions as $\chi^\theta(\xi) = \chi_1^\theta(\xi) \times \chi_2^\beta(\xi)$ for technical convenience. With this notation, the two operators,

$$\begin{aligned}
(T_1 u)(x) &= \mathcal{F}^{-1}\left((\chi_1^\theta \widehat{u}) \circ \xi^\theta \right)(x), \\
(T_2 v)(x) &= \mathcal{F}^{-1}\left([(\chi_2^\theta \widehat{v}) \circ \xi^\theta] \times \left| \frac{d\xi^\theta}{d\Xi} \right| \right)(x),
\end{aligned} \tag{98}$$

are both bounded in $B_{(p+1)/2}$ by (87). Dropping the index θ to enhance

legibility, we then get for each fixed θ,

$$\left| \int_{\mathbb{R}^d} \frac{\widehat{R}^\varepsilon_z(\xi^\tau)}{n_0^2 - \xi^2 + i\varepsilon\alpha_\varepsilon} \widehat{S}(\xi)\overline{\widehat{v}(\xi)}\chi(\xi)d\xi \right|$$
$$= \left| \int_{\mathbb{R}^d} \frac{\widehat{R}^\varepsilon_z(\Xi^\tau)}{-\Xi_{p+1} + i\varepsilon\alpha_\varepsilon} (\widehat{S}\chi_1\overline{\widehat{v}\chi_2})(\xi(\Xi)) \left|\frac{d\xi}{d\Xi}\right| d\Xi \right|,$$

and, upon using (93) again together with the basic identity (65) to Fourier transform $1/(-\Xi_{p+1} + i\varepsilon\alpha_\varepsilon)$,

$$\leq C\|A\|_{L^\infty} \int_{\mathbb{R}^{p+1}} \left| e^{-\varepsilon\alpha_\varepsilon x_{p+1}} \mathbf{1}(x_{p+1} > 0)\ (T_1 S \star_x \overline{T_2 v})(-x^\tau, -x_{p+1}, 0) \right| dx^\tau dx_{p+1}$$
$$\leq C\|A\|_{L^\infty} \int_{\mathbb{R}^{p+1}} \left| (T_1 S \star_x \overline{T_2 v})(x^\tau, x_{p+1}, 0) \right| dx^\tau dx_{p+1}$$
$$\leq C\|A\|_{L^\infty} \|S\|_{B_{\frac{p+1}{2}}} \|v\|_{B_{\frac{p+1}{2}}}, \tag{99}$$

where the last inequality is obtained using (84) as in the previous case.

Case 3: Contribution of the set $|n_0 - |\xi|| \leq \delta/4,\ |\xi^\nu| \leq \delta/4$.

The key property of this set is that the sphere $\xi^2 = n_0^2$ and the linear space $\xi^\tau = \nabla^\tau\phi(x)$ do not intersect, thanks to the assumption (71) of a uniformly non-characteristic phase, see (102) below.

For this case, we only need to use the new variable $\Xi_1 = \xi^2 - n_0^2$. Since the Jacobian matrix $d(\xi^2 - n_0^2)/d\xi$ has rank one on the set under consideration, there exists a finite open cover indexed by θ, and finitely many changes of variables $\Xi^\theta(\xi)$, with inverse $\xi^\theta(\Xi)$, such that the first coordinate satisfies $\Xi_1^\theta(\xi) = \xi^2 - n_0^2$. Letting $\chi^\theta(\Xi) = \chi_1^\theta(\xi) \times \chi_2^\theta(\xi)$ be the collection of localizing functions, we introduce the two operators,

$$T_3 u = \mathcal{F}^{-1}((\chi_1^\theta \widehat{R}^\varepsilon_z \widehat{u}) \circ \xi^\theta), \qquad T_4 v = \mathcal{F}^{-1}\left((\chi_2^\theta \widehat{v}) \circ \xi^\theta \left|\frac{d\xi^\theta}{d\Xi}\right| \right). \tag{100}$$

It is clear from (87) that T_4 is bounded on the spaces B_s, and we prove below in Lemma 1 that the source $\widehat{R}^\varepsilon_z(\xi)$ has enough regularity (i.e. roughly speaking it has "one derivative") to ensure the boundedness of T_3 on $B_{1/2}$ by (87).

With this information, and after dropping the index θ for convenience,

we get for any fixed θ,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \frac{\widehat{R}^\varepsilon_z(\xi^\tau)}{n_0^2 - \xi^2 + i\varepsilon\alpha_\varepsilon} \widehat{S}(\xi)\overline{\widehat{v}(\xi)}\chi(\xi) d\xi \right| \\
&= \left| \int_{\mathbb{R}^d} \frac{1}{-\Xi_1 + i\varepsilon\alpha_\varepsilon} (\widehat{S}\widehat{R}^\varepsilon_z \chi_1 \overline{\widehat{v}\chi_2})(\xi(\Xi)) \left| \frac{d\xi}{d\Xi} \right| d\Xi \right| \\
&= (2\pi)^{d+1} \left| \int_{\mathbb{R}} e^{-\varepsilon\alpha_\varepsilon x_1} \mathbf{1}(x_1 > 0)(T_3 S \star_x \overline{T_4 v})(-x_1, 0) dx_1 \right| \\
&\leq C \int_{\mathbb{R}} \left| (T_3 S \star \overline{T_4 v})(x_1, 0) \right| dx_1 \\
&\leq C \|T_3 S\|_{B_{1/2}} \|T_4 v\|_{B_{1/2}} \\
&\leq C \|S\|_{B_{\frac{1}{2}}} \|v\|_{B_{\frac{1}{2}}}. \qquad (101)
\end{aligned}$$

It remains to show that T_3 is bounded on $B_{1/2}$, which by (87) boils down to proving that the multiplier $\chi_1 \widehat{R}^\varepsilon_z \in C^1_c(\mathbb{R}^d)$ uniformly in ε and z. This is an easy consequence of Lemma 1 in view of the bound from below

$$|\nabla\phi(x) - \xi^\tau| \geq \sqrt{\xi^2 - (\xi^\nu)^2} - |\nabla\phi|(x) \geq \frac{\delta}{2}, \qquad (102)$$

which follows from (71) together with $|n_0 - |\xi|| \leq \delta/4$ and $|\xi_n| \leq \delta/4$.

Conclusion

Together with Property 2 in Section 3.2, estimates (95), (99), and (101) show that,

$$|\langle w^\varepsilon_z, v \rangle| \leq C \|S\|_{B_{\frac{p+1}{2}}} \|v\|_{B_{\frac{p+1}{2}}}, \qquad (103)$$

and Theorem 1 is proved. □

We have left to show

Lemma 1. *Fix $z \in$ supp $A(\gamma(t))$. Assume that $\Omega \subset \mathbb{R}^d$ is open and that,*

$$0 < \eta := \inf_{(\xi^\tau_z, \xi^\nu_z) \in \Omega} |\nabla\phi(\gamma(z)) - \xi^\tau_z| \leq 1. \qquad (104)$$

Also, let $\chi \in C^\infty_c(\Omega)$. Finally, let $k \in \mathbb{N}$ and assume that $A(\gamma(t)) \in C^{p+1+k}_c(\mathbb{R}^p)$ and $\phi(\gamma(t)) \in C^{p+1+k}(\mathbb{R}^p)$. Then, the source $\widehat{R}^\varepsilon_z(\xi)$ defined in (90) has C^1 regularity in the sense that there exists a positive constant C

depending on the size of the support of A, but independent of ε and z, such that,

$$\chi\widehat{R}^{\varepsilon}_{z} \in C^{1}_{c}(\Omega), \qquad \textit{with} \quad ||\chi\widehat{R}^{\varepsilon}_{z}||_{C^{1}_{b}} \leq \varepsilon^{k}\frac{C}{\eta^{2(p+1+k)}}||\chi||_{C^{1}_{b}}||A||_{C^{p+1+k}_{b}}. \quad (105)$$

If $A = \text{const}$, $\gamma(x^{\tau}, x^{\nu}) = (x^{\tau}, 0)$ and $\nabla\phi = \text{const}$, then $\chi\widehat{R}^{\varepsilon}_{z} \equiv 0$.

Proof. Using the definition (90) we write,

$$\begin{aligned}\chi(\xi)\widehat{R}^{\varepsilon}_{z}(\xi) &= \chi(\xi)\int_{\mathbb{R}^{p}} A(\gamma(z+\varepsilon t)) \\ &\qquad \exp\left(-i\frac{(\phi(\gamma(z)) - \phi(\gamma(z+\varepsilon t))) + \langle\xi, \gamma(z+\varepsilon t) - \gamma(z)\rangle}{\varepsilon}\right) dt \\ &= \chi(\xi)\exp\left(i\frac{\langle\xi,\gamma(z)\rangle - \phi(\gamma(z))}{\varepsilon}\right) \times \frac{1}{\varepsilon^{p}}\int_{\mathbb{R}^{p}} A(\gamma(t))\exp\left(i\frac{\psi(t)}{\varepsilon}\right) dt\ ,\end{aligned}$$

where we defined $\psi(t) := \phi(\gamma(t)) - \langle\xi, \gamma(t)\rangle$, and we used the change of variables $t \mapsto t/\varepsilon$. In the easy case where the amplitude A is constant on a linear manifold, we compute,

$$\frac{1}{\varepsilon^{p}}\int_{\mathbb{R}^{p}} A(\gamma(t))e^{i\frac{\psi(t)}{\varepsilon}}dt = A\int_{\mathbb{R}^{p}} e^{-i\langle\xi^{\tau} - \nabla^{\tau}\phi, t\rangle}dt = \delta(\xi^{\tau} - \nabla^{\tau}\phi), \qquad (106)$$

and the result follows directly from (104).

Under the first assumption, $A(\gamma(t)) \in C^{p+1+k}_{c}(\mathbb{R}^{p})$ and $\psi(\gamma(t)) \in C^{p+1+k}(\mathbb{R}^{p})$, the *non-stationary* phase method can be used (see e.g. [1]): For a given non-negative integer k, there is a constant C, depending on the support of A, such that,

$$\begin{aligned}&\frac{1}{\varepsilon^{p+k}}\left|\int_{\mathbb{R}^{p}} A(\gamma(t))e^{i\frac{\psi(t)}{\varepsilon}}dt\right| \leq \\ &C\sum_{|\alpha|\leq p+k}\|D^{\alpha}A(\gamma(t))\|_{L^{\infty}}\left(\inf_{t\in\text{supp}(A\circ\gamma)}|\nabla\psi(\gamma(t))|\right)^{|\alpha|-2(p+k)}\ . \quad (107)\end{aligned}$$

Now, since $|\nabla\psi| = |\nabla\phi - \xi^{\tau}_{z}| > \eta > 0$, we obtain the final bound,

$$|\chi\widehat{R}^{\varepsilon}_{z}| \leq \varepsilon^{k}\frac{C}{\eta^{2(p+k)}}\|\chi\|_{L^{\infty}}||A||_{C^{p+k}_{b}}\ . \qquad (108)$$

The bound on $|D(\chi\widehat{R}^{\varepsilon}_{z})|$ follows from the same technique, since for any $j = 1, \ldots, p$, we have,

$$\begin{aligned}&\chi(\xi)\partial_{\xi_{j}}\widehat{R}^{\varepsilon}_{z}(\xi) = \chi(\xi)e^{i\frac{\langle\xi,\gamma(z)\rangle - \phi(\gamma(z))}{\varepsilon}} \\ &\left(\frac{\gamma_{j}(z)}{\varepsilon^{p+1}}\int_{\mathbb{R}^{p}} A(\gamma(t))e^{i\frac{\psi(t)}{\varepsilon}}dt\ + \frac{i}{\varepsilon^{p}}\int_{\mathbb{R}^{p}} t_{j}A(\gamma(t))e^{i\frac{\psi(t)}{\varepsilon}}dt\right)\ , \qquad (109)\end{aligned}$$

and we conclude as above, using that $|\gamma(z)|$ is bounded for $z \in \mathrm{supp}(A \circ \gamma)$, and $A(\gamma(t)) \in C_c^{p+1+k}(\mathbb{R}^p)$ as well as $t_j A(\gamma(t)) \in C_c^{p+k}(\mathbb{R}^p)$.

Finally, for fixed ε, z the functions $\chi \widehat{R}_z^\varepsilon$ and $D(\chi \widehat{R}_z^\varepsilon)$ are continuous since $A(\gamma(t)))$ and $t_j A(\gamma(t)))$ are L^1 functions and Lebesgue's convergence theorem applies. □

4.2 Proof of Theorem 2 and Theorem 3

Both theorems are easy consequences of Theorem 1.

Proof of Theorem 2

We begin by noting that for $s > 0$ the norm B_s^* is in fact equivalent to the norm,

$$\|u\|_{B_s^*} \leq \sup_{R>1} \frac{1}{R^s} \left(\int_{|x|<R} |u|^2 dx \right)^{1/2} \leq C||u||_{B_s^*}. \tag{110}$$

Indeed, the left inequality is trivial while the right one follows from,

$$\begin{aligned}
&\sup_{R>1} \frac{1}{R^s} \left(\int_{|x|<R} |u|^2 dx \right)^{1/2} \leq \sup_{R>1} \sum_{|2^j| \leq R} 2^{-js} \left(\int_{C_j} |u|^2 dx \right)^{1/2} \\
&\leq \sup_{R>1} \sup_{\substack{j \geq -1 \\ 2^j \leq R}} 2^{-js} \left(\int_{C_j} |u|^2 dx \right)^{1/2} \times \sum_{2^k \leq R} 2^{-ks} \\
&\leq C \sup_{j \geq -1} 2^{-js} \left(\int_{C_j} |u|^2 dx \right)^{1/2},
\end{aligned}$$

since the series in the last line converges for $s > 0$. The equivalence (110) allows us to turn the estimate (74) on w_z^ε into a bound on u^ε,

$$\begin{aligned}
&\sup_{R>1} \frac{1}{R^s} \left(\int_{|x|<R} |u^\varepsilon|^2 dx \right)^{1/2} \\
&\leq \varepsilon^{d/2} \sup_{R>1} \frac{1}{R^s} \left(\int_{\varepsilon|x|<R+|\gamma|} |u^\varepsilon(\gamma(z)+\varepsilon x)|^2 dx \right)^{1/2} \\
&\leq C\varepsilon^{(p+1)/2} \sup_{R>1} \left(\frac{\varepsilon \langle \gamma(z) \rangle}{\varepsilon(R+|\gamma|)} \right)^s \left(\int_{\varepsilon|x|<R+|\gamma|} |w_z^\varepsilon|^2 dx \right)^{1/2} \\
&\leq C\varepsilon^{(p+1)/2-s} \langle \gamma(z) \rangle^s \sup_{R>1} \frac{1}{R^s} \left(\int_{|x|<R} |w_z^\varepsilon|^2 dx \right)^{1/2},
\end{aligned} \tag{111}$$

and taking $s = (p+1)/2 - \lambda$ gives the result. □

Proof of Theorem 3

The proof is given in two steps.

First step. We prove that f^ε is uniformly bounded in $X_{p+1+\lambda}$, for any $\lambda > 0$.
This is an easy consequence of the estimate (75) on u^ε. Indeed, we take $\lambda > 0$ and set $t = (p+1+\lambda)/2$. We also choose a test function ψ in X_{2t}, and use the notation $\widehat{\psi}(x,y) \equiv \mathcal{F}_{\xi\to y}\psi(x,\xi)$. Now we consider the duality product,

$$\begin{aligned}\langle f^\epsilon, \psi\rangle &= \langle \mathcal{F}_{\xi\to y} f^\epsilon, \mathcal{F}_{\xi\to y}\psi\rangle \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u\left(x+\frac{\varepsilon}{2}y\right)\overline{u\left(x-\frac{\varepsilon}{2}y\right)}\widehat{\psi}(x,y)dxdy. \qquad (112)\end{aligned}$$

On the other hand, the uniform boundedness of $u^\varepsilon \in B^*_{\frac{p+1}{2}}$ readily gives,

$$\|\langle x\rangle^{-t} u^\varepsilon\|_{L^2} = \|u^\varepsilon\|_{L^2_{-t}} \le C_\lambda \|u^\varepsilon\|_{B^*_{\frac{p+1}{2}}} \le C_\lambda \|S\|_{B_{\frac{p+1}{2}}}\ , \qquad (113)$$

for some positive constant C_λ, since $t > (p+1)/2$. (As is well-known the constant in the continuous embedding $B^*_s \subset L^2_{-(s+\lambda)}$ only depends on λ.) Therefore, also using the estimate $\langle x+y\rangle\langle x-y\rangle \le \langle |x|+|y|\rangle^2$ in (112), we get,

$$\begin{aligned}&|\langle f^\epsilon, \psi\rangle| \\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x+\frac{\varepsilon}{2}y)\overline{u(x-\frac{\varepsilon}{2}y)}|}{\langle x+\frac{\varepsilon}{2}y\rangle^\tau \langle x-\frac{\varepsilon}{2}y\rangle^\tau}\left\langle x+\frac{\varepsilon}{2}y\right\rangle^\tau\left\langle x-\frac{\varepsilon}{2}y\right\rangle^\tau |\widehat{\psi}(x,y)|dxdy \\ &\le \|u\|^2_{L^2_{-t}}\int_{\mathbb{R}^d}\sup_{x\in\mathbb{R}^d}\left\langle x+\frac{\varepsilon}{2}y\right\rangle^\tau\left\langle x-\frac{\varepsilon}{2}y\right\rangle^\tau |\widehat{\psi}(x,y)|dy \\ &\le C_\lambda ||S||^2_{B_{\frac{p+1}{2}}}\int_{\mathbb{R}^d}\sup_{x\in\mathbb{R}^d}\langle |x|+|y|\rangle^{2t}|\widehat{\psi}(x,y)|dy. \qquad (114)\end{aligned}$$

Hence, $|\langle f^\epsilon, \psi\rangle| \le C_\lambda \|S\|^2_{B_{\frac{p+1}{2}}}\|\psi\|_{X_{p+1+\lambda}}$ for any $\lambda > 0$, which proves the uniform bound,

$$\|f^\epsilon\|_{X^*_{p+1+\lambda}} \le C_\lambda \|S\|^2_{B_{\frac{p+1}{2}}}\ . \qquad (115)$$

Second step. The space $X^*_{p+1+\lambda}$ is a Banach space with predual $X_{p+1+\lambda}$. This, together with (115), gives the existence of a subsequence, still denoted by f^ε, which converges in the weak-* topology of $X^*_{p+1+\lambda}$ towards some limiting distribution $f \in X^*_{p+1+\lambda}$. The non-negativity of f is a general property

of Wigner measures, see for instance [7] and [10]. The fact that $f \geq 0$ as a distribution implies that f is in fact a locally bounded measure. Finally, the more precise localisation property (78) is obtained by choosing the test function $\psi_\tau(x,\xi) = \exp(-\tau\xi^2)/\langle x\rangle^{p+1+\lambda}$. Indeed, using (114) gives,

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} \frac{f(x,\xi)}{\langle x\rangle^{p+1+\lambda}} dx d\xi &= \lim_{\tau\to 0} \langle f, \psi_\tau\rangle \\
&\leq \lim_{\tau\to 0} C_\lambda ||S||^2_{B_{\frac{p+1}{2}}} \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} \langle |x|+|y|\rangle^{p+1+\lambda} \frac{e^{-|y|^2/\tau}}{\tau^{d/2}\langle x\rangle^{p+1+\lambda}} dy \\
&= \lim_{\tau\to 0} C_\lambda ||S||^2_{B_{\frac{p+1}{2}}} \int_{\mathbb{R}^d} \langle y\rangle^{p+1+\lambda} \frac{e^{-|y|^2/\tau}}{\tau^{d/2}} dy \\
&= \lim_{\tau\to 0} C_\lambda ||S||^2_{B_{\frac{p+1}{2}}} \int_{\mathbb{R}^d} \left\langle \sqrt{\tau} y\right\rangle^{p+1+\lambda} e^{-|y|^2} dy \\
&= C_\lambda ||S||^2_{B_{\frac{p+1}{2}}} \int_{\mathbb{R}^d} e^{-|y|^2} dy. \qquad (116)
\end{aligned}$$

□

4.3 Proof of Theorem 4

The proof is given in several steps.

First step: Preliminary reduction

In order to show that f satisfies (83), we take a test function $R \in \mathcal{D}(\mathbb{R}^{2d}\backslash\{\xi = 0\})$ and let g_ε be the solution to,

$$-\alpha_\varepsilon g_\varepsilon + \xi\cdot\nabla_x g_\varepsilon = R(x,\xi). \qquad (117)$$

By duality, it is clear from (14) that,

$$\langle Q_\varepsilon, g_\varepsilon\rangle = \langle f^\epsilon, R\rangle \qquad (= \alpha_\varepsilon \langle f^\epsilon, g_\varepsilon\rangle + \langle \xi\cdot\nabla_x f^\epsilon, g_\varepsilon\rangle)\ . \qquad (118)$$

Hence proving the relation (83) reduces to proving the following convergence results,

$$\lim_{\varepsilon\to 0} \langle Q_\varepsilon, g_\varepsilon\rangle = \langle Q, g\rangle\ , \qquad (119)$$

together with,

$$\lim_{\varepsilon\to 0} \langle f^\epsilon, R\rangle = \langle f, R\rangle\ , \qquad (120)$$

where g and Q are defined in the theorem.

Since $R \in X_{p+1+\lambda}$ for $\lambda > 0$, the convergence (120) immediately follows from the weak-* convergence of f^ϵ. Moreover, by (16) and the computation (24), the left hand side of (119) satisfies

$$\begin{aligned}\langle Q_\varepsilon, g_\varepsilon \rangle = & \text{ Re } \left(\frac{i}{\varepsilon^d} \int_{\mathbb{R}^{2d+p}} dxdydtA(t)S\left(\frac{x-t}{\varepsilon} + \frac{y}{2}\right) \overline{w_t^\varepsilon\left(\frac{x-t}{\varepsilon} - \frac{y}{2}\right)} \widehat{g}_\varepsilon(x,y) \right) \\ & = - \text{ Im } \left(\int_{\mathbb{R}^{2d+p}} dxdydtA(t)S(x+y)\overline{w_t^\varepsilon(x)}\widehat{g}_\varepsilon(t+\varepsilon(x+y/2),y) \right) ,\end{aligned}$$

where $\widehat{g}_\varepsilon(x,y) := (\mathcal{F}^{-1}_{\xi \to y} g_\varepsilon)(x,y)$, and we decompose,

$$\begin{aligned} & = - \text{ Im } \int_{\mathbb{R}^{2d+p}} dxdydtA(t)S(x+y)\overline{w_t^\varepsilon(x)}(\widehat{g}_\varepsilon(t+\varepsilon(x+y/2),y) - \widehat{g}_\varepsilon(t,y)) \\ & - \text{ Im } \int_{\mathbb{R}^{2d+p}} dx\ dy\ dt\ A(t)S(x+y)\overline{w_t^\varepsilon(x)}(\widehat{g}_\varepsilon(t,y) - \widehat{g}(t,y)) \\ & - \text{ Im } \int_{\mathbb{R}^{2d+p}} dxdydtA(t)S(x+y)\overline{w_t^\varepsilon(x)}\widehat{g}(t,y) \\ \equiv & \ I_\varepsilon + II_\varepsilon + III_\varepsilon. \end{aligned} \tag{121}$$

The rest of the analysis is devoted to proving that I_ε and II_ε go to zero, as well as computing the limit of III_ε.

Second step: Preliminary estimates on $\widehat{g}_\varepsilon(x,y)$

The study of I_ε and II_ε is performed along the same lines as in [2]. The key point lies in proving that the (Fourier transformed) test function $\widehat{g}_\varepsilon(x,y)$ decays sufficiently fast in the y variable at infinity to balance the lack of integrability of the prefactor $S(x+y)w_t^\varepsilon(x)$ in x and y. This is the reason why we estimate $\widehat{g}_\varepsilon(x,y)$ in the present step.

Firstly, one readily obtains from (117) the explicit value,

$$\begin{aligned} g_\varepsilon(x,\xi) & = -\int_{s=0}^{+\infty} \exp(-\alpha_\varepsilon s)\ R(x+\xi s,\xi)\ ds\ , \\ & = -\frac{1}{|\xi|}\ \int_{s=0}^{+\infty} \exp(-\alpha_\varepsilon |\xi|^{-1} s)\ R\left(x + \frac{\xi}{|\xi|}s, \xi\right)\ ds\ , \end{aligned} \tag{122}$$

after changing variables $s \to s/|\xi|$, and using the fact that the support of R does not meet the set of vanishing velocities $\{\xi = 0\}$. Let us introduce the constants r_0 and R_0 for the support of R,

$$(x,\xi) \in \text{supp } R \qquad \Rightarrow \qquad 0 < r_0 \leq |\xi| \leq R_0, \qquad |x| \leq R_0. \tag{123}$$

Note then that the integral in $s \in [0, +\infty[$ in the last formula actually ranges over a subset of the compact set $s \in [|x|-R_0, |x|+R_0]$. For this reason, we are able to upper bound $\widehat{g}_\varepsilon(x,y)$ in the following way. Let M be a non-negative even integer and estimate the moment,

$$\left|\langle y\rangle^M \widehat{g}_\varepsilon(x,y)\right| =$$

$$= \left| \langle y\rangle^M \int_{\mathbb{R}^d} \int_{s=0}^{+\infty} \exp(-iy\xi) \exp(-\alpha_\varepsilon |\xi|^{-1} s)\, \frac{1}{|\xi|}\, R\left(x + \frac{\xi}{|\xi|} s, \xi\right)\, ds\, d\xi \,\right|$$

$$= \left| \int_{\mathbb{R}^d} \int_{s=0}^{+\infty} \exp(-iy\xi) \langle i\partial_\xi\rangle^M \left\{ \exp(-\alpha_\varepsilon |\xi|^{-1} s)\, \frac{1}{|\xi|}\, R\left(x + \frac{\xi}{|\xi|} s, \xi\right)\right\}\, ds\, d\xi \,\right|$$

$$\leq C \int_{s=0}^{+\infty} \sup_{r_0 \leq |\xi| \leq R_0} \left| \langle i\partial_\xi\rangle^M \left\{ \frac{\exp(-\alpha_\varepsilon |\xi|^{-1} s)}{|\xi|}\, R\left(x + \frac{\xi}{|\xi|} s, \xi\right)\right\}\right| \, ds$$

$$\leq C \int_{s=0}^{+\infty} \sup_{|\xi| \leq R_0} \langle s\rangle^M \exp(-\alpha_\varepsilon |\xi|^{-1} s) \mathbf{1}\left(\left|s - |x|\right| \leq R_0\right)\, ds\ , \qquad (124)$$

for some constant $C > 0$ depending on M together with the profile R. Hence,

$$\langle y\rangle^M \left|\widehat{g}_\varepsilon(x,y)\right| \leq C \langle x\rangle^M \exp(-C\alpha_\varepsilon |x|)\ . \qquad (125)$$

In other words, we arrive at the bound,

$$\langle y\rangle^M \left|\widehat{g}_\varepsilon(x,y)\right| \leq C\, [\langle x\rangle^M \wedge {\alpha_\varepsilon}^{-M}]\ , \qquad (126)$$

where we use the usual notation $a \wedge b := \min(a,b)$. The result trivially extends to all real $M \geq 0$. Note also that the first bound, (125), remains true for $\widehat{g}$ with $\alpha = 0$. By the same argument as in (124), we also have the following equicontinuity,

$$\langle y\rangle^M |\widehat{g}_\varepsilon(x_1,y) - \widehat{g}_\varepsilon(x_2,y)| =$$

$$\leq C \int_{s=0}^{+\infty} \sup_{|\xi| \leq R_0} \langle s\rangle^M e^{-\alpha_\varepsilon |\xi|^{-1} s} \left| R\left(x_1 + \frac{\xi}{|\xi|} s, \xi\right) - R\left(x_2 + \frac{\xi}{|\xi|} s, \xi\right)\right|$$

$$\leq C|x_1 - x_2|\ \langle |x_1| + |x_2|\rangle^M\ . \qquad (127)$$

Third step: Convergence of III_ε

We begin by showing that any subsequence of $\{w_t^\varepsilon\}$ converges to the outgoing solution w_t of (29). We introduce the solution $\widetilde{w}_t^\varepsilon$ to the auxiliary equation,

$$i\varepsilon\alpha_\varepsilon\widetilde{w}_t^\varepsilon + \Delta\widetilde{w}_t^\varepsilon + \widetilde{w}_t^\varepsilon = A(t)\int_{\mathbb{R}^p} e^{-i\langle t',\nabla\phi(t)\rangle} S\left(x^\tau - t', x^\nu\right) dt'. \tag{128}$$

As is well-known, any subsequence of $\{\widetilde{w}_t^\varepsilon\}$ converges weakly to w_t in $B^*_{(p+1)/2}$ as $\varepsilon \to 0$, since the right-hand-side of (129) belongs to $B_{(p+1)/2}$. Now, define,

$$V^\varepsilon(t') = A(t+\varepsilon t')\exp\left(i\frac{\phi(t+\varepsilon t')-\phi(t)}{\varepsilon}\right) - A(t)\exp\left(i\left\langle t', \nabla\phi(t)\right\rangle\right). \tag{129}$$

The difference $q_\varepsilon = w_t^\varepsilon - \widetilde{w}_t^\varepsilon$ satisfies,

$$i\varepsilon\alpha_\varepsilon q_\varepsilon + \Delta q_\varepsilon + q_\varepsilon = \int_{\mathbb{R}^p} V^\varepsilon(t') S\left(x^\tau - t', x^\nu\right) dt'. \tag{130}$$

We prove the weak convergence of q_ε to zero. To this aim, we take a test function $v \in B_{(p+1)/2}$. The same procedure as in Section 4.1 gives, for all $k > 0$,

$$\begin{aligned} |\left\langle q_\varepsilon, v\right\rangle| \quad \leq \quad & \int_{\mathbb{R}^p} \left|V^\varepsilon(x^\tau)(\overline{TS} \star v)(x^\tau, 0)\right| dx^\tau \\ & + \int_{\mathbb{R}^{p+1}} \left|V^\varepsilon(x^\tau)(\overline{T_1 S} \star_x T_2 v)(x^\tau, x_{p+1}, 0)\right| dx^\tau dx_{p+1} \\ & + \varepsilon^k \frac{C}{\delta^{2(p+1+k)}} ||A||_{C_b^{p+1+k}} ||S||_{B_{1/2}} ||v||_{B_{1/2}}. \end{aligned} \tag{131}$$

The first two terms converge to zero by the dominated convergence theorem, and the last term also tends to zero by (72). Hence $w_t^\varepsilon \overset{*}{\rightharpoonup} w_t$ in $B^*_{(p+1)/2}$.

In order to derive a bound on the integrand, we take advantage of the compactness of the support of A, defining A_0 to be a constant such that $t \in \operatorname{supp} A$ implies $|t| \leq A_0$. By (68), Theorem 1, and by using the Cauchy-Schwarz inequality in the x variable, we get,

$$\begin{aligned} & \left|\int_{\mathbb{R}^{d+p}} A(t)S(x+y)\overline{w_t^\varepsilon(x)}\widehat{g}(t,y)dxdt\right| \\ & \leq \quad C \sup_{|t|\leq A_0} \int_{\mathbb{R}^d} \left\langle x+y\right\rangle^N |S(x+y)| \frac{|w_t^\varepsilon(x)|\left\langle x\right\rangle^{\frac{p+1+0}{2}}}{\left\langle x\right\rangle^{\frac{p+1+0}{2}}\left\langle x+y\right\rangle^N} |\widehat{g}(t,y)|dx \\ & \leq \quad C||S||_{L^2_N}||S||_{B_{\frac{p+1}{2}}} \sup_{x\in\mathbb{R}^d,\ |t|\leq A_0} \frac{\left\langle |x|+|y|\right\rangle^{\frac{p+1+0}{2}}}{\left\langle x\right\rangle^N} |\widehat{g}(t,y)|, \end{aligned} \tag{132}$$

where we also used the decay assumption $S \in L^2_N$. Here and in the sequel the symbol $+0$ means some small positive number whose actual value is irrelevant. Through (68) and (125) there are therefore constants C and M such that we can bound (132) by,

$$\cdots \leq \frac{C}{\langle y \rangle^{M - \frac{p+1+0}{2}}} \in L^1(\mathbb{R}^d), \qquad M > \frac{p+1}{2} + d. \tag{133}$$

Consequently, we can use the dominated convergence theorem to obtain,

$$\begin{aligned} III_\varepsilon &= -\operatorname{Im} \int_{\mathbb{R}^{d+p}} A(t) \left\langle S(\cdot + y), \overline{w_t^\varepsilon} \right\rangle \widehat{g}(t,y) dt dy \\ &\to -\operatorname{Im} \int_{\mathbb{R}^{d+p}} A(t) \left\langle S(\cdot + y), \overline{w_t} \right\rangle \widehat{g}(t,y) dt dy, \end{aligned} \tag{134}$$

since $S(x+y) \in B_{(p+1)/2}$ for all fixed y. Hence, $III_\varepsilon \to \langle Q, g \rangle$ by (34).

Fourth step: Convergence of II_ε

From the result in the second and third steps the convergence of II_ε follows easily. By the same procedure as in (132) we immediately get,

$$|II_\varepsilon| \leq C \int_{\mathbb{R}^d} \sup_{\substack{x \in \mathbb{R}^d, \\ |t| \leq A_0}} \left[\frac{\langle |x| + |y| \rangle^{\frac{p+1+0}{2}}}{\langle x \rangle^N} \left| \widehat{g}_\varepsilon(t,y) - \widehat{g}(t,y) \right| \right] dy. \tag{135}$$

Here and in the rest of this paragraph, C denote various constants depending on the natural norms of A, S, and the test function R. The last term is easily estimated by,

$$\leq C \sup_{y \in \mathbb{R}^d, \, |t| \leq A_0} \langle y \rangle^{\frac{p+1+0}{2} + M} \left| \widehat{g}_\varepsilon(t,y) - \widehat{g}(t,y) \right|, \tag{136}$$

up to choosing $M > d$. Reasoning as we did in establishing (124), we conclude that the right-hand-side of (136) is bounded by,

$$\leq C \int_{s=0}^{A_0 + R_0} \langle s \rangle^{\frac{p+1+0}{2} + M} \sup_{|\xi| \leq R_0} \left| e^{-\alpha_\varepsilon |\xi|^{-1} s} - e^{-\alpha |\xi|^{-1} s} \right| ds, \tag{137}$$

and the dominated convergence theorem allows us to conclude that $II_\varepsilon \to 0$.

Fifth step: Convergence of I_ε

As in the preceding steps, we get,

$$|I_\varepsilon| \leq C \int_{\mathbb{R}^d} \sup_{\substack{x\in\mathbb{R}^d, \\ |t|\leq A_0}} \left[\frac{\langle |x|+|y| \rangle^{\frac{p+1+0}{2}}}{\langle x \rangle^N} |\widehat{g}_\varepsilon(t+\varepsilon(x+\frac{y}{2}),y) - \widehat{g}_\varepsilon(t,y)| \right] dy. \quad (138)$$

It becomes therefore natural to estimate the right hand side of (138) differently on the following three sets,

$$D_1 := \{x,y \ / \ |x+\frac{y}{2}| \leq \varepsilon^{-1+0}\} \ , \ D_2 := \{x,y \ / \ |x+\frac{y}{2}| \geq \varepsilon^{-1+0}, \ |x| \geq \frac{|y|}{4}\} \ ,$$

$$D_3 := \{x,y \ / \ |x+\frac{y}{2}| \geq \varepsilon^{-1+0}, \ |x| \leq \frac{|y|}{4}\} \ . \quad (139)$$

On the first set, we are able to use (127), the uniform continuity of $\widehat{g}_\varepsilon(x,y)$ as mentioned in the second step above. Hence the corresponding contribution to I_ε obviously goes to zero with ε.

On the second set, we cannot use the continuity of $\widehat{g}_\varepsilon$ anymore, so we now rely on (126) to upper-bound the corresponding contribution. By also using the fact that $|x| \geq C\varepsilon^{-1+0}$ on D_2, we get

$$\begin{aligned} \cdots &\leq C \int_{\mathbb{R}^d} \sup_{x\geq C\varepsilon^{-1+0}} \left[\frac{\langle |x|+|y| \rangle^{\frac{p+1+0}{2}}}{\langle x \rangle^N} \ \frac{\langle \varepsilon(|x|+|y|) \rangle^M \wedge {\alpha_\varepsilon}^{-M}}{\langle y \rangle^M} \right] dy \\ &\leq C \int_{\mathbb{R}^d} \frac{1}{\langle y \rangle^M} dy \times \sup_{x\geq C\varepsilon^{-1+0}} \left[\langle x \rangle^{\frac{p+1+0}{2}-N} \ (\langle \varepsilon x \rangle^M \wedge {\alpha_\varepsilon}^{-M}) \right] \\ &\leq C \ \varepsilon^{-[\beta+1][\frac{p+1+0}{2}-N]-\beta M} \to 0 \ , \end{aligned} \quad (140)$$

up to choosing $M > d$ but close to d, and using the assumptions (79, 80). (In addition we assumed that $N < (p+1)/2 + d$ which is no restriction.)

We argue in the same way on the set D_3 and obtain,

$$\begin{aligned} \cdots &\leq C \int_{|y|\geq C\varepsilon^{-1+0}} |y|^{\frac{p+1+0}{2}} \ \frac{\langle \varepsilon y \rangle^M \wedge {\alpha_\varepsilon}^{-M}}{\langle y \rangle^M} \ dy \\ &\leq C \ \varepsilon^{M-[\beta+1][\frac{p+1+0}{2}+d]} \to 0 \ , \end{aligned} \quad (141)$$

up to choosing M large enough ($M > (\beta+1)(d+(p+1)/2)$ will do) and using (79).

Last step: Conclusion

We have now proved the relation (83). The fact that f satisfies the Liouville equation in the sense of distribution follows easily from (83) by setting $R = -\alpha\psi + \xi \cdot \nabla_x \psi$ for testfunctions $\psi \in \mathcal{D}(\mathbb{R}^{2d})$ and noting that since decay estimates are trivial in this case, the support of R can include $\{\xi = 0\}$. This ends the proof of Theorem 4. □

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