ANALYSIS OF HETEROGENEOUS MULTISCALE METHODS FOR LONG TIME WAVE PROPAGATION PROBLEMS

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Abstract. In this paper, we analyze a multiscale method developed under the heterogeneous multiscale methods (HMM) framework for numerical approximation of multiscale wave propagation problems in periodic media. In particular, we are interested in the long time $O(\varepsilon^{-2})$ wave propagation, where ε represents the size of the microscopic variations in the media. In large time scales, the solutions of multiscale wave equations exhibit $O(1)$ dispersive effects which are not observed in short time scales. A typical HMM has two main components: a macro model and a micro model. The macro model is incomplete and lacks a set of local data. In the setting of multiscale PDEs, one has to solve for the full oscillatory problem over local microscopic domains of size $\eta = O(\varepsilon)$ to upscale the parameter values which are missing in the macroscopic model. In this paper, we prove that if the micro problems are consistent with the macroscopic solutions, HMM approximates the unknown parameter values in the macro model up to any desired order of accuracy in terms of ε/η.

Key words. Multiscale wave equation, Long time wave equation, Homogenization

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1. Introduction. A decade ago, E and Engquist, [9], proposed the heterogeneous multiscale methods (HMM) as a general framework for treating multiscale and possibly multiphysics problems. HMM is often useful when we have the full microscopic model which is not affordable to use throughout the entire domain. The basic idea is that one starts with assuming a macro model in which some missing data are upscaled from local microscopic simulations, where the micro model is forced to be consistent with the coarse scale/macroscopic data. The HMM framework has been successfully applied to many disciplines of sciences. To name a few applications here, we refer the reader to [2, 10] for homogenization problems, to [13] for applications to gas dynamics, and to [17] for complex fluids applications. For a more recent update about the improvements in the HMM based approaches see e.g. [3].

Our main goal in this paper is to mathematically investigate the properties of a HMM type multi-scale algorithm for approximating the solution of the following initial boundary value problem modelling long time wave propagation

$$\begin{aligned} &\partial_{tt} u^\varepsilon(t,x) - \nabla\cdot(A(x/\varepsilon)\nabla u^\varepsilon(t,x)) = 0, \quad \text{in} \quad [0,T^\varepsilon]\times\Omega \\ &u^\varepsilon(0,x) = q(x), \quad \partial_t u^\varepsilon(0,x) = z(x), \quad \text{on} \quad \{t=0\}\times\Omega, \end{aligned} \tag{1.1}$$

where $A(y) \in \mathbb{R}^{d\times d}$ is a 1-periodic, symmetric and uniformly positive definite matrix, $\Omega \subset \mathbb{R}^d$, with $|\Omega| = 1$, $\varepsilon \ll 1$ represents the size of the periodic microstructures in the media, and $T^\varepsilon \approx O(\varepsilon^{-2})$. We assume that the above equation is equipped with suitable boundary data. Since ε is small, the coefficients $A(x/\varepsilon)$ varies rapidly, and hence a direct numerical simulation of the problem (1.1) is infeasible due to the need to resolve the small scale variations in the media. However, for small ε the multiscale problem (1.1) can be replaced by an effective equation. For short time scales $T^\varepsilon = T \approx O(1)$, the classical homogenization theory reveals the limiting behavior of the multi-scale problem. As $\varepsilon \to 0$, the solution of (1.1) with $T^\varepsilon = T$

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tends to a solution u^0 which has no dependence on the small scale parameter ε. In this setting, the solution u^0 satisfies

$$
\begin{aligned}
&\partial_{tt} u^0(t,x) - \nabla \cdot \left(\hat{A} \nabla u^0(t,x) \right) = 0, \quad \text{in} \quad [0,T] \times \Omega \\
&u^0(0,x) = q(x), \quad \partial_t u^0(0,x) = z(x), \quad \text{on} \quad \{t=0\} \times \Omega,
\end{aligned} \tag{1.2}
$$

where the homogenized coefficient $\hat{A}$ is a constant matrix, the computation of which involves solving another set of non-oscillatory periodic elliptic problems called cell problems. For details regarding convergence rate of the homogenized solution $u^0(t,x)$ to the multiscale solution $u^\varepsilon(t,x)$ and other technical issues about homogenization we refer the reader to the seminal book by Bensoussan, et. al. [8]. For heterogeneous multiscale methods based on finite differences and finite elements which approximate the solution of the short time equation (1.2) see [12] (FD-HMM) and [6] (FE-HMM). For time scales $T^\varepsilon = O(\varepsilon^{-2})$, the solution $u^\varepsilon(t,x)$ starts to exhibit $O(1)$ dispersive effects which are not present in the short time homogenized solution. In the long time case, Symes and Santosa, [18], derived an effective equation for the multiscale problem (1.1). In one dimension, the effective equation has the form

$$
\begin{aligned}
&\partial_{tt} \hat{u}(t,x) - \partial_x \left(\hat{a} \partial_x \hat{u}(t,x) + \varepsilon^2 \beta \partial_{xxx} \hat{u}(t,x) \right) = 0, \quad \text{in} \quad [0,T^\varepsilon] \times \Omega \\
&\hat{u}(0,x) = q(x), \quad \partial_t \hat{u}(0,x) = z(x), \quad \text{on} \quad \{t=0\} \times \Omega,
\end{aligned} \tag{1.3}
$$

where β is a complicated functional of A (formula (5.6) in this paper), and $\hat{a}$ is the homogenized coefficient in one dimension. The solution $\hat{u}(t,x)$ of the long time effective equation (1.3) will then approximate the multiscale solution $u^\varepsilon(t,x)$ over large timescales. Note that the problem (1.3) is ill-posed in the sense that high frequency initial data will result in blow up in the solution.

From a numerical point of view, it is important to develop cheap numerical methods which approximate the effective solution $\hat{u}(t,x)$, and hence are able to capture the long time dispersive effects which do not appear in the homogenized limit $u^0(t,x)$. In 2011, Engquist et. al., [11], developed a method based on finite difference HMM (FD-HMM). Their method approximated the solution $\hat{u}(t,x)$ of the effective equation (1.3) in three steps:

Step 1: *The macro model and macro solver*: The HMM assumes the macro model:

$$
\partial_{tt} u(t,x) = \nabla \cdot F, \tag{1.4}
$$

where the flux F is the unknown data in the model. In one dimension, for instance, the macro model can be discretized with the following high order scheme

$$
u_i^{n+1} = 2u_i^n - u_i^{n-1} + \frac{\triangle t^2}{24 \triangle x} \left(F^n_{i-\frac{3}{2}} - 27 F^n_{i-\frac{1}{2}} + 27 F^n_{i+\frac{1}{2}} - F^n_{i+\frac{3}{2}} \right). \tag{1.5}
$$

Step 2: *Micro model*: To find the flux F^n at a point x_0, we solve the full oscillatory problem

$$
\begin{aligned}
&\partial_{tt} u^\varepsilon(t,x) - \nabla \cdot (A(x/\varepsilon) \nabla u^\varepsilon(t,x)) = 0, \quad \text{in} \quad [0,\tau] \times \Omega_{\eta,x_0} \\
&u^\varepsilon(0,x) = \bar{u}(x), \quad \partial_t u^\varepsilon(0,x) = 0, \quad \text{on} \quad \{t=0\} \times \Omega_{\eta,x_0},
\end{aligned} \tag{1.6}
$$

where $\bar{u}(x)$ is a third order polynomial, $\Omega_{\eta,x_0} = [x_0 - L_\eta, x_0 + L_\eta]$, $L_\eta = \eta + \tau\sqrt{|A|_\infty}$, where $\eta = O(\varepsilon)$ represents the size of the spatial domain and $\tau = O(\varepsilon)$ stands for the microscopic final time. The initial data $\bar{u}(x)$ has the following property: suppose that we interpolate the macroscopic data $\{u_l^n\}$ (the subscript l is used to indicate the discrete macroscopic solutions located around $x = x_0$) with a third order interpolant $\hat{u}(x)$ around $x = x_0$. Then the initial data $\bar{u}(x)$ is such that the average of the micro solution $u^\varepsilon(t,x)$ over the subset $[-\eta+x_0, \eta+x_0]\times(0,\tau]$ of the micro box $\Omega_{\eta,x_0}\times(0,\tau]$ agrees with the interpolant $\hat{u}(x)$ up to high orders in ε. The consistent initial data $\bar{u}(x)$ for a given macroscopic state $\hat{u}(x)$ can be obtained numerically by an algorithm presented in [14]. Mathematically speaking, we say that $\bar{u}(x)$ is consistent with $\hat{u}(x)$ up to $O(\varepsilon^q)$ if

$$(1.7)\quad \int_{-\tau}^{\tau}\int_{-\eta+x}^{\eta+x} K_\tau(t)K_\eta(\tilde{x}-x)u^\varepsilon(t,\tilde{x})dtd\tilde{x} = \hat{u}(x)+O(\varepsilon^q), \quad x\in[-\eta+x_0,\eta+x_0],$$

where $K_\tau(t)$ and $K_\eta(x)$ are averaging kernels (in time and space respectively) with compact supports of sizes τ and η. Further details about averaging kernels are discussed in Section 2.3.

Step 3. *Upscaling* : The last step of the HMM is to average the local microscopic flux $A(x/\varepsilon)\partial_x u^\varepsilon(t,x)$. Using the notation $F_{HMM}(x_0)$ instead of $F^n(x_0)$, the HMM flux is then computed by

$$(1.8)\qquad F_{HMM}(x_0) := \int_{-\tau}^{\tau}\int_{-\eta+x_0}^{\eta+x_0} K_\tau(t)K_\tau(x)A(x/\varepsilon)\partial_x u^\varepsilon(t,x)dtdx,$$

where the HMM flux $F_{HMM}(x_0)$ approximates the macroscopic flux $\hat{F}$ defined as

$$(1.9)\qquad \hat{F}(x_0) = \hat{a}\partial_x\hat{u}(x_0) + \varepsilon^2\beta\partial_{xxx}\hat{u}(x_0).$$

We want to mention here that for a wellposed effective model for the long time wave equation we refer the reader to [16]. Furthermore, for a FE-HMM approximating the solution of the mentioned well-posed model we refer to [5]. In principle, the accuracy of the HMM described above depends on the accuracy of the upscaling procedure. This is because the macro model (1.4) has exactly the same form as the long time effective equation (1.3), and if, in addition, the upscaling procedure gives the right macrosopic flux then the HMM is equivalent to a finite dimensional approximation of the long-time effective equation (1.3). For a preliminary incomplete analysis of the upscaling error we refer the reader to [14].

In this paper, we give a theoretical foundation of the (FD-HMM) [11], by proving that HMM indeed computes the correct flux for the long time multiscale wave problem (1.1). With suitable macroscale discretization parameters, it will therefore capture the $O(1)$ dispersive effects in (1.3). More precisely, let F_{HMM} be the flux computed by HMM when the micro problem (1.6) is given initial data $\bar{u}(x)$ consistent (up to $O\left((\varepsilon/\eta)^q\right)$) with a third order polynomial $\hat{u}(x)$, then

$$\left|F_{HMM}(x_0) - \hat{F}(x_0)\right| \le C\left((\varepsilon/\eta)^q + \eta\,(\varepsilon/\eta)^{q-1}\right),$$

where q is a parameter associated with the smoothness of the averaging kernels, which in principle can be chosen arbitrarily large. Moreover, as a part of our analysis, we give a surprisingly simple expression for the parameter β which was known before to equal a very complicated functional of A, [18]. We prove that it is simply

$$\beta = \hat{a}\|\chi\|^2_{L^2[0,1]},$$

where χ is the well-known periodic cell solution from homogenization theory. In our proof we use two new ideas; the first idea is to look at the solutions of periodic wave equations with a special form of data known as quasi polynomials, where the polynomial coefficients are replaced by periodic functions. This is useful in unfolding the spatial structure of the solution as well as expressing the locally periodic solutions in terms of combination of much simpler purely periodic functions. Next, we look at the local time averages of solutions of hyperbolic PDEs and provide general statements which might potentially be applicable to much broader areas. With the help of these two ideas we are able to fully understand the crucial role consistency plays in HMM type algorithms. Finally we present numerical results to support our theoretical statements.

This paper is organized as follows. In Section 2, we present quasi-polynomials and we introduce the general purpose averaging kernels. In Section 3, we present an energy estimate regarding coupled wave equations. In Section 4, we introduce the local time averaging of oscillatory solutions of hyperbolic PDEs. Section 5 is devoted to the analysis and contains the main statement. Finally we conclude our paper by providing numerical examples to verify our theoretical claims.

2. Preliminaries.

2.1. Notation. In this section we introduce the notation that we use throughout this paper. We write $Y := [0,1]^d$ to denote the d-dimensional unit cube. The notation $\overline{f(t,\cdot)} := \int_Y f(t,y)dy$ stands for the spatial average of a function f over the unit cube Y. If f is time independent, we will simply use $\overline{f} := \int_Y f(y)dy$. We write

$$A \in M(c_1, c_2, \Omega), \tag{2.1}$$

to mean that $A = \{a_{ij}\}_{i,j=1}^d \in (C^\infty(\Omega))^{d\times d}$ is a smooth, uniformly positive definite, symmetric and bounded matrix function on Ω s.t

$$c_1|\zeta|^2 \leq \zeta^T A(x)\zeta, \quad |A(x)\zeta| \leq c_2|\zeta|, \quad \forall \zeta \in \mathbb{R}^d, \text{ and } \forall x \in \Omega.$$

We will use the notation a instead of A for 1-dimension. We say that a function f defined on Y is Y-periodic if $f(y+e_j) = f(y)$ for all $y \in Y$, where e_j is the standard canonical basis in jth direction. For simplicity we will say that a function is periodic to mean that it is Y-periodic. We will also use the one-dimensional periodic operators:

$$L[w] = \partial_x(a\partial_x w), \quad M[w] = a\partial_x w + \partial_x(aw), \quad N[w] = aw. \tag{2.2}$$

where $a \in M(c_1, c_2, Y)$ is periodic and w is a smooth, periodic function. The d-dimensional version of the Y-periodic operator L is defined as $L := \nabla \cdot A\nabla$.

2.2. Quasi-polynomials. Given a macroscopic state $\hat{u}(x) = s_0 + s_1 x + s_2 x^2 + s_3 x^3$, the HMM solves the micro-problem (1.6). For the analysis, we assume that the micro problem is posed over the entire $\mathbb{R}$:

$$\begin{aligned} &\partial_{tt}u^\varepsilon(t,x) = \partial_x(a(x/\varepsilon)\partial_x u^\varepsilon(t,x)), \\ &u^\varepsilon(0,x) = \bar{u}, \quad \partial_t u^\varepsilon(0,x) = 0, \end{aligned} \tag{2.3}$$

with a polynomial $\bar{u}$ consistent with $\hat{u}$.

As equation (2.3) is equipped with a polynomial initial data, we use the notion of quasi-polynomials from [7] to understand the structure of periodic wave equations with polynomial initial data. In particular, using the theory from [7] we can prove that the solution of the periodic wave problem (2.3) has a special structure known as quasi-polynomials.

DEFINITION 2.1. *A function $P(x, y) : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ belongs to the set $\mathbb{P}_n$ of quasi-polynomials of degree n if*

$$P(x, y) = p_0(y) + p_1(y)x + p_2(y)x^2 + \cdots + p_n(y)x^n,$$

where $p_i(y)$ are infinitely differentiable 1-periodic functions, named the coefficients functions of P.

In [7], we proved that the solution of a periodic wave equation with quasi-polynomial data belongs to the class of quasi-polynomials as well. We present the one-dimensional version of the theorem from [7].

THEOREM 2.1. *Assume that $Q, Z, P(t, \cdot, \cdot) \in \mathbb{P}_n$ and that $u(t, x)$ solves*

$$\begin{aligned} &u_{tt} = L[u] + P(t, x, x), \\ &u(0, x) = Q(x, x), \quad u_t(0, x) = Z(x, x), \end{aligned} \tag{2.4}$$

where the operator L is defined in (2.2). Then there is a family of quasi-polynomial $U(t, \cdot, \cdot) \in \mathbb{P}_n$ such that the solution to (2.4) is given as $u(t, x) = U(t, x, x)$. The coefficient functions of U solve the forced wave equations

$$\begin{aligned} \partial_{tt} u_j \quad &= L[u_j] + p_j + f_j, \\ u_j(0, x) \quad &= q_j(x), \quad \partial_t u_j(0, x) = z_j(x), \end{aligned}$$

where p_j, q_j, z_j are the coefficient functions of P, Q, Z, and

$$f_j(t, x) = \begin{cases} 0 & j = n, \\ nM[u_n] & j = n - 1, \\ (j + 1)M[u_{j+1}] + (j + 2)(j + 1)N[u_{j+2}] & j \leq n - 2, \end{cases}$$

and the operators M and N are given in (2.2).

2.2.1. Splitting. In general u_j does not have zero average in space. Here we split the coefficient function u_j into an average zero part $\tilde{u}_j(t, x) := u_j(t, x) - \overline{u_j(t, \cdot)}$, where $\overline{u(t, \cdot)} = \int_0^1 u(t, y)dy$, and a space independent part $g_j(t) := \overline{u_j(t, \cdot)}$ such that

$$u_j(t, x) = \tilde{u}_j(t, x) + g_j(t).$$

Then we can rewrite the solution u of (2.4) as

$$u(t, x) = \sum_{j=0}^{n} x^j u_j(t, x) = \sum_{j=0}^{n} x^j \tilde{u}_j(t, x) + \sum_{j=0}^{n} x^j g_j(t).$$

Now the coefficient functions $\tilde{u}_j$ and the space average g_j satisfy

$$\begin{aligned} \partial_{tt} \tilde{u}_j \quad &= L[\tilde{u}_j] + p_j(t, x) - \overline{p_j(t, \cdot)} + f_j(t, x) - \overline{f_j(t, \cdot)}, \\ \tilde{u}_j(0, x) \quad &= q_j(x) - \overline{q_j(\cdot)}, \quad \partial_t \tilde{u}_j(0, x) = z_j(x), -\overline{z_j(\cdot)}, \end{aligned}$$

and

$$\begin{aligned} g_j''(t) \quad &= \overline{p_j(t, \cdot)} + \overline{f_j(t, \cdot)}, \\ g_j(0) \quad &= \overline{q_j(\cdot)}, \quad g_j'(0) = \overline{z_j(\cdot)}, \end{aligned}$$

where

$$f_j(t,x)=\begin{cases}0 & j=n,\\ nM[\tilde{u}_n]+ng_n(t)a_x & j=n-1,\\ (j+1)\left(M[\tilde{u}_{j+1}]+g_{j+1}(t)a_x\right)+(j+2)(j+1)\left(N[\tilde{u}_{j+2}]+g_{j+2}(t)a\right) & j\le n-2.\end{cases}$$

$$\overline{f_j(t,\cdot)}=\begin{cases}0 & j=n,\\ n\overline{a\partial_x\tilde{u}_n} & j=n-1,\\ (j+1)\overline{a\partial_x\tilde{u}_{j+1}}+(j+1)(j+2)\left(\overline{a\tilde{u}_{j+2}}+g_{j+2}(t)\bar{a}\right) & j\le n-2.\end{cases}$$

Now we present a corollary which follows immediately from the above discussions. The result will give the exact form of the equations for $\{\tilde{u}_j\}_{j=0}^n$ when a wave equation is equipped with polynomial initial data and zero forcing.

COROLLARY 2.1. *Suppose that v solves the one dimensional problem*

$$\begin{aligned}&\partial_{tt}v=L[v],\\ &v(0,x)=r_0+r_1x+\cdots+r_nx^n,\quad \partial_t v(0,x)=0.\end{aligned}\tag{2.5}$$

where r_i are constant numbers and $0\le n\le 3$, then

$$v(t,x)=\tilde{v}_0(t,x)+x\tilde{v}_1(t,x)+\cdots+x^n\tilde{v}_n(t,x)+g_0(t)+xg_1(t)+\cdots+x^ng_n(t).$$

where $\tilde{v}_j(t,x)$s are 1-periodic with zero spatial average $\overline{\tilde{v}_j(t,\cdot)}=0$ and satisfy

$$\begin{aligned}&\partial_{tt}\tilde{v}_{n-i}=L[\tilde{v}_{n-i}]+P_{n-i}-\overline{P_{n-i}(t,\cdot)}+f_{n-i}(t,x)-\overline{f_{n-i}(t,\cdot)},\\ &\tilde{v}_{n-i}(0,x)=\partial_t\tilde{v}_{n-i}(0,x)=0\\ &g''_{n-i}(t)=\overline{P_{n-i}(t,\cdot)}+\overline{f_{n-i}(t,\cdot)},\quad g_{n-i}(0)=r_{n-i},\quad g'_{n-i}(0)=0,\quad i=0,1,\cdots,n,\end{aligned}$$

with

$$P_{n-i}=\begin{cases}0 & i=0\\ nM[\tilde{v}_n] & i=1,\quad n\ge1,\\ (n-1)M[\tilde{v}_{n-1}]+n(n-1)a\tilde{v}_n & i=2,\quad n\ge2,\\ (n-2)M[\tilde{v}_{n-2}]+(n-1)(n-2)a\tilde{v}_{n-1} & i=3,\quad n=3\end{cases}$$

$$f_{n-i}=\begin{cases}0 & i=0\\ ng_n(t)a_x & i=1,\quad n\ge1,\\ (n-1)g_{n-1}(t)a_x+n(n-1)g_n(t)a & i=2,\quad n\ge2,\\ (n-2)g_{n-2}(t)a_x+(n-1)(n-2)g_{n-1}(t)a & i=3,\quad n=3.\end{cases}$$

2.3. Averaging kernels. The upscaling procedure in HMM requires solving a multi-scale problem with a fixed ε and averaging out the small scale fluctuations to obtain effective data which are then used in the macroscopic model. Indeed, the accuracy of the HMM is associated with how fast these local averaging converges to the effective macroscopic data. At this point it is vital to have high-order convergence rates, in terms of the small scale parameter ε, for averages of the oscillatory data involved in the computations. In principle, it is possible to improve the convergence rate to an arbitrary order using the idea of averaging kernels. We start with introducing the space of smoothing kernels $\mathbb{K}^{p,q}$.

DEFINITION 2.2. *A normalized function K is in the space $\mathbb{K}^{p,q}$ if*

- $K^{(q+1)} \in BV(\mathbb{R})$ *and* K *has compact support in* $[-1,1]$
- K *has* p *vanishing moments:*

$$\int_{-1}^{1} K(t)t^r dt = \begin{cases} 1 & r = 0 \\ 0 & 0 < r \leq p. \end{cases}$$

EXAMPLE 2.1. *Two examples of kernels* $K \in \mathbb{K}^{p,q}$ *are* $\chi_{[-1,1]} \in \mathbb{K}^{1,-1}$*, and* $\frac{3}{4}\chi_{[-1,1]}(1-t^2) \in \mathbb{K}^{1,0}$*. In the former, the kernel itself has bounded variation while in the latter the derivative of the kernel has bounded variation.* In HMM, we average out oscillatory functions in localized domains, sizes of which should be such that the kernels capture at least a few oscillations of the integrand. The size of these localized domains are represented by η which in practice is $O(\varepsilon)$. Therefore, for averaging over these domains, we use the η-scaled version of $K(t)$:

$$K_\eta(t) = \frac{1}{\eta} K(\frac{t}{\eta}).$$

Our aim in this section is to apply these scaled kernels to functions of the form $f(t,s) = t^r b(s)$, where f is multiplicatively seperable in terms of fast and slow variations. We consider only symmetric kernels and define the convolution as

$$\left(K * f\right)(t) = \int_{\mathbb{R}} K(t-s)f(s)ds = \int_{\mathbb{R}} K(s-t)f(s)ds.$$

An immediate consequence of the above definition is the following Lemma.

LEMMA 2.1. *Suppose* $f(t) = t^r$ *with* $r \in \mathbb{Z}^+$ *and* $K \in \mathbb{K}^{p,q}$ *with* $p \geq r > 0$*, then*

$$\left(K_\eta * f\right)(t) = t^r.$$

We also have that

LEMMA 2.2. *Suppose that* $H \in BV(\mathbb{R})$ *with supp*$H \subset [-1,1]$*. Let* g *be a continuous 1-periodic function with* $\overline{g} = 0$*. Then, with* $0 < \alpha < 1$ *we have*

$$\left| \int_{-1}^{1} g(t/\alpha)H(t)dt \right| \leq 3\alpha |g|_\infty Var(H), \tag{2.6}$$

where $Var(H)$ *is the total variation of* H*.*

Proof. Let $1/\alpha = N + \gamma$ where N stands for the integer part and γ stands for the fractional part of $1/\alpha$. Then

$$\begin{aligned} \int_{-1}^{1} g(t/\alpha)H(t)dt &= \alpha \int_{-1/\alpha}^{1/\alpha} g(t)H(\alpha t)dt \\ &= \alpha \left(\left\{ \int_{-N-\gamma}^{-N} + \int_{N}^{N+\gamma} + \int_{-N}^{N} \right\} g(t)H(\alpha t)dt \right) \\ &= \alpha \left(R_1 + R_2 + R_3\right). \end{aligned}$$

Since H has compact support it follows that $|H|_\infty \leq Var(H)$. Hence

$$|R_1| := \left| \int_{-N-\gamma}^{-N} g(t)H(\alpha t)dt \right| \leq |g|_\infty |H|_\infty \leq |g|_\infty Var(H).$$

Similarly $|R_2| \leq |g|_\infty Var(H)$. It remains to bound $|R_3|$. Since g is periodic and $\overline{g} = 0$

$$\begin{aligned} |R_3| &= \left| \sum_{j=-N}^{N-1} \int_0^1 g(t)H\left(\alpha\left(j+t\right)\right)dt \right| \\ &= \left| \sum_{j=-N}^{N-1} \int_0^1 g(t)\left[H\left(\alpha\left(j+t\right)\right) - H(\alpha j)\right]dt \right| \leq |g|_\infty Var(H). \end{aligned}$$

This shows (2.6). □

We consider now functions of the form $g(t,s) = t^r f(s)$, for a periodic function f and prove the following Lemma.

LEMMA 2.3. *Let f be a 1-periodic continuous function and $K \in \mathbb{K}^{p,q}$. Then, with $\alpha = \varepsilon/\eta$, and $\bar{f} = \int_0^1 f(s)ds$*

$$\left| \int_{\mathbb{R}} K_\eta(t) f(t/\varepsilon) dt - \bar{f} \right| \leq C |f|_\infty \alpha^{q+2},$$

and when $r \in \mathbb{Z}^+$

$$\left| \int K_\eta(t) t^r f(t/\varepsilon) dt \right| \leq C \begin{cases} |f|_\infty \alpha^{q+2} \eta^r & 1 \leq r \leq p \\ |f|_\infty \alpha^{q+2} \eta^r + |\bar{f}| \eta^r & r > p, \end{cases}$$

where the constant C does not depend on ε, η, f or s but may depend on K, p, q, r.

Proof. Let $F^{[0]}(t) = f(t) - \bar{f}$ and

$$F^{[n+1]}(t) = \int_0^t F^{[n]}(s) - \int_0^1 \int_0^s F^{[n]}(\tau) d\tau ds.$$

Then $F^{[n]}(t)$ is 1-periodic and has zero average for all n. Moreover,

$$\frac{d^n}{dt^n} F^{[n]}(t) = f(t) - \bar{f}.$$

Also $|F^{[n]}|_\infty \leq 2^n |f|_\infty$. Hence with $\alpha = \varepsilon/\eta$,

$$\begin{aligned} \int K_\eta(t) t^r f(t/\varepsilon) dt &= \eta^r \int_{-1}^1 K(t) t^r f(t/\alpha) dt \\ &= \alpha^{q+1} \eta^r \int_{-1}^1 K(t) t^r \frac{d^{q+1}}{dt^{q+1}} F^{[q+1]}(t/\alpha) dt + \eta^r \bar{f} \int K(t) t^r dt \\ &= (-1)^{q+1} \alpha^{q+1} \eta^r \int_{-1}^1 F^{[q+1]}(t/\alpha) \frac{d^{q+1}}{dt^{q+1}} \left(K(t) t^r \right) dt + \eta^r \bar{f} \int K(t) t^r dt. \end{aligned}$$

Since $K^{(q+1)} \in BV(\mathbb{R})$ and K has compact support we have that $K^{(s)} \in BV(\mathbb{R})$ for all $0 \leq s \leq q$. From here it follows that $\frac{d^{q+1}}{dt^{q+1}} \left(K(t) t^r \right) \in BV(\mathbb{R})$. Furthermore, by the fact that F^n is 1-periodic and has zero average, an application of Lemma 2.2 gives

$$\left| \int_{-1}^1 F^{[q+1]}(t/\alpha) \frac{d^{q+1}}{dt^{q+1}} \left(K(t) t^r \right) dt \right| \leq 3\alpha |F^{[q+1]}|_\infty Var((Kt^r)^{q+1}) \leq C_{K,q,r} \alpha |f|_\infty.$$

On the other hand

$$\int_{-1}^1 K(t) t^r dt = \begin{cases} 1 & r = 0 \\ 0 & 1 \leq r \leq p \\ C_r & r > p. \end{cases}$$

This completes the proof of the Lemma. □

Finally, we present a lemma showing that when we apply K to a quasi-polynomial, the result is again a quasi-polynomial. This will be used later in Theorem 5.1.

LEMMA 2.4. *Suppose that $b \in \mathbb{P}_n$ such that $b(x) = \sum_{j=0}^n x^j b_j(x) \in \mathbb{P}_n$, and $K \in \mathbb{K}^{p,q-2}$ with $p \geq n$, and $q \geq 2$. Then*

$$\varepsilon^n \left(K_\eta * b(\cdot/\varepsilon) \right)(x) = Q^\varepsilon(x, x/\varepsilon),$$

where $Q^\varepsilon(x,y) = \sum_{j=0}^n x^j Q_j^\varepsilon(x) \in \mathbb{P}_n$*. Moreover, if* $p \geq n$ *and* $\overline{b_j} = 0$ *then*

$$|Q_k^\varepsilon(x)| \leq C\alpha^q \sum_{r=k}^n \varepsilon^{n-r} |b_r|_\infty$$

where C *depends only on* K, p, q *and* n.

Proof. We have with $c_{r,j} = (-1)^{j-r}\binom{j}{r}$

$$\begin{aligned}
\varepsilon^n \left(K_\eta * b(\cdot/\varepsilon)\right)(x) &= \varepsilon^n \int K_\eta(x-s)b(s/\varepsilon)ds = \varepsilon^n \textstyle\sum_{j=0}^n \int K_\eta(x-s)\frac{s^j}{\varepsilon^j} b_j(\frac{s}{\varepsilon})ds \\
&= \varepsilon^n \textstyle\sum_{j=0}^n \sum_{r=0}^j c_{r,j}\varepsilon^{-j}x^r \underbrace{\int K_\eta(s)s^{j-r}b_j(\frac{x-s}{\varepsilon})ds}_{:=\psi_{r,j}(x/\varepsilon)} \\
&= \varepsilon^n \textstyle\sum_{j=0}^n \sum_{r=0}^j c_{r,j}\varepsilon^{-j}x^r \psi_{r,j}(x/\varepsilon) \\
&= \textstyle\sum_{k=0}^n x^k \underbrace{\sum_{r=k}^n c_{k,r}\varepsilon^{n-r}\psi_{k,r}(x/\varepsilon)}_{Q_k^\varepsilon(x/\varepsilon)}.
\end{aligned}$$

Note that $\psi_{k,r}(y)$ is 1-periodic since $b_r(y)$ is 1-periodic. Therefore, $Q_k(y)$ is also 1-periodic. This proves the first part of the Lemma. For proving the second claim, first using Lemma 2.3 we estimate $|\psi_{k,r}(y)|$

$$|\psi_{k,r}(y)| = \left|\int K_\eta(s)s^{r-k}b_r(y-s/\varepsilon)ds\right| \leq C\alpha^q |b_r|_\infty$$

since $p \geq n \geq r-k$ and $\overline{b_j} = 0$. Next we have the desired estimate as follows

$$|Q_k^\varepsilon(y)| \leq C\sum_{r=k}^n \varepsilon^{n-r}|\psi_{k,r}(y)| \leq C\alpha^q \sum_{r=k}^n \varepsilon^{n-r}|b_r|_\infty .$$

□

3. Energy estimates for solutions of coupled wave equations. The main result of this section is Theorem 3.2 which shows the polynomial time growth of solutions of coupled wave equations. The equations are coupled in the sense that starting with a hyperbolic PDE with a smooth forcing term, the solution of this initial PDE enters as the forcing term in the next equations and new equations are forced with all other preceding solutions. This coupling introduces a resonance effect leading to polynomial growth in the solution. We emphasize that some of the results in this section hold under weaker assumptions on data. However, we assume smoothness to make the exposition simpler. We start with some intermediate lemmas.

LEMMA 3.1. *Let* $f \in H^k(Y)$, $\overline{f} = 0$ *and* $A \in M(c_1, c_2, Y)$ *be* Y*-periodic. Then there exists a unique* Y*-periodic function* $u \in H^{k+2n}$ *satisfying*

$$(-1)^n L^n[u] = f, \quad \overline{u} = 0,$$

for any positive integer n*. In addition, the following stability estimate holds*

$$\|u\|_{H^{k+2n}(Y)} \leq C\|f\|_{H^k(Y)},$$

where C *depends only on* k*, the domain* Y *and the coefficient* A.

Proof. For $n = 1$ the result is classical. Now suppose that $n > 1$, then with $u_1 = u$ we write

$$-Lu_1 = u_2, \quad -Lu_2 = u_3, \cdots, -Lu_{n-1} = u_n, \quad -Lu_n = f.$$

Furthermore, note that u_js are Y-periodic and $\overline{u_j} = 0, j = 1, \cdots, n$. This follows from the fact that u_1 is an average zero Y-periodic function and that Lu_1 is 1-periodic with zero average. Then we obtain the final result by induction. ◻ Now we present a well-known Theorem which we use later in the analysis.

THEOREM 3.1. *Let $f \in C^\infty([0,T] \times Y)$, $f(t,y)$ be periodic in y, and $\overline{f(t,\cdot)} = 0$. Moreover, let $g, h \in C^\infty(Y)$ be 1-periodic functions with $\overline{g} = \overline{h} = 0$. Then there is a unique solution $u \in C^\infty([0,T] \times Y)$, with $\overline{u(t,\cdot)} = 0$, solving*

$$\begin{aligned} &\partial_{tt} u = L[u] + f(t,y), \\ &u(0,y) = g, \quad \partial_t u(0,y) = h. \end{aligned}$$

Moreover, there exists a constant C independent of t such that

$$\|u(t,\cdot)\|_{H^1(Y)} \le CE_u^{1/2}(0) + C \begin{cases} \int_0^t \|f(s,\cdot)\|_{L^2(Y)} ds, & f \text{ is time dependent} \\ \|f\|_{L^2(Y)} & f \text{ is time independent.} \end{cases} \tag{3.1}$$

where the energy $E_u(t)$ is defined as

$$E_u(t) := \frac{1}{2} \int_Y |u_t(t,\cdot)|^2 + A\nabla u(t,\cdot) \cdot \nabla u(t,\cdot) dy.$$

Proof. The existence and uniqueness is classical. The proof of the estimate (3.1) when f is time dependent follows from the standard energy estimate

$$E_u^{1/2}(t) \le E_u^{1/2}(0) + \int_0^t \|f(s,\cdot)\|_{L^2(Y)} ds,$$

and the Poincaré inequality. The proof of the estimate for time independent f is obtained by considering $v = u + \psi$, where $L[\psi] = f$. ◻

Now we are ready to state the main result of this section.

THEOREM 3.2. *Suppose $\{u_j\}_{j=0}^n$ solve the one-dimensional periodic wave equations*

$$\begin{aligned} &\partial_{tt} u_j = L[u_j] + P_j(u_{j+1}, u_{j+2}, \cdots, u_n) + f_j(t,x), \\ &u_j(0,x) = \partial_t u_j(0,x) = 0. \end{aligned}$$

Here

$$P_j(u_{j+1}, \cdots, u_n) = \sum_{i=j+1}^{n} \alpha_{j,i} \left(M[u_i] - \overline{M[u_i]} \right) + \beta_{j,i} \left(N[u_i] - \overline{N[u_i]} \right),$$

where L, M, N are defined in (2.2)*. $\alpha_{j,i}, \beta_{j,i}$ are real bounded constants, and $f_j \in C^\infty([0,T] \times Y)$ with $\overline{f_j(t,\cdot)} = 0$. Then*

$$\|u_0(t,\cdot)\|_{H^1(Y)} \le C \sum_{j=0}^{n} \left(1 + t^{j+1}\right) \max_{0 \le s \le t} \|f_j(s,\cdot)\|_{L^2(Y)},$$

where C is independent of t but may depend on Y and a.

Proof. Let $\{u_{j,i}\}_{i=j+1}^{n+1}$ be the solution of the wave equation with the forcing terms

$$F_{j,i} = \begin{cases} \alpha_{j,i}\left(M[u_i] - \overline{M[u_i]}\right) + \beta_{j,i}\left(N[u_i] - \overline{N[u_i]}\right), & j+1 \le i < n+1 \\ f_j, & i = n+1, \end{cases}$$

and zero initial data, then

$$u_j(t,y) = \sum_{i=j+1}^{n+1} u_{j,i}(t,y).$$

Note that by triangle inequality we have

$$\|u_j\|_{H^1(Y)} \le \sum_{i=j+1}^{n+1} \|u_{j,i}\|_{H^1(Y)}.$$

In addition, since $\overline{F_{j,i}} = 0$ we have also that $\overline{u_{j,i}(t,\cdot)} = 0$. We can then employ Theorem 3.1 and the fact

$$\|M[u] - \overline{M[u]}\|_{L^2(Y)} \le C\|u\|_{H^1(Y)}$$

and

$$\|N[u] - \overline{N[u]}\|_{L^2(Y)} \le C\|u\|_{L^2(Y)}$$

to see that

$$\|u_{j,i}\|_{H^1(Y)} \le Ct \max_{0\le s\le t} \begin{cases} \|u_i(s,\cdot)\|_{H^1(Y)} & \text{if } j+1 \le i \le n \\ \|f_j(s,\cdot)\|_{L^2(Y)}, & \text{if } i = n+1. \end{cases}$$

Therefore for $j = 0, \cdots, n$ we have

$$\|u_j\|_{H^1(Y)} \le Ct\left(\left(\sum_{i=j+1}^{n} \max_{0\le s\le t} \|u_i(s,\cdot)\|_{H^1(Y)}\right) + \max_{0\le s\le t} \|f_j(s,\cdot)\|_{L^2(Y)}\right).$$

From here a simple induction leads to the desired result. ◻

4. Time averaging of wave equations. The HMM flux is computed by averaging the microscopic flux $A(x/\varepsilon)\nabla u^\varepsilon$ in time and space in a domain of size $\eta = O(\varepsilon)$. We introduce the time averaged solution

$$d^\varepsilon(x) := (K_\tau * u^\varepsilon(\cdot, x))(0).$$

Then the HMM flux F_{HMM} can be rewritten as

$$\begin{aligned} F_{HMM}(x_0) &= (K_\tau * (K_\eta * a(\cdot/\varepsilon)\partial_x u^\varepsilon(\cdot,\cdot))(x_0))(0) \\ &= (K_\eta * a(\cdot/\varepsilon)\partial_x (K_\tau * u^\varepsilon(\cdot,\cdot))(0))(x_0) \\ &= (K_\eta * a(\cdot/\varepsilon)\partial_x d^\varepsilon(\cdot))(x_0). \end{aligned}$$

With the above motivation, we present a theorem which gives an explicit equation for the time averages of solutions of wave equations.

Before stating the result we introduce eigenfunction expansions which will be used in the proof. Let $\{\lambda_j^2, \varphi_j(y)\}_{j=0}^{\infty}$ be the eigenvalue eigenfunction pair corresponding to the operator $-L$. Then, except for the zero eigenvalue $\lambda_0^2 = 0$, all other eigenvalues are strictly positive such that, [15],

$$0 = \lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots .$$

The eigenfunctions $\varphi_j \in C^\infty$ are periodic with zero average $\overline{\varphi_j(\cdot)} = 0$ except for $\varphi_0 = 1$ and they form an orthonormal basis for the space of periodic and $L^2(Y)$ integrable functions such that every periodic function $f \in L^2(Y)$ can be written as

$$f(y) = \sum_{j=0}^{\infty} f_j \varphi_j(y).$$

THEOREM 4.1. *Suppose* $\alpha = \frac{\varepsilon}{\tau}$ *where* $0 < \varepsilon \leq \tau$. *Let* $f \in C^\infty([0, \alpha^{-1}], Y)$ *be a* Y*-periodic function with* $\overline{f(t, \cdot)} = 0$, *and* $K \in \mathbb{K}^{p,q}$ *with an even* q. *Furthermore, let* w *be the solution of the problem*

$$\begin{aligned} &\partial_{tt} w = L[w] + f(t, x), \\ &w(0, x) = \partial_t w(0, x) = 0. \end{aligned} \tag{4.1}$$

Then the local time average $K_\tau * \partial_t^{2k} w(\cdot/\varepsilon, x)(0)$ *satisfies*

$$K_\tau * \partial_t^{2k} w(\cdot/\varepsilon, x)(0) = \sum_{\ell=k}^{q/2-1} L^k \psi_\ell + \alpha^q R_k(x), \quad k = 0, 1, \cdots, q/2 - 1,$$

where ψ_ℓ *is the zero average solution of the problem*

$$L^{\ell+1} \psi_\ell = -K_\tau * \partial_t^{2\ell} f(\cdot/\varepsilon, x)(0).$$

Moreover R_k *is* Y*-periodic with zero average* ($\overline{R_k} = 0$) *and*

$$\|R_k\|_{H^1(Y)} \leq C_{p,q} \max_{-1 \leq s \leq 1} \|w(s/\alpha, \cdot)\|_{L^2(Y)}, \quad L[R_{k-1}] = R_k, \quad k = 1, 2, \cdots, q/2 - 1.$$

REMARK 4.1. *Later in the main Theorem we will write down equations for the time averages of solutions of wave equations. To ease the readability later, we note that the time average* $d(x) := K_\tau * w(\cdot/\varepsilon, x)(0)$ *satisfies*

$$L[d](x) = -\sum_{\ell=0}^{q/2-1} L^{-\ell} K_\tau * \partial_t^{2\ell} f(\cdot/\varepsilon, x)(0) + \alpha^q R_1(x).$$

Moreover, later in the analysis the right hand side functions f *will be even functions of* t. *This implies that the solution* w *is extended evenly for* $t < 0$. *Therefore, we equivalently write*

$$\max_{-1 \leq s \leq 1} \|w(s/\alpha, \cdot)\|_{L^2(Y)} = \max_{0 \leq s \leq 1} \|w(s/\alpha, \cdot)\|_{L^2(Y)}.$$

Proof. Since $\overline{f(t, \cdot)} = 0$, by Theorem 3.1 $\overline{w(t, \cdot)} = 0$, and we write

$$w(t, x) = \sum_{j=1}^{\infty} w_j(t) \varphi_j(x).$$

Now we replace $f(t,x) = \sum_{j=1}^{\infty} f_j(t)\varphi_j(x)$ in the equation (4.1) and exploit the orthogonality of the eigenfunctions to see that $w_j(t)$ satisfies

$$w_j''(t) + \lambda_j^2 w_j(t) = f_j(t), \quad j > 0.$$

Setting $\alpha = \frac{\varepsilon}{\tau}$, and then using the above ODE we obtain:

$$w_j(t) = \frac{(-1)^{q/2}}{\lambda^q} w_j^{(q)}(t) + \frac{1}{\lambda_j^2} \sum_{m=0}^{q/2-1} \frac{(-1)^m}{\lambda_j^{2m}} f_j^{(2m)}(t).$$

Therefore

$$\begin{aligned} K_\tau * w_j(\cdot/\varepsilon)(0) &= \textstyle\int_{-\tau}^{\tau} K_\tau(s) w_j(s/\varepsilon) ds = \int_{-1}^{1} K(s) w_j(s/\alpha) ds \\ &= \textstyle\frac{(-1)^{q/2}}{\lambda_j^q} \int_{-1}^{1} K(s) w_j^{(q)}(s/\alpha) ds + \frac{1}{\lambda_j^2} \sum_{m=0}^{q/2-1} \frac{(-1)^m}{\lambda_j^{2m}} \int_{-1}^{1} K(s) f_j^{(2m)}(s/\alpha) ds \\ &= \textstyle\frac{\alpha^q (-1)^{q/2}}{\lambda_j^q} \int_{-1}^{1} K^{(q)}(s) w_j(s/\alpha) ds + \frac{1}{\lambda_j^2} \sum_{m=0}^{q/2-1} \frac{(-1)^m}{\lambda_j^{2m}} K_\tau * f_j^{(2m)}(\cdot/\varepsilon)(0). \end{aligned}$$

But

$$\begin{aligned} K_\tau * w(\cdot/\varepsilon, x) := \textstyle\sum_{j=1}^{\infty} K_\tau * w_j(\cdot/\varepsilon)(0)\varphi_j(x) = \sum_{j=1}^{\infty} \frac{\alpha^q (-1)^q}{\lambda_j^q} \int_{-1}^{1} K^{(q)}(s) w_j(s/\alpha) ds \varphi_j(x) \\ + \textstyle\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} \sum_{m=0}^{q/2-1} \frac{(-1)^m}{\lambda_j^{2m}} K_\tau * f_j^{(2m)}(\cdot/\varepsilon)(0)\varphi_j(x). \end{aligned}$$

Now let us define ψ_m by

$$\psi_m := -\sum_{j=1}^{\infty} \frac{(-1)^{m+1}}{\lambda_j^{2(m+1)}} K_\tau * f_j^{(2m)}(\cdot/\varepsilon)(0)\varphi_j(x), \tag{4.2}$$

and v by

$$v(t,x) = \sum_{j=1}^{\infty} \frac{(-1)^q}{\lambda_j^q} w_j(t)\varphi_j(x). \tag{4.3}$$

Then

$$K_\tau * w(\cdot/\varepsilon, x) = \sum_{m=0}^{q/2-1} \psi_m + \alpha^q \int_{-1}^{1} K^{(q)}(s) v(s/\alpha, x) ds.$$

Note that $v(t,x)$ in (4.3) is 1-periodic with respect to x, and is an average zero $\overline{v(t,\cdot)} = 0$ solution of the problem

$$L^{q/2} v(t,x) = (-1)^{q/2} \sum_{j=1}^{\infty} w_j(t)\varphi_j(x) = (-1)^{q/2} w(t,x).$$

By Lemma 3.1 we have

$$\|v(t,\cdot)\|_{H^1(Y)} \le C \|w(t,\cdot)\|_{L^2(Y)}.$$

Therefore we obtain

$$\|K_\tau * w(\cdot/\varepsilon, x)(0) - \sum_{\ell=0}^{q/2-1} \psi_\ell\|_{H^1(Y)} \le C_{K,p,q} \alpha^q \max_{-1 \le s \le 1} \|w(s/\alpha, \cdot)\|_{L^2(Y)}.$$

Furthermore from (4.2) we have

$$L^{m+1}\psi_m = -\sum_{j=1}^{\infty} K_\tau * f_j^{(2m)}(\cdot/\varepsilon)(0)\varphi_j(x) = -K_\tau * \partial_t^{2m} f(\cdot/\varepsilon, x)(0).$$

This proves the Theorem for $k = 0$. To prove the remaining part ($k > 0$) we first introduce

$$\begin{aligned} R_0(x) &:= K^{(q)} * v(\cdot/\alpha, x)(0) \\ R_k(x) &:= L^k[R_0], \quad k = 0, 1, \cdots, q/2 - 1. \end{aligned}$$

Again by precisely the same arguments as above we have

$$\|R_k\|_{H^1(Y)} \leq C_{K,p,q} \max_{-1\leq s\leq 1} \|w(s/\alpha, \cdot)\|_{L^2(Y)}. \tag{4.4}$$

Now applying the operator K_τ to (4.1) and using the result of the Theorem for $k = 0$, and the fact that $L[\psi_0] = -K_\tau * f(\cdot/\varepsilon, x)(0)$ we obtain

$$\begin{aligned} K_\tau * \partial_t^2 w(\cdot/\varepsilon, x)(0) &= L[K_\tau * w(\cdot/\varepsilon, x)(0)] + K_\tau * f(\cdot/\varepsilon, x)(0) \\ &= L[\textstyle\sum_{\ell=0}^{q/2-1} \psi_\ell + \alpha^q R_0] + K_\tau * f(\cdot/\varepsilon, x)(0) \\ &= \textstyle\sum_{\ell=0}^{q/2-1} L[\psi_\ell] + \alpha^q R_1(x) + K_\tau * f(\cdot/\varepsilon, x)(0) \\ &= \textstyle\sum_{\ell=1}^{q/2-1} L[\psi_\ell] + \alpha^q R_1(x). \end{aligned}$$

Now assume that the result holds for $k = j$. We take the $2j$-th derivative of equation (4.1) and apply the kernel and use the fact that $L^{j+1}[\psi_j] = -K_\tau * \partial_t^{2j} f(\cdot/\varepsilon, x)(0)$ to see

$$\begin{aligned} K_\tau * \partial_t^{2(j+1)} w(\cdot/\varepsilon, x)(0) &= L[K_\tau * \partial_t^{2j} w(\cdot/\varepsilon, x)(0)] + K_\tau * \partial_t^{2j} f(\cdot/\varepsilon, x)(0) \\ &= L[\textstyle\sum_{\ell=j}^{q/2+1} L^j[\psi_\ell] + \alpha^q R_j(x)] + K_\tau * \partial_t^{2j} f(\cdot/\varepsilon, x)(0) \\ &= \textstyle\sum_{\ell=j}^{q/2+1} L^{j+1}[\psi_\ell] + \alpha^q R_{j+1}(x) + K_\tau * \partial_t^{2j} f(\cdot/\varepsilon, x)(0) \\ &= \textstyle\sum_{\ell=j+1}^{q/2+1} L^{j+1}[\psi_\ell] + \alpha^q R_{j+1}(x). \end{aligned}$$

Finally R_{j+1} satisfies (4.4). This finishes the proof of the Theorem. □

5. Main statements. In HMM, the upscaled quantity is the flux $F_{HMM}(x_0)$. It is computed by solving the oscillatory wave equation (1.6) in the microscopic domain $\Omega_{\eta,x_0} \times [0, \tau]$, where $\Omega_{\eta,x_0} := [-L_\eta + x_0, L_\eta + x_0]$ and $L_\eta = \eta + \tau\sqrt{|A|_\infty}$. One also has to make the micro-problem consistent with the current macroscopic quantity. This is done by choosing an appropriate initial data $\bar{u}(x)$ based on the coarse data and a suitable boundary condition (ex: $u^\varepsilon - \bar{u}$ is 1-periodic) for the micro-problem. In the context of long time wave propagation one needs to use a third-order interpolant of the coarse data to capture the $O(1)$ dispersive behaviors appearing in the long time. From now on we restrict ourselves to one dimension. We represent the interpolant of the coarse scale solutions by

$$\hat{u}(x) = s_0 + s_1(x - x_0) + s_2(x - x_0)^2 + s_3(x - x_0)^3. \tag{5.1}$$

Furthermore, let $\bar{u}(x)$ be a third degree polynomial such that when used as an initial data for the micro-problem

$$\begin{aligned} &\partial_{tt} u^\varepsilon(t, x) - \partial_x \left(a(x/\varepsilon)\partial_x u^\varepsilon(t, x)\right) = 0, \quad \text{in} \quad [0, \tau] \times \Omega_{\eta,x_0} \\ &u^\varepsilon(0, x) = \bar{u}(x), \quad \partial_t u^\varepsilon(0, x) = 0, \quad \text{on} \quad \{t = 0\} \times \Omega_{\eta,x_0}, \end{aligned} \tag{5.2}$$

the time and space average of the oscillatory solution u^ε is equal to $\hat{u}(x)$ up to high orders in $\alpha := \varepsilon/\eta$ (assume here $\eta = \tau$)

$$(\mathcal{K}_{\tau,\eta} * u^\varepsilon)(0,x) = \hat{u}(x) + O(\alpha^q), \quad x \in \Omega_{\eta,x_0}, \tag{5.3}$$

where the averaging operator $\mathcal{K}$ is defined as follows:

$$(\mathcal{K}_{\tau,\eta} * f)(t,x) := \int_{-\tau+t}^{\tau+t} \int_{-\eta+x}^{\eta+x} K_\tau(\tilde{t}-t) K_\eta(\tilde{x}-x) f(\tilde{t},\tilde{x}) d\tilde{x} d\tilde{t}.$$

We say that $\bar{u}$ is *consistent* with $\hat{u}$ up to $O(\alpha^q)$ if (5.3) holds. Finding such an appropriate $\bar{u}$ has already been observed to be essential in approximating the homogenized flux [11].

The final step of the HMM is to compute the flux by

$$F_{HMM}(x_0) = (\mathcal{K}_{\tau,\eta} * a(x/\varepsilon)\partial_x u^\varepsilon(t,x))(0,x_0). \tag{5.4}$$

The aim in this section is to estimate the error between the HMM flux (5.4) and the effective macroscopic flux

$$\hat{F}(x_0) = \hat{a}\partial_x \hat{u}(x_0) + \varepsilon^2 \beta \partial_{xxx} \hat{u}(x_0), \tag{5.5}$$

when the coefficient a is periodic. The coefficient $\hat{a}$ is the homogenized coefficient in one dimension and β is the same coefficient appearing in the formula (1.3). An explicit representation for β was given by Santosa and Symes [18]:

$$\begin{aligned} \beta &= \frac{\hat{a}}{12} - \hat{a}^2 \int_0^1 \int_0^y \int_0^s a^{-1}(s) dr ds dy - \hat{a}^3 \int_0^1 \int_0^y \int_0^s a^{-1}(y) a^{-1}(r) dr ds dy \\ &\quad + \hat{a}^2 \int_0^1 \int_0^y a^{-1}(r) dr dy - \hat{a}^3 \left(\int_0^1 \int_0^y a^{-1}(r) dr dy \right)^2. \end{aligned} \tag{5.6}$$

Now suppose that we are given the microscopic problem with an initial data

$$\bar{u}(x) = r_0(\varepsilon) + r_1(\varepsilon)(x-x_0) + r_2(\varepsilon)(x-x_0)^2 + r_3(\varepsilon)(x-x_0)^3$$

consistent (up to $O(\alpha^q)$) with the macro state $\hat{u}(x)$ in (5.1). Note that to achieve this, the coefficients of the polynomial $\bar{u}$ depend on ε. Then the micro-problem reads

$$\begin{aligned} \partial_{tt} u^\varepsilon(t,x) &= \partial_x (a(x/\varepsilon)\partial_x u^\varepsilon), \\ u^\varepsilon(0,x) = \bar{u}(x), &\quad \partial_t u^\varepsilon(0,x) = 0. \end{aligned} \tag{5.7}$$

We decompose the solution u^ε into four parts:

$$u^\varepsilon(t,x) = s_0 u_0^\varepsilon(t,x) + s_1 u_1^\varepsilon(t,x) + s_2 u_2^\varepsilon(t,x) + s_3 u_3^\varepsilon(t,x),$$

where u_n^ε will be chosen such that the initial data of u_n^ε is consistent with x^n up to $O(\alpha^q)$. Then, by linearity the consistent initial data of the microscopic solution u^ε will be given by combining the initial data of these solutions. Furthermore, let us denote the flux contribution of the term $u_n^\varepsilon(t,x)$ by

$$F_{n,HMM}^\varepsilon(x_0) := (\mathcal{K}_{\tau,\eta} * a(x/\varepsilon)\partial_x u_n^\varepsilon)(0,x_0). \tag{5.8}$$

Then the HMM flux (5.4) can be written as

$$F_{HMM}(x_0) = \sum_{j=0}^{n} s_j F^{\varepsilon}_{j,HMM}(x_0). \tag{5.9}$$

Our main result is Theorem 5.1 showing that given the third order macro state (5.1), the HMM flux (5.4) approximates the macroscopic flux (5.5) up to high orders in α.

THEOREM 5.1. *Suppose that* $\{u^{\varepsilon}_n(t,x)\}_{n=0}^{3}$ *solve the one dimensional periodic wave equation*

$$\begin{aligned} &\partial_{tt} u^{\varepsilon}_n(t,x) = \partial_x \left(a(x/\varepsilon) \partial_x u^{\varepsilon}_n(t,x) \right), \\ &u^{\varepsilon}_n(0,x) = \bar{u}_n(x), \quad \partial_t u^{\varepsilon}_n(0,x) = 0, \quad n = 0,1,2,3, \end{aligned} \tag{5.10}$$

where the initial data $\bar{u}_n(x)$ *is an* n*-th order polynomial. Then* $\left(\mathcal{K}_{\tau,\eta} * u^{\varepsilon}_n\right)(0,x) \in \mathbb{P}_n$ *is a quasi-polynomial and we assume that*

$$\left(\mathcal{K}_{\tau,\eta} * u^{\varepsilon}_n\right)(0,x) = x^n + \alpha^q E^{\varepsilon}(x, x/\varepsilon), \tag{5.11}$$

where $E^{\varepsilon}(x,y) \in \mathbb{P}_n$ *is a quasi-polynomial with coefficients* E^{ε}_k *uniformly bounded in* ε*. Suppose that* $\alpha = \frac{\varepsilon}{\eta}$ *and* $K \in \mathbb{K}^{p,q-2}$ *with* $p,q > 3$ *and* $\tau = \eta$*. Furthermore, let* $\hat{F}(x_0)$ *and* $F_{HMM}(x_0)$ *be given by* (5.5) *and* (5.9) *respectively. Then*

$$\left| F_{HMM}(x_0) - \hat{F}(x_0) \right| \le C \max_j |s_j| \left(\alpha^q + \eta \alpha^{q-1} \right), \tag{5.12}$$

where C *is independent of* $x_0, \varepsilon,$ *and* η *but may depend on* $a, p, q,$ *and* n*.*

To prove Theorem 5.1 we first prove below in Theorem 5.2 that $\beta = \hat{a} \|\chi\|^2_{L^2(Y)}$, where χ solves

$$\begin{aligned} &L[\chi] = -\partial_x a(x), \\ &\chi \text{ is 1-periodic}, \quad \overline{\chi} = 0. \end{aligned} \tag{5.13}$$

Then we prove (5.12) by showing that

$$\left| F_{HMM}(x_0) - \hat{a}\partial_x \hat{u}(x_0) - \varepsilon^2 \hat{a} \|\chi\|^2_{L^2(Y)} \partial_{xxx} \hat{u}(x_0) \right| \le C \max_j |s_j| \left(\alpha^q + \eta \alpha^{q-1} \right).$$

Replacing $\beta = \hat{a}\|\chi\|^2_{L^2(Y)}$ in (5.5), one can see that this inequality is the same as (5.12). Throughout the analysis we will use the fact that in the one-dimensional case the homogenized coefficient $\hat{a}$ is explicitely given by

$$\hat{a} = \left(\int_0^1 a^{-1}(y)dy \right)^{-1} = \int_0^1 a(y) + a(y)\chi'(y)dy.$$

5.1. Equivalence with Santosa and Symes's formula. In 1991 Santosa and Symes, [18], derived a rather complicated formula (see (5.6)) for the coefficient β in the effective flux (5.5). We prove that this can be expressed simply as $\beta = \hat{a}\|\chi\|^2_{L^2(Y)}$.

THEOREM 5.2. *Let* β *be given as in* (5.6)*, and* χ *be the zero average cell solution solving* (5.13)*. Then*

$$\beta = \hat{a}\|\chi\|^2_{L^2(Y)},$$

where $Y = [0,1]$*, and* $\hat{a}$ *is the homogenized coefficient in* 1*-dimension.*

Proof. To simplify the proof we define,

$$b(y) = \int_0^y \frac{\hat{a}}{a(s)} ds - y, \qquad \int_Y b(y) dy = \bar{b}.$$

Then $b(y)$ is periodic with $b(0) = b(1) = 0$ and

$$\chi(y) = b(y) - \bar{b}.$$

We consider first $\|\chi\|^2_{L^2(Y)}$,

$$\|\chi\|^2_{L^2(Y)} = \int_Y b(y)^2 - 2b(y)\bar{b} + \bar{b}^2 dy = \int_Y b(y)^2 - \bar{b}^2 dy.$$

For the integrals in the expression for β we get

$$\hat{a} \int_Y \int_0^y \frac{1}{a(r)} dr dy = \int_Y \int_0^y [b'(r) + 1] dr dy = \int_Y [b(y) + y] dy = \bar{b} + \frac{1}{2},$$

and

$$\begin{aligned}
\hat{a} \int_Y \int_0^y \int_0^s \frac{1}{a(s)} dr ds dy + \hat{a}^2 \int_Y \int_0^y \int_0^s \frac{1}{a(y)a(r)} dr ds dy \\
&= \int_Y \int_0^y \int_0^s b'(s) + 1 + (b'(y) + 1)(b'(r) + 1) dr ds dy \\
&= \int_Y \int_0^y s(b'(s) + 1) + (b'(y) + 1)(b(s) + s) ds dy \\
&= \int_Y \int_0^y [(sb(s))' + 2s] ds + b'(y) \int_0^y [b(s) + s] ds dy \\
&= \int_Y y b(y) + y^2 - b(y)(b(y) + y) dy \\
&= \frac{1}{3} - \int_Y b(y)^2 dy.
\end{aligned}$$

Hence, from (5.6),

$$\frac{\beta}{\hat{a}} = \frac{1}{12} - \frac{1}{3} + \int_Y b(y)^2 dy + \bar{b} + \frac{1}{2} - \left(\bar{b} + \frac{1}{2}\right)^2 = \int_Y b(y)^2 dy - \bar{b}^2 = \|\chi\|^2_{L^2(Y)}.$$

□

5.2. Proof of Theorem 5.1. Without loss of generality we will assume that $x_0 = 0$. For $x_0 \neq 0$, we can replace $a(x/\varepsilon)$ with $a((x_0 + x)/\varepsilon)$ and use the same initial data as we used for $x_0 = 0$. The analysis will be the same and the constant C will be independent of x_0 since a is periodic. We prove this Theorem in five steps:

1. Rescale the solution and write it as a quasi-polynomial.
2. Reformulate the consistency condition (5.11) in terms of the rescaled variable.
3. Find the energy of the coefficient functions of the solution.
4. Write down equations for time averages of the coefficient functions.
5. Find the fluxes for every $0 \leq n \leq 3$.

Step 1. The solution u_n^ε oscillates at a wavelength $O(\varepsilon)$. To bring these oscillations back to $O(1)$ we introduce the scaled solution $\varepsilon^n v(t/\varepsilon, x/\varepsilon) = u_n^\varepsilon(t,x)$. Then v satisfies

$$\begin{aligned} \partial_{tt} v(t,x) &= \partial_x \left(a\left(x\right) \partial_x v\left(t,x\right)\right), \\ v(0,x) &= g_0(0) + g_1(0)x + \cdots + g_n(0)x^n, \quad \partial_t v(0,x) = 0. \end{aligned}$$

The choice of the notation g_j for initial data is due to the fact that by theory of quasi-polynomials we can write

$$\begin{aligned} v(t,x) &= v_0(t,x) + x v_1(t,x) + \cdots + x^n v_n(t,x) \\ &= \tilde{v}_0(t,x) + g_0(t) + x\left(\tilde{v}_1(t,x) + g_1(t)\right) + \cdots + x^n \left(\tilde{v}_n(t,x) + g_n(t)\right), \end{aligned}$$

where $\{\tilde{v}_j\}$s are enforced with zero initial data and are 1-periodic with zero average in space. We note here that the functions $v, \tilde{v}_j, g_j$ also depend on n. We consider a fixed n and drop it in the notation to improve the readability. By Corollary 2.1, the coefficient functions $\{\tilde{v}_{n-i}(t,x)\}_{i=0}^n$ are 1-periodic with zero spatial average and they satisfy

$$\begin{aligned} \partial_{tt}\tilde{v}_{n-i}(t,x) &= L[\tilde{v}_{n-i}] + \tilde{P}_{n-i} + \tilde{f}_{n-i}, \\ \tilde{v}_{n-i}(0,x) &= \partial_t \tilde{v}_{n-i}(0,x) = 0, \end{aligned} \tag{5.14}$$

where

$$\tilde{f}_{n-i} = \begin{cases} 0, & i=0 \\ n g_n(t) a_x, & i=1, \quad n \geq 1, \\ (n-1) g_{n-1}(t) a_x + n(n-1) g_n(t)(a - \bar{a}), & i=2, \quad n \geq 2, \\ (n-2) g_{n-2}(t) a_x + (n-1)(n-2) g_{n-1}(t)(a-\bar{a}), & i=3, \quad n=3. \end{cases}$$

and

$$\tilde{P}_{n-i} = P_{n-i} - \overline{P}_{n-i}.$$

where

$$P_{n-i} = \begin{cases} 0, & i=0 \\ nM[\tilde{v}_n], & i=1, \quad n \geq 1, \\ (n-1)M[\tilde{v}_{n-1}] + n(n-1)a\tilde{v}_n, & i=2, \quad n \geq 2, \\ (n-2)M[\tilde{v}_{n-2}] + (n-1)(n-2)a\tilde{v}_{n-1}, & i=3, \quad n=3. \end{cases}$$

Furthermore, the functions $\{g_{n-i}\}_{i=0}^n$ solve

$$g''_{n-i}(t) = \begin{cases} 0, & i=0 \\ n\overline{a\partial_x\tilde{v}_n}, & i=1, \quad n \geq 1 \\ (n-1)\overline{a\partial_x\tilde{v}_{n-1}} + n(n-1)\left(\overline{a\tilde{v}_n} + g_n(t)\bar{a}\right), & i=2, \quad n \geq 2 \\ (n-2)\overline{a\partial_x\tilde{v}_{n-2}} + (n-1)(n-2)\left(\overline{a\tilde{v}_{n-1}} + g_{n-1}(t)\bar{a}\right), & i=3, \quad n=3, \end{cases}$$

together with some initial data $g_{n-i}(0) = r_{n-i}$, and $g'_{n-i}(0) = 0$. In the above equations, the operators L, M are defined in (2.2). We will later choose the initial data r_j of g_j such that the consistency condition (5.11) is satisfied.

Before proceeding with the remaining steps we simplify the above hierarchy of equations. Throughout the analysis we refer the reader to the simplified equations given below.

Simplifications:

- $\tilde{v}_n = 0$ since $\tilde{P}_n = \tilde{f}_n = 0$, and that $\tilde{v}_n$ has zero initial data.
- $g_n(t) = r_n$ is constant since $g_n''(t) = 0$, $g_n(0) = r_n$, and $g_n'(0) = 0$.
- $g_{n-1}(t) = r_{n-1}$ is constant since $g_n''(t) = n\overline{\tilde{v}_n} = 0$, $g_{n-1}(0) = r_{n-1}$, and $g'(0) = 0$.

Employing these simplifications the above equations can be rewritten as

$$\tilde{f}_{n-i} = \begin{cases} 0, & i = 0 \\ n r_n a_x, & i = 1, \quad n \geq 1, \\ (n-1) r_{n-1} a_x + n(n-1) r_n (a - \bar{a}), & i = 2, \quad n \geq 2, \\ (n-2) g_{n-2}(t) a_x + (n-1)(n-2) r_{n-1}(a - \bar{a}), & i = 3, \quad n = 3. \end{cases}$$

and

$$P_{n-i} = \begin{cases} 0, & i = 0 \\ 0, & i = 1, \quad n \geq 1, \\ (n-1) M[\tilde{v}_{n-1}], & i = 2, \quad n \geq 2, \\ (n-2) M[\tilde{v}_{n-2}] + (n-1)(n-2) a \tilde{v}_{n-1}, & i = 3, \quad n = 3. \end{cases}$$

Moreover,

(5.15)
$$g_{n-i}''(t) = \begin{cases} 0, & i = 0 \\ 0, & i = 1, \quad n \geq 1 \\ (n-1)\overline{M[\tilde{v}_{n-1}]} + n(n-1) r_n \bar{a}, & i = 2, \quad n \geq 2 \\ (n-2)\overline{M[\tilde{v}_{n-2}]} + (n-1)(n-2)\left(\overline{a\tilde{v}_{n-1}} + r_{n-1}\bar{a}\right), & i = 3, \quad n = 3, \end{cases}$$

Step 2. In this step we show how the consistency condition (5.11) gives an estimate of $K_\tau * g_k(\cdot/\varepsilon)$. For $k = n$ and $k = n-1$ we will use the fact that $g_n(t) = r_n$ and $g_{n-1}(t) = r_{n-1}$ are constants by the simplifications made in step 1. In this step, we will prove that

(5.16)
$$\begin{aligned} &|r_n - 1| \leq C\alpha^q \\ &|r_{n-1}| \leq C\varepsilon^{-1}\alpha^q \left(1 + \varepsilon \max_{0 \leq s \leq 1} \|\tilde{v}_{n-1}(s/\alpha, \cdot)\|_{H^1(Y)}\right), \quad 1 \leq n. \\ &|K_\tau * g_{n-k}(\cdot/\varepsilon)(0)| \leq C\varepsilon^{-k}\alpha^q \left(1 + \sum_{r=n-k}^{n-1} \varepsilon^{n-r} \max_{0 \leq s \leq 1} \|\tilde{v}_r(s/\alpha, \cdot)\|_{H^1(Y)}\right), \quad 2 \leq k \leq n. \end{aligned}$$

Expanding v as in the previous step, and using the fact that $\tilde{v}_n = 0$, the consistency condition (5.11) can be rewritten as

$$\varepsilon^n \left(\mathcal{K}_{\eta,\tau} * \sum_{j=0}^{n-1} \frac{x^j}{\varepsilon^j} \tilde{v}_j(t/\varepsilon, x/\varepsilon)\right)(0,x) + \varepsilon^n \left(\mathcal{K}_{\eta,\tau} * \sum_{j=0}^{n} \frac{x^j}{\varepsilon^j} g_j(t/\varepsilon)\right)(0,x) = x^n + \alpha^q E^\varepsilon(x, x/\varepsilon).$$

Furthermore, putting $q(x) = \sum_{j=0}^{n-1} x^j K_\tau * \tilde{v}_j(\cdot/\varepsilon, x)(0)$, and using Lemma 2.1 since $p \geq n$, we can rewrite the above equation as

$$\varepsilon^n \left(K_\eta * q(\cdot/\varepsilon)\right)(x) + \varepsilon^n \left(\sum_{j=0}^{n} \frac{x^j}{\varepsilon^j} K_\tau * g_j(\cdot/\varepsilon)(0)\right) = x^n + \alpha^q E^\varepsilon(x, x/\varepsilon). \tag{5.17}$$

By Lemma 2.4 we know that $\varepsilon^n\left(K_\eta * q(\cdot/\varepsilon)\right)(x) = Q^\varepsilon(x, x/\varepsilon)$, where $Q^\varepsilon \in \mathbb{P}_{n-1}$ is a quasi-polynomial with

$$|Q_k^\varepsilon| \leq C\alpha^q \sum_{r=k}^{n-1} \varepsilon^{n-r}\,|q_r|_\infty \leq C\alpha^q \sum_{r=k}^{n-1} \varepsilon^{n-r} \max_{0\leq s\leq 1} \|\tilde{v}_r(s/\alpha,\cdot)\|_{H^1(Y)}, \tag{5.18}$$

since $q_r(y) = K_\tau * \tilde{v}_r(\cdot/\varepsilon, y)(0) = K * \tilde{v}_r(\cdot/\alpha, y)(0)$.

Next to prove (5.16) we equate the equal powers of x^{n-k} in equation (5.17). First equating the coefficients in front of x^n and using $g_n(t) = r_n$ we have

$$K_\tau * r_n = 1 + \alpha^q E_n^\varepsilon(x/\varepsilon).$$

Furthermore, by the boundedness of E_n^ε and since r_n is constant we have

$$|r_n - 1| = |K_\tau * r_n - 1| \leq C\alpha^q.$$

Next we equate the coefficients of x^{n-k} in (5.17) and obtain

$$Q_{n-k}^\varepsilon(x/\varepsilon) + \varepsilon^{-k} K_\tau * g_{n-k}(\cdot/\varepsilon)(0) = \alpha^q E_{n-k}^\varepsilon(x/\varepsilon).$$

By estimate (5.18) we know that $\left|Q_{n-k}^\varepsilon\right| \leq C\alpha^q \sum_{r=n-k}^{n-1} \varepsilon^{n-r} \max_{0\leq s\leq 1} \|\tilde{v}_r(s/\alpha,\cdot)\|_{H^1(Y)}$. Hence, from the boundedness of E_{n-k}^ε and by the fact that $g_{n-1}(t) = r_{n-1}$ the last two estimates in (5.16) follows.

Step 3. In the inequalities (5.16), we need to estimate $\|\tilde{v}_{n-i}\|_{H^1(Y)}$ in order to give an upperbound in terms of ε and η only. By Theorem 3.2, we can get this estimate by bounding $\|\tilde{f}_{n-i}\|_{L^2(Y)}$. In this step, we prove the following precise statement

Lemma 5.1. *Suppose $\alpha = \dfrac{\varepsilon}{\eta} < 1$ and that the assumptions of Theorem 5.1 hold, then*

$$\max_{0\leq s\leq 1} \|\tilde{v}_{n-i}(s/\alpha,\cdot)\|_{H^1(Y)} \leq C \begin{cases} 1, & i=1, \quad n\geq 1, \\ \alpha^{-2} + \varepsilon^{-1}\alpha^{q-1} & i=2, \quad n\geq 2, \\ \alpha^{-3} + \varepsilon^{-2}\alpha^{q-1} + \varepsilon^{-1}\alpha^{q-2}, & i=3, \quad n=3, \end{cases} \tag{5.19}$$

where C is independent of ε, η and α but may depend on $p, q, K,$ and a. Moreover we have

$$|r_{n-1}| \leq C\alpha^q \varepsilon^{-1}, \quad n\geq 1, \tag{5.20}$$

and

$$|K_\tau * g_{n-i}(\cdot/\varepsilon)(0)| \leq C \begin{cases} \alpha^q\varepsilon^{-2}, & i=2, \quad n\geq 2 \\ \alpha^q\varepsilon^{-3}, & i=3, \quad n=3. \end{cases} \tag{5.21}$$

1. $\boldsymbol{i = 1, n \geq 1}$: We know that $\tilde{f}_{n-1} = n r_n a_x$. Moreover, using the first estimate in (5.16) it follows that

$$\|\tilde{f}_{n-1}(t,\cdot)\|_{L^2(Y)} \leq C\left(1 + \alpha^q\right) \leq C.$$

Next since $\tilde{P}_{n-1} = 0$, and $\tilde{f}_{n-1}$ is time independent, Theorem 3.1 gives that

$$\|\tilde{v}_{n-1}(t,\cdot)\|_{H^1(Y)} \le C\|\tilde{f}_{n-1}(t,\cdot)\|_{L^2(Y)} \le C. \tag{5.22}$$

This shows (5.19) for $i = 1$. Using the second estimate in (5.16) and the inequality (5.22) we have

$$|r_{n-1}| \le C\varepsilon^{-1}\alpha^q \left(1 + \varepsilon \max_{0\le s\le 1} \|\tilde{v}_{n-1}(s/\alpha,\cdot)\|_{H^1(Y)}\right) \le C\alpha^q\varepsilon^{-1}.$$

This proves (5.20).

2. $\boldsymbol{i = 2, n \ge 2}$: We know that

$$\tilde{f}_{n-2}(t,x) = (n-1)r_{n-1}a_x + n(n-1)r_n\,(a - \bar{a}).$$

Therefore, $\tilde{f}_{n-2}$ is time independent. Moreover, using the first inequality in (5.16), and (5.20) we can bound the L^2 norm of $\tilde{f}_{n-2}(t,x)$.

$$\|\tilde{f}_{n-2}(t,\cdot)\|_{L^2(Y)} \le C\left(1 + \varepsilon^{-1}\alpha^q\right).$$

Next by Theorem 3.2 we have

$$\begin{aligned}\|\tilde{v}_{n-2}(t,\cdot)\|_{H^1(Y)} &\le C\left((1+t)\max_{0\le s\le t}\|\tilde{f}_{n-2}(s,\cdot)\|_{L^2(Y)} + \left(1+t^2\right)\max_{0\le s\le t}\|\tilde{f}_{n-1}(s,\cdot)\|_{L^2(Y)}\right)\\ &\le C\left((1+t)\left(1+\varepsilon^{-1}\alpha^q\right) + \left(1+t^2\right)(1+\alpha^q)\right).\end{aligned}$$

Then putting $t = 1/\alpha$ we obtain

$$\max_{0\le s\le 1}\|\tilde{v}_{n-2}(s/\alpha,\cdot)\|_{H^1(Y)} \le C\left(\alpha^{-2} + \varepsilon^{-1}\alpha^{q-1}\right). \tag{5.23}$$

This proves (5.19) for $i = 2$. The estimates (5.16), (5.23), (5.22) gives

$$|K_\tau * g_{n-2}(\cdot/\varepsilon)(0)| \quad\le\quad C\varepsilon^{-2}\alpha^q\left(1 + \sum_{r=n-2}^{n-1}\varepsilon^{n-r}\max_{0\le s\le 1}\|\tilde{v}_r(s/\alpha,\cdot)\|_{H^1(Y)}\right) \le C\varepsilon^{-2}\alpha^q. \tag{5.24}$$

Now we want to bound the term $|g_{n-2}(t)|$ since we will need this in the upcoming case. In (5.15), we are given that

$$g''_{n-2}(t) = (n-1)\overline{M[\tilde{v}_{n-1}]} + n(n-1)r_n\bar{a}.$$

Using the initial data $g'_{n-2}(0) = 0$ and (5.22) we can write

$$\begin{aligned}\left|g'_{n-2}(t)\right| &\le \int_0^t \left|g''_{n-2}(s)\right| ds \le C\left(\int_0^t \left|\overline{M[\tilde{v}_{n-1}(s,\cdot)]}\right| ds + t\right)\\ &\le C\left(\int_0^t \|\tilde{v}_{n-1}(s,\cdot)\|_{H^1(Y)} ds + t\right) \le Ct\end{aligned}$$

Next we integrate again to have an estimate for $|g_{n-2}(t)|$ as follows:

$$|g_{n-2}(t)| \le |g_{n-2}(0)| + \int_0^t \left|g'_{n-2}(s)\right| ds \le |r_{n-2}| + Ct^2,$$

since $g_{n-2}(0) = r_{n-2}$. It remains to bound $|r_{n-2}|$. To do this, we write the exact formula for $g_{n-2}(t)$

$$g_{n-2}(t) = r_{n-2} + (n-1)\int_0^t \int_0^\zeta \int_0^1 a(x)\partial_x \tilde{v}_{n-1}(s,x)dxdsd\zeta + \frac{n(n-1)}{2}t^2 \bar{a} r_n.$$

The term r_{n-2} can then be obtained from inequality (5.24)). For this we apply the averaging kernel K_τ with $p \geq 2$ in the last equation and see that

$$r_{n-2} = \left[-(n-1)K_\tau * \int_0^{t/\varepsilon} \int_0^\zeta \int_0^1 a(x)\partial_x \tilde{v}_{n-1}(s,x)dxdsd\zeta + K_\tau * g_{n-2}(\cdot/\varepsilon) \right](0).$$

From (5.22) we get $\left| \int_0^\zeta \int_0^1 a(x)\partial_x \tilde{v}_{n-1}(s,x)dxds \right| \leq C\zeta$, and

$$\begin{aligned} |r_{n-2}| &\leq C\int_{-\tau}^{\tau} |K_\tau(t)| \int_0^{t/\varepsilon} \zeta d\zeta dt + C\varepsilon^{-2}\alpha^q \\ &= C\int_{-1}^{1} |K(t)| \int_0^{t/\alpha} \zeta d\zeta dt + C\varepsilon^{-2}\alpha^q \leq C\left(\alpha^{-2} + \varepsilon^{-2}\alpha^q\right). \end{aligned}$$

Hence,

$$|g_{n-2}(t)| \leq |r_{n-2}| + Ct^2 \leq C\left(\alpha^{-2} + \varepsilon^{-2}\alpha^q + t^2\right).$$

3. $\boldsymbol{i = 3, n = 3}$: We have $\tilde{f}_{n-3} = (n-2)g_{n-2}(t)a_x + (n-1)(n-2)r_{n-1}(a - \bar{a})$. Therefore,

$$\|\tilde{f}_{n-3}(t)\|_{L^2(Y)} \leq C\left(|g_{n-2}(t)| + \varepsilon^{-1}\alpha^q\right) \leq C\left(\alpha^{-2} + \varepsilon^{-2}\alpha^q + t^2\right).$$

Next employing Theorem 3.2 we bound the norm $\|\tilde{v}_{n-3}(t)\|_{H^1(Y)}$ as follows

$$\begin{aligned} \|\tilde{v}_{n-3}(t,\cdot)\|_{H^1(Y)} &\leq C\sum_{j=0}^{2}\left(1 + t^{j+1}\right)\max_{0\leq s\leq t}\|\tilde{f}_{n-3+j}(s,\cdot)\|_{L^2(Y)} \\ &\leq C\left((1+t)\left(\alpha^{-2} + \varepsilon^{-2}\alpha^q + t^2\right) + \left(1+t^2\right)\left(1+\varepsilon^{-1}\alpha^q\right) + \left(1+t^3\right)\right). \end{aligned}$$

Putting $t = 1/\alpha$, we complete the proof of (5.19) for $i = 3$. Finally the proof of (5.21) for $i = 3, n = 3$ follows readily from (5.16) and the last estimate in (5.19).

Step 4. In this step we derive equations for the time averages of the coefficient functions $\tilde{v}_{n-i}$. We denote the time averages by

$$\tilde{d}_k(x) := K_\tau * \tilde{v}_k(\cdot/\varepsilon, x)(0).$$

Clearly $\{\tilde{d}_k\}_{k=0}^3$ have zero average since $\overline{\tilde{v}_k(t,\cdot)} = 0$. Furthermore, let us define the operator

$$\tilde{M}[u] = M[u] - \overline{M[u]}. \tag{5.25}$$

In this step we will show that $\{\tilde{d}_{n-i}\}_{i=0}^3$ satisfy

$$L[\tilde{d}_{n-i}] = \phi_{n-i} + \alpha^q Z_{n-i} \tag{5.26}$$

where

$$\phi_{n-i}(x) = \begin{cases} 0 & i = 0 \\ -na_x & i = 1, n \geq 1 \\ -(n-1)\left(\tilde{M}[\tilde{d}_{n-1}] + n\left(a - \bar{a}\right)\right) & i = 2, n \geq 2 \\ -(n-2)\left(\tilde{M}[\tilde{d}_{n-2}] + (n-1)\left(a\tilde{d}_{n-1} - \overline{a\tilde{d}_{n-1}}\right)\right) + n(n-1)\hat{a}\chi(x) & i = 3, n = 3 \end{cases}$$

and $\{Z_k\}_{k=0}^3$ are 1-periodic functions with $\overline{Z_k} = 0$ such that $Z_n = 0$ and

$$\|Z_{n-1}\|_{H^1(Y)} \leq C, \quad \|Z_{n-2}\|_{H^1(Y)} \leq C\left(\varepsilon^{-1} + \alpha^{-2}\right), \quad \|Z_{n-3}\|_{H^1(Y)} \leq C\left(\varepsilon^{-2} + \alpha^{-3}\right).$$

To simplify the notation we introduce

$$h_j^{\{k\}}(x) := K_\tau * \partial_t^k \left(\tilde{P}_j + \tilde{f}_j\right)(\cdot/\varepsilon, x)(0), \quad \tilde{d}_j^{\{k\}}(x) := K_\tau * \partial_t^k \tilde{v}_j(\cdot/\varepsilon, x)(0).$$

Note that $\tilde{d}_j^{\{0\}}$ is equal to $\tilde{d}_j$. Then by Theorem 4.1 we have

$$L[\tilde{d}_j] = -\sum_{\ell=0}^{q/2-1} L^{-\ell} h_j^{\{2\ell\}} + \alpha^q R_{j,1}, \tag{5.27}$$

and

$$\tilde{d}_j^{\{2m\}}(x) = -\sum_{\ell=m}^{q/2-1} L^{-\ell-1+m} h_j^{\{2\ell\}}(x) + \alpha^q R_{j,m}(x), \tag{5.28}$$

where $\{R_{j,m}\}_{m=0}^{q/2-1}$ are 1-periodic and have zero average. Moreover, they satisfy the relation $L^m[R_{j,0}] = R_{j,m}$ and all are bounded as follows

(5.29)

$$\|R_{n-j,m}\|_{H^1(Y)} \leq C \max_{0 \leq s \leq 1} \|\tilde{v}_{n-j}(s/\alpha, \cdot)\|_{L^2(Y)} \leq C \begin{cases} 0 & j = 0 \\ 1 & j = 1, \quad n \geq 1 \\ \alpha^{-2} + \varepsilon^{-1}\alpha^{q-1} & j = 2, \quad n \geq 2, \\ \alpha^{-3} + \varepsilon^{-2}\alpha^{q-1} + \varepsilon^{-1}\alpha^{q-2}, & j = 3, \quad n = 3, \end{cases}$$

where we used (5.19) for the second inequality. In order to deal with $O(1)$ quantities we will also use the first estimate in (5.16) and the estimate (5.20) and write

$$r_n = 1 + \alpha^q \delta_n, \quad r_{n-1} = \varepsilon^{-1}\alpha^q \delta_{n-1}, \quad \text{where } |\delta_n| \leq C, |\delta_{n-1}| \leq C. \tag{5.30}$$

1. $\boldsymbol{i = 0}$: Clearly $\tilde{d}_n = 0$ since $\tilde{v}_n = 0$. Hence $L[\tilde{d}_n] = 0$.
2. $\boldsymbol{i = 1, n \geq 1}$: It follows from (5.30) that $\tilde{P}_{n-1} + \tilde{f}_{n-1} = n r_n a_x = n a_x + \alpha^q \delta_n n a_x$. Since F_{n-1} is time independent we have

$$h_{n-1}^{\{2m\}}(x) = \begin{cases} n a_x + \alpha^q \delta_n n a_x, & m = 0 \\ 0, & m \geq 1. \end{cases}$$

Then by (5.27)

$$L[\tilde{d}_{n-1}](x) = -n a_x - \alpha^q \delta_n n a_x + \alpha^q R_{n-1,1}(x) =: -n a_x + \alpha^q Z_{n-1}(x), \tag{5.31}$$

where $Z_{n-1} := -\delta_n n a_x + R_{n-1,1}$. The boundedness of Z_{n-1} follows from (5.29). The proof of (5.26) for $\tilde{d}_{n-1}$ is completed. We finish this step by giving an expression for the time averages of the derivatives of the coefficient function $\tilde{v}_{n-1}$. This result will be used in the upcoming case and follows from (5.28).

$$\tilde{d}_{n-1}^{\{2m\}}(x) = \alpha^q R_{n-1,m}(x), \quad m \geq 1. \tag{5.32}$$

3. $\boldsymbol{i=2, n\geq 2}$: The right hand side reads

$$\begin{aligned}\tilde{P}_{n-2}+\tilde{f}_{n-2} &= (n-1)\left(M[\tilde{v}_{n-1}]-\overline{M[\tilde{v}_{n-1}]}\right)+n(n-1)r_n\,(a-\bar{a})+(n-1)r_{n-1}a_x\\ &=: (n-1)\left(\tilde{M}[\tilde{v}_{n-1}]+n\,(a-\bar{a})\right)+\alpha^q\varepsilon^{-1}\delta_{n-2}(x).\end{aligned}$$

Here we define $\delta_{n-2}(x)$ by

$$\delta_{n-2}(x) := (n-1)\left(\varepsilon n\delta_n(a-\bar{a})+\delta_{n-1}a_x\right).$$

Clearly $\overline{\delta_{n-2}}=0$. Furthermore,

$$\|\delta_{n-2}\|_{H^1(Y)}\leq C. \tag{5.33}$$

Now applying the time averaging kernel to $\tilde{P}_{n-2}+\tilde{f}_{n-2}$ and using (5.32) we obtain

$$h_{n-2}^{\{2m\}}(x)=\begin{cases}(n-1)\left(\tilde{M}[\tilde{d}_{n-1}]+n\,(a-\bar{a})\right)+\alpha^q\varepsilon^{-1}\delta_{n-2}(x), & m=0\\ \alpha^q(n-1)\tilde{M}[R_{n-1,m}], & 1\leq m\leq q/2-1.\end{cases}$$

Then by (5.27) we get

$$\begin{aligned}L[\tilde{d}_{n-2}](x) &= -h_{n-2}^{\{0\}}(x)-\textstyle\sum_{\ell=1}^{q/2-1}L^{-\ell}h_{n-2}^{\{2\ell\}}(x)+\alpha^qR_{n-2,1}(x)\\ &=: -(n-1)\left(\tilde{M}[\tilde{d}_{n-1}]+n\,(a-\bar{a})\right)+\alpha^q\varepsilon^{-1}\delta_{n-2}(x)+\alpha^qL[\tilde{R}_{n-2,0}]\\ &=: -(n-1)\left(\tilde{M}[\tilde{d}_{n-1}]+n\,(a-\bar{a})\right)+\alpha^qZ_{n-2}(x),\end{aligned}$$

where we used the fact that $L[R_{n-2,0}]=R_{n-2,1}$ and defined $\tilde{R}_{n-2,0}(x)$ as

$$\tilde{R}_{n-2,0}(x)=(n-1)\sum_{\ell=1}^{q/2-1}L^{-\ell-1}\tilde{M}[R_{n-1,l}]+R_{n-2,0}(x).$$

$Z_{n-2}(x)$ is also defined as

$$Z_{n-2}(x)=\varepsilon^{-1}\delta_{n-2}(x)+L[\tilde{R}_{n-2,0}].$$

Clearly Z_{n-2} is 1-periodic and has zero average. Before proceeding further, we present an expression for the time averages of the higher order derivatives of $\tilde{v}_{n-2}$. The following result will be employed in the next case and is a consequence of (5.28): for $m\geq 1$

$$\begin{aligned}\tilde{d}_{n-2}^{\{2m\}}(x) &= -\alpha^q(n-1)\textstyle\sum_{\ell=m}^{q/2-1}L^{m-1}L^{-\ell}\tilde{M}[R_{n-1,\ell}]+\alpha^qR_{n-2,m}(x)\\ &=: \alpha^q\tilde{R}_{n-2,m}(x).\end{aligned} \tag{5.34}$$

It remains to estimate Z_{n-2}. To do this estimation first we note that $\{R_{n-2,m}\}_{m=0}^{q/2-1}$ are 1-periodic and have zero average. Then we use the inequality (5.33), elliptic regularity Lemma 3.1, and the estimate (5.29) to see that

$$
\begin{aligned}
\|Z_{n-2}\|_{H^1(Y)} &\le \varepsilon^{-1}\|\delta_{n-2}\|_{H^1(Y)} + \|L[\tilde{R}_{n-2,0}]\|_{H^1(Y)} \\
&\le \varepsilon^{-1}C + C\sum_{\ell=1}^{q/2-1} \|L^{-\ell}\tilde{M}[R_{n-1,\ell}]\|_{H^1(Y)} + \|R_{n-2,1}\|_{H^1(Y)} \\
&\le C\left(\varepsilon^{-1} + \sum_{\ell=1}^{q/2-1} \|\tilde{M}[R_{n-1,\ell}]\|_{L^2(Y)} + \alpha^{-2} + \varepsilon^{-1}\alpha^{q-1}\right) \\
&\le C\left(\varepsilon^{-1} + \sum_{\ell=1}^{q/2-1} \|R_{n-1,\ell}\|_{H^1(Y)} + \alpha^{-2} + \varepsilon^{-1}\alpha^{q-1}\right) \\
&\le C\left(\varepsilon^{-1} + 1 + \alpha^{-2} + \varepsilon^{-1}\alpha^{q-1}\right) \le C\left(\varepsilon^{-1} + \alpha^{-2}\right).
\end{aligned}
$$

This proves the statement for $i = 2, n \ge 2$. Before continuing with the last case we give an estimate for the remainder term $\tilde{R}_{n-2,m}$, for $m = 0, \cdots, q/2$ which we will need in the next case. First we note that $\overline{\tilde{R}_{n-2,m}} = 0$. Next using the elliptic regularity Lemma 3.1, and the estimate (5.29) we readily obtain:

$$
\|\tilde{R}_{n-2,m}\|_{H^1(Y)} \le C\left(\alpha^{-2} + \varepsilon^{-1}\alpha^{q-1}\right), \quad m = 0, 1, \cdots, q/2-1. \tag{5.35}
$$

4. $\boldsymbol{i = 3, n = 3}$: Using (5.30) we have

$$
\begin{aligned}
\tilde{P}_0 + \tilde{f}_0 &= \tilde{M}[\tilde{v}_1] + 2\left(a\tilde{v}_2 - \overline{a\tilde{v}_2}\right) + g_1(t)a_x + r_2 2\,(a - \bar{a}) \\
&= \tilde{M}[\tilde{v}_1] + 2\left(a\tilde{v}_2 - \overline{a\tilde{v}_2}\right) + g_1(t)a_x + \varepsilon^{-1}\alpha^q\delta_2 2\,(a - \bar{a}),
\end{aligned}
$$

where $|\delta_2| \le C$. To simplify the exposition we introduce

$$
G^{\{m\}} := K_\tau * \partial_t^m g_1(\cdot/\varepsilon)(0).
$$

Then from (5.21) we have that $G^{\{0\}} = \alpha^q\varepsilon^{-2}\delta_1$ where $|\delta_1| \le 1$.
Now we apply the averaging kernel to $\tilde{P}_0 + \tilde{f}_0$ and the higher derivatives of it, and use (5.32) and (5.34) to see that

$$
h_0^{\{2m\}}(x) = \begin{cases} \tilde{M}[\tilde{d}_1] + 2\left(a\tilde{d}_2 - \overline{a\tilde{d}_2}\right) + \alpha^q\varepsilon^{-2}\delta_1 a_x + \alpha^q\varepsilon^{-1}\delta_2 2\,(a - \bar{a}), & m = 0 \\ G^{\{2m\}}a_x + \alpha^q\tilde{M}[\tilde{R}_{1,m}] + \alpha^q 2\left(aR_{2,m} - \overline{aR_{2,m}}\right), & m \ge 1. \end{cases}
$$

Now first we note from (5.31) that $\tilde{d}_2(x) = 3\chi(x) + \alpha^q s(x)$, where $L[s] = Z_2$. This is because

$$
L[\tilde{d}_2] = 3L[\chi] + \alpha^q L[s] = -3a_x + \alpha^q Z_2(x).
$$

Furthermore, $\|s\|_{H^1(Y)} \le \|Z_2\|_{L^2(Y)} \le C$. Then we use (5.30) and get

$$
G^{\{2\}} = 2\overline{a\partial_x\tilde{d}_2} + 6\bar{a} + \alpha^q\delta_3 6\bar{a} = 6\hat{a} + \alpha^q C_1,
$$

where C_1 is a constant independent of ε and η. Furthermore, for the higher derivatives we use (5.32) and obtain

$$
G^{\{2m\}} = 2\overline{a\partial_x\tilde{d}_2^{\{2(m-1)\}}} = \alpha^q 2\overline{a\partial_x R_{2,m-1}}, \quad 2 \le m \le q/2-1.
$$

Then by (5.27) and since $\chi(x) = -L^{-1}a_x$ we have

$$\begin{aligned} L[\tilde{d}_0] &= -h_0^{\{0\}}(x) - L^{-1}h_0^{\{2\}}(x) - \textstyle\sum_{\ell=2}^{q/2-1} L^{-\ell} h_0^{2\ell}(x) + \alpha^q R_{0,1}(x) \\ &= -\tilde{M}[\tilde{d}_1] - 2\left(a\tilde{d}_2 - \overline{a\tilde{d}_2}\right) + 6\hat{a}\chi(x) + \alpha^q Z_0(x), \end{aligned}$$

where

$$\begin{aligned} &Z_0(x) = -\varepsilon^{-2}\delta_1 a_x - \varepsilon^{-1}\delta_2 2(a-\bar{a}) - L^{-1}\left(\tilde{M}[\tilde{R}_{1,1}] + 2\left(aR_{2,1} - \overline{aR_{2,1}}\right)\right) \\ &+C_1\chi - \textstyle\sum_{\ell=2}^{q/2-1} L^{-\ell}\left(2\overline{a\partial_x R_{2,\ell-1}}a_x + \left(\tilde{M}[\tilde{R}_{1,\ell}] + 2\left(aR_{2,\ell} - \overline{aR_{2,\ell}}\right)\right)\right) + R_{0,1}. \end{aligned}$$

Note that $R_{0,1}$ is 1-periodic and have zero average. From this it follows that Z_0 is also 1-periodic with zero average. The boundedness of $\|Z_0\|$ follows by using the elliptic regularity Lemma 3.1, and the estimates(5.29) and (5.35):

$$\begin{aligned} \|Z_0\|_{H^1(Y)} &\le C\left(\varepsilon^{-2} + \varepsilon^{-1} + 1 + \sum_{\ell=1}^{q/2-1} \|\tilde{R}_{1,\ell}\|_{H^1(Y)} + \|R_{2,\ell}\|_{H^1(Y)} + \|R_{0,1}\|_{H^1(Y)}\right) \\ &\le C\left(\varepsilon^{-2} + \left(\alpha^{-2} + \varepsilon^{-1}\alpha^{q-1}\right) + 1 + \left(\alpha^{-3} + \varepsilon^{-2}\alpha^{q-1} + \varepsilon^{-1}\alpha^{q-2}\right)\right) \\ &\le C\left(\varepsilon^{-2} + \alpha^{-3}\right). \end{aligned}$$

This finishes the proof of the case $i = 3, n = 3$, and that of the statement of step 4.

Step 5. In this step we will prove our main estimate (5.12). Remember that by (5.9), the HMM flux can be written as

$$F_{HMM}(0) = \sum_{n=0}^{3} s_n F^{\varepsilon}_{n,HMM}(0)$$

where $F^{\varepsilon}_{n,HMM}$ is the flux corresponding to the initial data $\bar{u}_n$ consistent with the macro state x^n. Moreover, $\hat{F}$ is as given in (5.5) and by Theorem 5.2 it can be redefined as

$$\hat{F}(0) = \hat{a}\partial_x\hat{u}(0) + \varepsilon^2\hat{a}\|\chi\|^2_{L^2(Y)}\partial_{xxx}\hat{u}(0),$$

where $\hat{u}$ is given in (5.1). To prove the main estimate (5.12) we will equivalently show that

(5.36)

$$F^{\varepsilon}_{0,HMM}(0) = 0, \quad |F^{\varepsilon}_{1,HMM}(0) - \hat{a}| \le C\alpha^q, \quad |F^{\varepsilon}_{2,HMM}(0)| \le C\left(\alpha^q + \varepsilon\alpha^{q-2}\right),$$

$$|F^{\varepsilon}_{3,HMM}(0) - 6\varepsilon^2\hat{a}\|\chi\|^2_{L^2(Y)}| \le C\left(\alpha^q + \varepsilon^2\alpha^{q-3}\right).$$

These estimates imply the main estimate (5.12) since $\partial_x\hat{u}(0) = s_1$, $\partial_{xxx}\hat{u}(0) = 6s_3$, and $\varepsilon\alpha^{q-2} = \eta\alpha^{q-1}$.

Remember that since $\tilde{v}_n = 0$

$$v(t,x) = \sum_{j=0}^{n-1} x^j\tilde{v}_j(t,x) + \sum_{j=0}^{n} x^j g_j(t).$$

Moreover, let us introduce

$$d(x) := K_\tau * v(\cdot/\varepsilon, x)(0), \quad G_j := K_\tau * g_j(\cdot/\varepsilon)(0).$$

Then by the fact that $g_n(t) = r_n$ and $g_{n-1}(t) = r_{n-1}$, the equality (5.30) and the estimate (5.21), there exist constants $\{\delta_j\}_{j=0}^{n}$ uniformly bounded in ε and η such that

$$d(x) = \sum_{j=0}^{n-1} x^j \tilde{d}_j(x) + \sum_{j=0}^{n} x^j G_j = \sum_{j=0}^{n-1} x^j \tilde{d}_j(x) + x^n + \alpha^q \sum_{j=0}^{n} x^j \varepsilon^{j-n} \delta_j. \tag{5.37}$$

We furthermore note that $F^{\varepsilon}_{n,HMM}$ can be written in terms of d as follows

$$F^{\varepsilon}_{n,HMM}(0) := \varepsilon^n \left(K_\eta * a(\cdot/\varepsilon)\partial_x d(\cdot/\varepsilon)\right)(0).$$

Before proceeding with the flux calculation we present two utility lemmas.

Lemma 5.2. *Let*

$$d(x) = x^n + \sum_{j=0}^{n-1} x^j d_j(x) + \alpha^q \sum_{j=0}^{n} x^j \varepsilon^{j-n} \delta_j, \qquad F^{\varepsilon} = \varepsilon^n K_\eta * a(\cdot/\varepsilon)\partial_x d(\cdot/\varepsilon)(0), \tag{5.38}$$

where $\{d_j\}_{j=0}^{n-1}$ *are smooth,* 1*-periodic with zero average such that* $\overline{d_j(\cdot)} = 0$*,* $K_\eta \in \mathbb{K}^{p,q-2}$ *with* $p, q \geq n-1$*, and* a *is a* 1*-periodic smooth function. Then, with* $\alpha = \frac{\varepsilon}{\eta}$*, we have*

$$F^{\varepsilon} = 0, \ \text{if } n = 0,$$

$$|F^{\varepsilon} - \overline{a\,(1 + d_0')}| \leq C\alpha^q, \ \text{if } n = 1,$$

$$|F^{\varepsilon} - \varepsilon^{n-1}\overline{a\,(d_0' + d_1)}| \leq C\alpha^q \ \text{if } n > 1.$$

Proof of Lemma 5.2 is given at the end of section 5.

Lemma 5.3. *Suppose that* a *is a* 1*-periodic smooth, bounded and positive function. Let* $\chi(x)$ *be the solution of the cell problem* (5.13)*, then for any smooth and* 1*-periodic function* u *we have*

$$\int_0^1 \chi(x)\partial_x (au)(x)dx = \int_0^1 au dx - \hat{a}\bar{u},$$

where $\hat{a}$ *is the homogenized coefficient.*

Proof. The cell solution satisfies $\partial_x \chi = -1 + \hat{a}a^{-1}(x)$. Putting this into the given L^2 inner product and integrating by part we obtain the desired equality. □

We note first that by (5.37) the time average $d(x)$ will have the form in (5.38), and we can therefore use Lemma 5.2 for all n with $d_j = \tilde{d}_j$.

1. $\boldsymbol{n = 0}$: From (5.37) we have $d(x) = 1 + \alpha^q \delta_0$ and therefore, $F^{\varepsilon}_{0,HMM} = 0$ by Lemma 5.2.

2. $\boldsymbol{n = 1}$: By step 4, $\tilde{d}_0$ satisfies

$$L[\tilde{d}_0] = -a_x + \alpha^q Z_0(x), \quad \|Z_0(\cdot)\|_{H^1(Y)} \leq C, \tag{5.39}$$

and $Z_0(x)$ is 1-periodic and has zero average. Furthermore, by Lemma 5.2 we know that

$$\left| F^{\varepsilon}_{1,HMM} - \overline{a\left(1 + \partial_x \tilde{d}_0\right)} \right| \leq C\alpha^q. \tag{5.40}$$

Moreover, since by (5.39) we have $\partial_x\left(a\partial_x\tilde{d}_0+a\right)=\alpha^q Z_0$, an application of Lemma 5.3 gives

$$\left|\langle\chi,\partial_x\left(a\partial_x\tilde{d}_0+a\right)\rangle\right|=\left|\overline{a\left(\partial_x\tilde{d}_0+1\right)}-\hat{a}\right|=\alpha^q\left|\langle\chi,Z_0\rangle\right|\leq\alpha^q\|Z_0\|_{L^2(Y)}\|\chi\|_{L^2(Y)}\leq C\alpha^q.$$

Hence using the last inequality and the estimate (5.40) we get the desired result

$$\left|F^{\varepsilon}_{1,HMM}-\hat{a}\right|\leq\left|F^{\varepsilon}_{1,HMM}-\overline{a\left(\partial_x\tilde{d}_0+1\right)}\right|+\left|\overline{a\left(\partial_x\tilde{d}_0+1\right)}-\hat{a}\right|\leq C\alpha^q.$$

3. $\boldsymbol{n=2}$: From step 4, $\tilde{d}_0$ and $\tilde{d}_1$ satisfy

$$L[\tilde{d}_1]=-2a_x+\alpha^q Z_1(x) \tag{5.41}$$

$$L[\tilde{d}_0]=-\left(\left(M[\tilde{d}_1]-\overline{M[\tilde{d}_1]}\right)+2\left(a-\bar{a}\right)\right)+\alpha^q Z_0(x), \tag{5.42}$$

where

$$\|Z_1\|_{H^1(Y)}\leq C,\qquad \|Z_0\|_{H^1(Y)}\leq C\left(\varepsilon^{-1}+\alpha^{-2}\right).$$

By Lemma 5.2 we know that

$$|F^{\varepsilon}_{2,HMM}(0)-\varepsilon\overline{a\left(\partial_x\tilde{d}_0+\tilde{d}_1\right)}|\leq C\alpha^q. \tag{5.43}$$

Furthermore, taking the derivative of equation (5.42) and exploiting the relation (5.41) we obtain

$$\partial_x^2\left(a\partial_x\tilde{d}_0(x)+a\tilde{d}_1(x)\right)=\alpha^q\left(\partial_x Z_0(x)+Z_1(x)\right), \tag{5.44}$$

Now we define $h(x):=\int_0^x\chi(s)ds$ (which is periodic since $\bar{\chi}=0$), and take the innerproduct of the left hand side with h and apply Lemma 5.3 to see that

$$\begin{aligned}\left|\langle h,\partial_x^2\left(a\partial_x\tilde{d}_0+a\tilde{d}_1\right)\rangle\right|&=\left|-\langle\chi,\partial_x\left(a\partial_x\tilde{d}_0+a\tilde{d}_1\right)\rangle\right|=\left|\overline{a\left(\partial_x\tilde{d}_0+\tilde{d}_1\right)}\right|\\ =\alpha^q\left|\langle Z_1+\partial_x Z_0,h\rangle\right|&\leq\alpha^q\|Z_1+\partial_x Z_0\|_{L^2(Y)}\|h\|_{L^2(Y)}\leq C\left(\varepsilon^{-1}\alpha^q+\alpha^{q-2}\right).\end{aligned}$$

Using the last inequality and the estimate (5.43) we obtain

$$\begin{aligned}\left|F^{\varepsilon}_{2,HMM}(0)\right|&\leq\left|F^{\varepsilon}_{2,HMM}-\varepsilon\overline{a\left(\partial_x\tilde{d}_0+\tilde{d}_1\right)}\right|+\varepsilon\left|\overline{a\left(\partial_x\tilde{d}_0+\tilde{d}_1\right)}\right|\\&\leq C\alpha^q+C\varepsilon\left(\varepsilon^{-1}\alpha^q+\alpha^{q-2}\right)\leq C\left(\alpha^q+\varepsilon\alpha^{q-2}\right)\end{aligned}$$

4. $\boldsymbol{n=3}$: By step 4, $\tilde{d}_0$, $\tilde{d}_1$, and $\tilde{d}_2$ satisfy

$$L[\tilde{d}_2]=-3a_x+\alpha^q Z_2(x) \tag{5.45}$$

$$L[\tilde{d}_1]=-\left(2\left(M[\tilde{d}_2]-\overline{M[\tilde{d}_2]}\right)+6\left(a-\bar{a}\right)\right)+\alpha^q Z_1(x) \tag{5.46}$$

$$L[\tilde{d}_0]=-\left(\left(M[\tilde{d}_1]-\overline{M[\tilde{d}_1]}\right)+2\left(a\tilde{d}_2-\overline{a\tilde{d}_2}\right)\right)+6\hat{a}\chi(x)+\alpha^q Z_0(x), \tag{5.47}$$

where

$$\|Z_2\|_{H^1(Y)}\leq C,\quad \|Z_1\|_{H^1(Y)}\leq C\left(\varepsilon^{-1}+\alpha^{-2}\right)\quad \|Z_0\|_{H^1(Y)}\leq C\left(\varepsilon^{-2}+\alpha^{-3}\right).$$

First by Lemma 5.2 we know that

$$|F^{\varepsilon}_{3,HMM}(0) - \varepsilon^2 \overline{a\left(\partial_x \tilde{d}_0 + \tilde{d}_1\right)}| \leq C\alpha^q. \tag{5.48}$$

Taking the derivative of equation (5.47) we obtain

$$\partial_x^2 \left(a\partial_x \tilde{d}_0 + a\tilde{d}_1\right) = 6\hat{a}\partial_x \chi(x) - \partial_x \left(a\partial_x \tilde{d}_1 + 2a\tilde{d}_2\right) + \alpha^q \partial_x Z_0. \tag{5.49}$$

Furthermore, taking the derivative of (5.46) and exploiting the relation (5.45) we have $\partial_x^2 \left(a\partial_x \tilde{d}_1 + 2a\tilde{d}_2\right) = \alpha^q \left(Z_2 + \partial_x Z_1\right)$. Moreover, by Poincaré inequality

$$\begin{aligned} \|\partial_x \left(a\partial_x \tilde{d}_1 + 2a\tilde{d}_2\right)\|_{L^2(Y)} &\leq C\|\partial_x^2 \left(a\partial_x \tilde{d}_1 + 2a\tilde{d}_2\right)\|_{L^2(Y)} \\ &= C\alpha^q \|Z_2 + \partial_x Z_1\|_{L^2(Y)} \leq C\alpha^q \left(\varepsilon^{-1} + \alpha^{-2}\right). \end{aligned} \tag{5.50}$$

Now we define $h(x) := \int_0^x \chi(s)ds$ as before and take the innerproduct of the left hand side of (5.49) with h and apply Lemma 5.3 to see that

$$\langle h, \partial_x^2 \left(a\partial_x \tilde{d}_0 + a\tilde{d}_1\right)\rangle = -\langle \chi, \partial_x \left(a\partial_x \tilde{d}_0 + a\tilde{d}_1\right)\rangle = -\overline{a\left(\partial_x \tilde{d}_0 + \tilde{d}_1\right)}.$$

But from equation (5.49) we have

$$\begin{aligned} \overline{a\left(\partial_x \tilde{d}_0 + \tilde{d}_1\right)} &= -\langle h, \partial_x^2 \left(a\partial_x \tilde{d}_0 + a\tilde{d}_1\right)\rangle \\ &= -6\hat{a}\langle h, \partial_x \chi\rangle + \langle \partial_x \left(a\partial_x \tilde{d}_1 + 2a\tilde{d}_2\right), h\rangle - \alpha^q \langle \partial_x Z_0, h\rangle \\ &= 6\hat{a}\|\chi\|^2_{L^2(Y)} + \langle \partial_x \left(a\partial_x \tilde{d}_1 + 2a\tilde{d}_2\right), h\rangle - \alpha^q \langle \partial_x Z_0, h\rangle. \end{aligned}$$

Hence using (5.50) and (5.47) we get

$$\begin{aligned} \left|\overline{a\left(\partial_x \tilde{d}_0 + \tilde{d}_1\right)} - 6\hat{a}\|\chi\|^2_{L^2(Y)}\right| &\leq \|h\|_{L^2(Y)} \left(\|\partial_x \left(a\partial_x \tilde{d}_1 + 2a\tilde{d}_1\right)\|_{L^2(Y)} + \alpha^q \|\partial_x Z_0\|_{L^2(Y)}\right) \\ &\leq C\alpha^q \left(\left(\varepsilon^{-1} + \alpha^{-2}\right) + \left(\varepsilon^{-2} + \alpha^{-3}\right)\right) \leq C\left(\varepsilon^{-2}\alpha^q + \alpha^{q-3}\right). \end{aligned}$$

Moreover, using the last inequality and the estimate (5.48) we get

$$\begin{aligned} \left|F^{\varepsilon}_{3,HMM} - \varepsilon^2 6\hat{a}\|\chi\|^2_{L^2(Y)}\right| &\leq \left|F^{\varepsilon}_{3,HMM} - \varepsilon^2 \overline{a\left(\partial_x \tilde{d}_0 + \tilde{d}_1\right)}\right| + \left|\varepsilon^2 \overline{a\left(\partial_x \tilde{d}_0 + \tilde{d}_1\right)} - \varepsilon^2 6\hat{a}\|\chi\|^2_{L^2(Y)}\right| \\ &\leq C\alpha^q + C\left(\alpha^q + \varepsilon^2\alpha^{q-3}\right) \leq C\left(\alpha^q + \varepsilon^2\alpha^{q-3}\right). \end{aligned}$$

This completes the proof of Theorem 5.1.

5.3. Proof of Lemma 5.2. First let $n = 0$, then we have $d(x) = 1 + \alpha^q \delta_0$, and since d is constant we get

$$F^{\varepsilon} = K_\eta * a(\cdot/\varepsilon)\partial_x d(\cdot/\varepsilon)(0) = 0.$$

Now assume that $n \geq 1$ then $d(x) = x^n + \sum_{j=0}^{n-1} x^j d_j(x) + \alpha^q \sum_{j=0}^{n} x^j \varepsilon^{j-n}\delta_j$. To simplify the notation let us introduce $d_n(x) = 1$, and for $1 \leq j \leq n-1$

$$F_0(y) = a(y)\left(d_0'(y) + d_1(y)\right), \quad F_j(x,y) = x^j a(y)\left(d_j'(y) + (j+1)d_{j+1}(y)\right).$$

From this we can write

$$a(y)\partial_y d(y) = F_0(y) + \sum_{j=1}^{n-1} F_j(y,y) + \alpha^q \sum_{j=1}^{n} y^{j-1}\varepsilon^{j-n}\delta_j.$$

Then

$$\begin{aligned} F^\varepsilon \quad &:= \varepsilon^n K_\eta * a(x/\varepsilon)\partial_x d(x/\varepsilon) = \varepsilon^{n-1} K_\eta * F_0(\cdot/\varepsilon)(0) + \varepsilon^{n-1} \textstyle\sum_{j=1}^{n-1} \varepsilon^{-j} K_\eta * F_j(\cdot,\cdot/\varepsilon)(0) \\ &+\alpha^q \textstyle\sum_{j=1}^{n} j\delta_j K_\eta * x^{j-1} a(x/\varepsilon)(0). \end{aligned}$$

By Lemma 2.3 we get

$$\left|K_\eta * F_0(\cdot/\varepsilon)(0) - \overline{F_0}\right| \le C\alpha^q.$$

Moreover, for the remaining terms we again employ Lemma 2.3 with $p \ge n$ to obtain

$$|K_\eta * F_j(\cdot,\cdot/\varepsilon)(0)| \le C\alpha^q, \quad \left|K_\eta * x^j a(x/\varepsilon)(0)\right| \le C\alpha^q, \quad j \ge 1.$$

This gives then

$$\begin{aligned} \left|F^\varepsilon - \varepsilon^{n-1}\overline{F_0}\right| \quad &\le \varepsilon^{n-1}\left|K_\eta * F_0(\cdot/\varepsilon) - \overline{F_0}\right| + \varepsilon^{n-1}\textstyle\sum_{j=1}^{n-1} \varepsilon^{-j}\left|K_\eta * F_j(\cdot,\cdot/\varepsilon)\right| \\ &+C\alpha^q \textstyle\sum_{j=1}^{n} \left|K_\eta * x^{j-1} a(x/\varepsilon)(0)\right| \le C\left(\varepsilon^{n-1}\alpha^q + \alpha^q + \alpha^{2q}\right) \le C\alpha^q. \end{aligned}$$

6. Numerical results.

(i) **Convergence of the HMM flux:** In this section we present some numerical results to validate our theoretical arguments. We test the theoretical argument made in Theorem 5.1. We solve the periodic wave equation (1.6) over the microscopic domain $\Omega_{\eta,0}$, with $A(y) = 1.1 + \sin(2\pi y + 2)$. We use consistent initial data obtained via the algorithm in [14]. We are interested in the difference between the HMM flux $F_{HMM}(x_0)$ given in (5.4) and the macroscopic flux $\hat{F}(x_0)$ given in (5.5) at $x_0 = 0$. We let our macro state $\hat{u}$ be as

$$\hat{u}(x) = x + x^3.$$

Then with $K \in \mathbb{K}^{p,q-2}$ where $p = 3$ and $q = 9$, the left plot in Figure 1 shows $O(\alpha^q)$ convergence rate for the HMM flux as predicted by Theorem 5.1. In the right plot we illustrate the consistency error $(K_\tau * u^\varepsilon)(0,x) - \hat{u}(x)$. The consistency error decreases to zero with the same rate,i. e., $O(\alpha^q)$.

(ii) **Correction terms in the consistent initial data:** We want to understand the problem: if we are given a macro state $\hat{u}(x) = x^3$ what would be exact scaling of the coefficients of the consistent initial data $\bar{u}(x)$? For this we use the algorithm from [14] to find a consistent initial data $\bar{u}$ of the form

$$\bar{u}(x) = r_0(\varepsilon) + r_1(\varepsilon)x + r_2(\varepsilon)x^2 + r_3(\varepsilon)x^3 + x^3.$$

Then again with $K \in \mathbb{K}^{p,q-2}$ where $p = 3$ and $q = 9$, Figure 2 shows that $r_0 = r_2 = r_3 = O(\alpha^q)$, and $r_1 = O(\varepsilon^2)$. Therefore, the consistent initial data for x^3 is $x^3 + \varepsilon^2 x$. This result illustrates the fact that, in general, the consistent initial data is not the same as the macroscopic state, and that one needs to add a correction term of order $O(\varepsilon^2)$ to the macroscopic state in order to obtain an initial data with which the microscopic problem captures the correct homogenized quantities.

REMARK 6.1. *The total error E in the full HMM approximation is*

$$E = E_{macro} + E_{micro} + E_{HMM},$$

where E_{macro} and E_{micro} are the discretization errors in micro and macro levels, and E_{HMM} is the upscaling error. Assuming that the computational cost of solving a wave equation in d-dimensions is proportional to the number of degrees of freedom, and that we have s_1 and s_2 orders of accuracy in micro and macro levels, the micro and the macro stepsizes h and H should be refined simultaneously as

$$H = (\varepsilon/\eta)^{q/s_1}, \quad h = (\varepsilon/\eta)^{q/s_2}$$

to reach a total accuracy of $O\left((\varepsilon/\eta)^q\right)$ over a fixed time interval. Moreover, with $\tau = \eta$, and assuming that the micro problems are solved only once, the computational cost to achieve this accuracy becomes

$$\textit{Computational Cost} = O\left(\left(\frac{\eta}{\varepsilon}\right)^{\frac{qd}{s_1}+\frac{q(d+1)}{s_2}} \eta^{d+1} + \left(\frac{\eta}{\varepsilon}\right)^{\frac{q(d+1)}{s_1}}\right).$$

The overall cost can be substantially reduced by using high order methods in macro and the micro levels. See [4], for an example of using a pseudo-spectral method for an elliptic micro problem to obtain an almost linear computational complexity in the number of degrees of freedom of macro discretization. We refer also the reader to [1] for an example of efficient high-order discretization of the macro problem.

7. Conclusion. In this paper, we have analyzed a multiscale method based on the HMM methodology for approximation of effective solutions of long time wave propagation problems in periodic media. In [12], it was shown that the HMM captures the homogenized/macroscopic fluxes in short time wave propagation problems. Here, we have proved that the HMM also captures the long time dispersive behaviors of the waves if the microscopic simulations are provided with appropriate initial data. The ideas used in the analysis are new and general in terms of dimension, although the final result is one-dimensional. To the best knowledge of the authors, this is the first paper which deals with the complete analysis of the upscaling error of an HMM based method for long time wave propagation problems. We believe that the ideas here will be useful also in understanding the behavior of the HMM based numerical methods for time dependent problems in locally periodic media where the media has fast and slow variations at the same time.

We want to emphasize that since the homogenized equation (1.3) is ill-posed, it does not make sense to carry out a convergence study of the full HMM solution u_{HMM} to the homogenized solution $\hat{u}$. The convergence analysis of the solution requires a regularized homogenized equation. It is, however, known that if H/ε is sufficiently large then the HMM gives a stable solution, see theoretical arguments and numerical evidences in [11]. Another problem, even with a regularized equation, is the possible accumulation of the upscaling error over large time scales. This error should grow at most polynomially in time in order to maintain high order approximation of the homogenized quantities. This is the topic of current interest which will be addressed in upcoming papers.

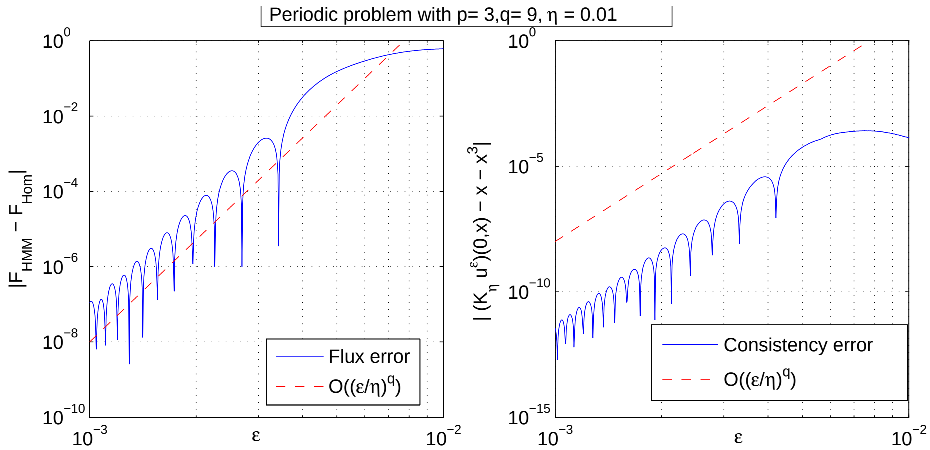

FIG. 1. *The error between the HMM flux and the homogenized flux (left plot), the consistency error between the time filtered microscopic solution and the macroscopic state $\hat{u}(x) = x + x^3$ (right plot). In this simulation we have chosen $A(y) = 1.1 + \sin(2\pi y + 2)$ and $K \in \mathbb{K}^{p,q-2}$ with $p = 3$ and $q = 9$. We clearly observe $O(\alpha^q)$ convergence rate for the flux and the filtered solution.*

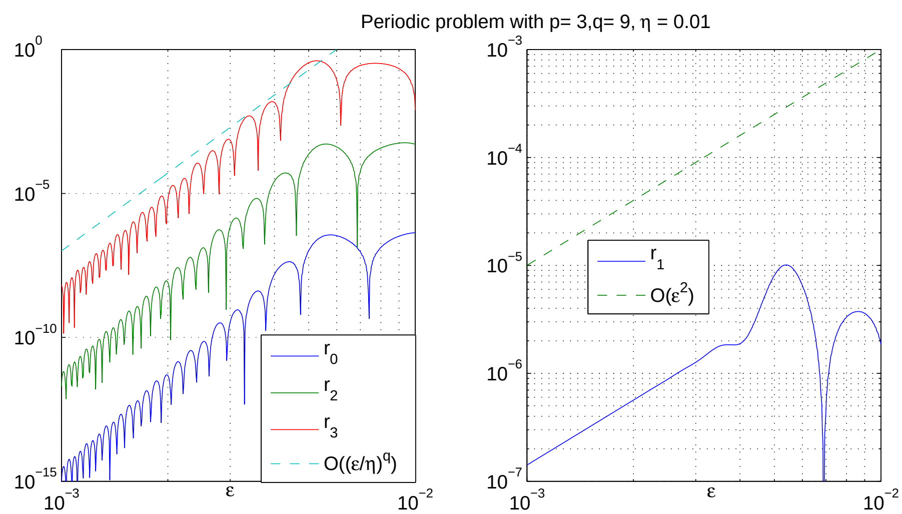

FIG. 2. *This result shows that the macro state $\hat{u}(x) = x^3$ gives a consistent initial data of the form $\bar{u}(x) = x^3 + \varepsilon^2 x$. We let $\bar{u} = r_0(\varepsilon) + r_1(\varepsilon)x + r_2(\varepsilon)x^2 + r_3(\varepsilon)x^3 + x^3$, then with a kernel $K \in \mathbb{K}^{p,q-2}$ where $p = 3$ and $q = 9$, the plots illustrate the rate at which the coefficients decrease to zero as $\varepsilon \longrightarrow 0$.*

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