Introduction to Normal Multiresolution Approximation

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Summary. A multiresolution analysis of a curve is normal if each wavelet detail vector with respect to a certain subdivision scheme lies in the local normal direction. In this paper we give an introduction to the analysis of normal approximations in [3]. We define the normal approximation in its basic form and show simplified proofs of the method's convergence, approximation quality and stability. We also explain how higher order approximations can be constructed using subdivision operators and give a brief summary of the corresponding results for these more general schemes.

Key words: subdivision, wavelet, normal mesh, normal multiresolution

1 Introduction

Finding representations of three-dimensional geometric data that allow for efficient computational processing have become an increasingly important problem, spurred by recent advances in shape acquisition technology such as laser scanners. It is now possible to sample real world three-dimensional objects with a very high level of detail; scanners can generate huge amounts of data typically in the form of triangular meshes with complex topology, [13]. The irregular format makes processing like compression, denoising, filtering and texturing, difficult. New ways of describing three-dimensional objects can lead to significantly improved compression algorithms and facilitate other processing. In addition, it is often desirable to support *progressive reconstruction*: a coarse version of the object is first quickly reconstructed and additional levels of detail are added as the reconstruction continues. This is useful for instance in streaming applications in networked environments.

Of particular interest for progressive reconstruction are multiresolution meshes, where the object is described through an hierarchy of increasingly detailed meshes. Each new mesh level is computed from the previous one by first *predicting* a new point, for instance by subdivision schemes like Butterfly or Loop [4, 14], and then *correcting* the predicted point by a *wavelet* (or detail) vector. Only the wavelet vectors need to be stored and because of the surface smoothness most wavelet vectors will be small, lending the representation well to compression.

The mathematical properties of wavelets are well understood in the so-called "functional setting", i.e., for the approximation of *functions* of one or more variables, which is the setting for image and sound processing. However, for the case of 1-D curves in the plane, or 2-D surfaces in 3-space, much less is known. Typically one takes a parameterization of the original curve or surface and ends up using wavelet analysis in each of the two or three components. This means the wavelet coefficients now become 2- or 3-vectors. It is important to choose an appropriate coordinate frame to describe these wavelet vectors. It is known that using an absolute coordinate frame for the wavelet or detail vectors leads to undesirable effects when editing curves; using a local coordinate frame defined by the normal works much better, as shown in [5, 6, 7, 15, 17].

In [8] the notion of normal approximation for curves or surfaces was introduced. A multiresolution approximation of a curve or surface is *normal* if all the wavelet vectors perfectly align with a locally defined normal direction which only depends on the coarser levels. Note that by the normal direction we mean a normal onto an approximation of the curve or surface. Given that this normal direction only depends on coarser levels, only a *single* scalar coefficient needs to be stored instead of the standard 2- or 3-vector. This is clearly extremely useful for compression applications, see [10, 11, 12]. In addition, [8] gives an algorithm to build normal mesh approximations of large complex scanned geometry.

Because they depend on the computation of a normal, these approximations lead to non-linear representations for which there is no general theory. However, a detailed study of the mathematical properties, such as convergence, regularity, and stability of normal multiresolution approximation for curves were made in [3]. In particular it was shown that these properties critically depend on the underlying subdivision scheme and that in general the convergence of the normal approximation of smooth curves equals the convergence of the subdivision scheme. See also [9] for the case of nonsmooth surfaces.

This paper is based on [3]. Here we give an introduction to the analysis of normal approximations in that article. In Sect. 2 we define the normal approximation in its basic form and show simplified proofs of the method's convergence, approximation quality and stability. In Sect. 3 we explain how higher order approximations can be constructed using subdivision operators and give a brief summary of the corresponding results for these more general schemes.

2 Basic Normal Multiresolution Analysis

2.1 Definitions and Notation

Fig. 1 illustrates the main idea from [8] in the case of a normal approximation based on midpoint subdivision. The original curve Γ is described by successively finer approximations, which are organized in different *multiresolution layers* indexed by j. We assume that Γ is a continuous, non intersecting curve in the plane, whose endpoints we shall take to be the zeroth level multiresolution points $v_{0,0}$ and $v_{0,1}$.

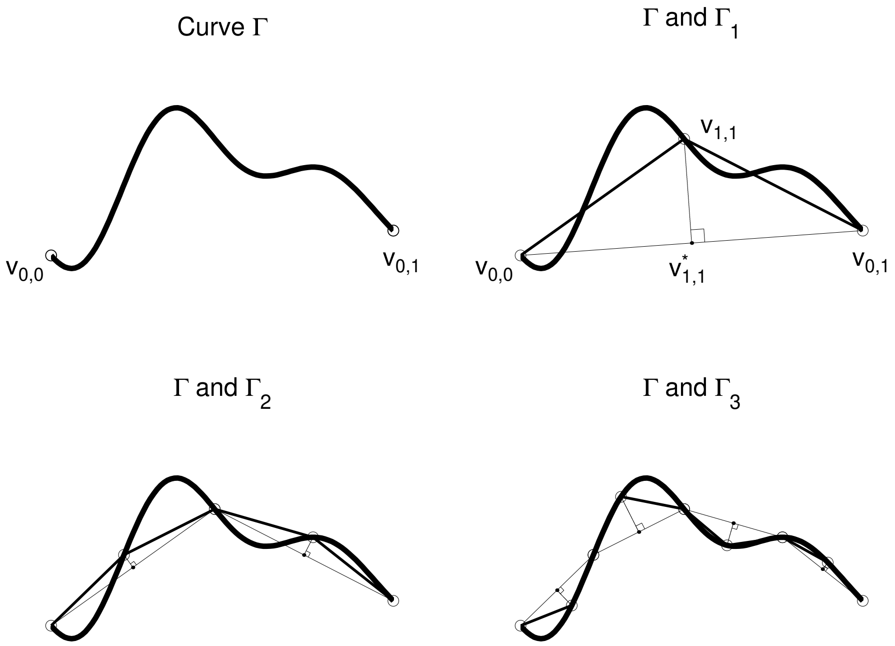

Fig. 1. Example of the normal mesh algorithm using the mean value of adjacent points as predictor

To construct the vertices at level $(j+1)$, we first set $v_{j+1,2k} = v_{j,k}$; this is what makes the construction interpolating. We also compute new points $v_{j+1,2k+1}$; each $v_{j+1,2k+1}$ lies in between the two old points $v_{j,k}$ and $v_{j,k+1}$. This is done by first computing a predicted or base point as the mean value of the old points, $v^*_{j+1,2k+1} = (v_{j,k} + v_{j,k+1})/2$. We next draw a line from $v^*_{j+1,2k+1}$ in the direction orthogonal to the line segment $(v_{j,k}, v_{j,k+1})$. This line is guaranteed to cross the curve segment between $v_{j,k}$ and $v_{j,k+1}$ at least once and we call one of those points $v_{j+1,2k+1}$. As this procedure continues, the polyline Γ_j, i.e. the piecewise linear curve connecting the points $v_{j,k}$ comes closer and closer to Γ. We can now think of this as a wavelet transformation similar to the notion of lifting [16]. Think of $v^*_{j+1,2k+1}$ as a prediction of the real point $v_{j+1,2k+1}$ computed based only on coarser information. Then the difference $v^*_{j+1,2k+1} - v_{j+1,2k+1}$ is a wavelet vector. Given that this vector points in a direction normal to a segment that again only depends on coarser data, we only need to store the length and one sign bit for this normal component to characterize it completely.

Before we go on to analyze this scheme, let us introduce some more notation. The norm $||\cdot||$ will always represent the euclidean norm in $\mathbb{R}^2$. We define the differences between breakpoints on Γ_j as $\Delta\Gamma_{j,k} = v_{j,k+1} - v_{j,k}$. We will always assume $\Gamma(s)$ is the arc length parameterization of Γ. This defines the corresponding breakpoints $s_{j,k}$ in parameter space: $\Gamma(s_{j,k}) = v_{j,k}$ for all j, k. The length along the curve between breakpoints is denoted $\Delta s_{j,k} = s_{j,k+1} - s_{j,k} > 0$. We also let $I_{j,k}$ be the interval $[s_{j,k}, s_{j,k+1}]$. Finally, we introduce the arc length ratio at which the scheme cuts the curve at each interval, $\beta_{j,k} = \Delta s_{j+1,2k}/\Delta s_{j,k}$.

Throughout the analysis we assume that $\Gamma(s)$ is twice continuously differentiable with bounded curvature. It can then be Taylor expanded

$$\Gamma(t) - \Gamma(s) = (t-s)\Gamma'(s) + \frac{1}{2}(t-s)^2 R(t,s) \ , \tag{1}$$

where the restterm is uniformly bounded in its arguments,

$$\sup_{s,t} ||R(t,s)|| \leq c_\gamma \ . \tag{2}$$

Most of the results are true with some modification also for less regular curves and we make a comment on this at the end of the section. For that case we introduce the Hölder spaces C^r; when $r = p + \kappa$, $p \in \mathbb{Z}$ and $0 < \kappa < 1$ we define C^r as the set of functions $f \in C^p$ for which the p-th derivative is Hölder continuous with exponent κ, i.e. $|f^{(p)}(t) - f^{(p)}(s)| \times |t-s|^{-\kappa}$ is bounded for all t, s.

2.2 Convergence

The construction of the normal scheme immediately begs the following question: how good an approximation to Γ is the polyline Γ_j? In particular, does the distance between Γ and the j-th level polyline decay to zero as j tends to ∞? In order to measure the distance between Γ and Γ_j we define the following parameterization for Γ_j,

$$\Gamma_j(s) = \frac{(s - s_{j,k})}{\Delta s_{j,k}} v_{j,k+1} + \frac{(s_{j,k+1} - s)}{\Delta s_{j,k}} v_{j,k} \ , \qquad s \in I_{j,k} \ . \tag{3}$$

With this parameterization we can prove uniform convergence $\Gamma_j \rightarrow \Gamma$. Before doing this, however, we need a lemma showing that the distance along the curve between the breakpoints $v_{j,k}$ on Γ_j decays exponentially to zero with j and that, in the limit, points at the next level will be added precisely in between points on the previous level.

Lemma 1. *Let $\Gamma(s) \in C^2([0,1]; \mathbb{R}^2)$ and $||\Gamma'(s)|| = 1$. Also assume that $\Gamma(s)$ does not cross itself. Then there are constants c_1 and c_2, independent of j, such that*

$$\sup_k \Delta s_{j,k} \leq c_1 2^{-j} \ . \tag{4}$$

and

$$\left| \beta_{j,k} - \frac{1}{2} \right| \leq c_2 \Delta s_{j,k} \ . \tag{5}$$

Proof. The proof is made in four steps.

1. Estimate difference between $||\Gamma(s+\Delta s)-\Gamma(s)||$ and Δs.
   Let $0 \le s < s+\Delta s \le 1$. Using (1, 2) we get

$$0 \le \Delta s - ||\Gamma(s)-\Gamma(s+\Delta s)|| = \Delta s||\Gamma'(s)|| - ||\Gamma(s)-\Gamma(s+\Delta s)||$$
$$\le ||\Gamma(s+\Delta s)-\Gamma(s)-\Delta s\Gamma'(s)|| = \frac{1}{2}(\Delta s)^2||R(s+\Delta s, s)|| \le \frac{1}{2}c_\gamma(\Delta s)^2.$$

2. Show that Γ^{-1} is Lipschitz.
   First, suppose $\Delta s \le 1/c_\gamma$. Then by the previous result,

$$||\Gamma(s+\Delta s)-\Gamma(s)|| \ge \Delta s - \frac{1}{2}c_\gamma(\Delta s)^2 \ge \frac{\Delta s}{2}\ .$$

   Second, for $1 \ge \Delta s \ge 1/c_\gamma$,

$$||\Gamma(s+\Delta s)-\Gamma(s)|| \ge \inf_{\substack{r\ge 1/c_\gamma \\ 0\le s\le s+r\le 1}} ||\Gamma(s+r)-\Gamma(s)|| \equiv c \ge c\Delta s\ .$$

   Since Γ is continuous and does not cross itself, $c > 0$. In conclusion,

$$||\Gamma(s+\Delta s)-\Gamma(s)|| \ge q\Delta s\ , \tag{6}$$

   where $q = \min(c, 1/2) > 0$.
3. Estimate deviation of $\beta_{j,k}$ from $1/2$.
   Since by the scheme's definition $||\Delta\Gamma_{j+1,2k}|| = ||\Delta\Gamma_{j+1,2k+1}||$, we have

$$\left|\beta_{j,k}-\frac{1}{2}\right| = \left|\frac{\Delta s_{j+1,2k}}{\Delta s_{j+1,2k}+\Delta s_{j+1,2k+1}} - \frac{1}{2}\right|$$
$$= \frac{1}{2}\left|\frac{\Delta s_{j+1,2k} - ||\Delta\Gamma_{j+1,2k}|| - \Delta s_{j+1,2k+1} + ||\Delta\Gamma_{j+1,2k+1}||}{\Delta s_{j,k}}\right|$$
$$\le \frac{1}{4}c_\gamma\frac{(\Delta s_{j+1,2k})^2+(\Delta s_{j+1,2k+1})^2}{\Delta s_{j,k}} \le \frac{1}{2}c_\gamma\Delta s_{j,k}\ .$$

   This shows (5). We also derive another estimate of this difference. Through (6) we get a lower bound on $\beta_{j,k}$,

$$\beta_{j,k} = \frac{\Delta s_{j+1,2k}}{\Delta s_{j,k}} \ge \frac{||\Delta\Gamma_{j,k}||}{2\Delta s_{j,k}} \ge \frac{q}{2}\ .$$

   In the same way we get $(1-\beta_{j,k}) \ge q/2$. Together, this yields

$$\left|\beta_{j,k}-\frac{1}{2}\right| \le \frac{1-q}{2}\ . \tag{7}$$

4. Show the convergence rate of $\Delta s_{j,k}$.
   We start by using (7),

$$\begin{aligned}\sup_k \Delta s_{j+1,k} &= \sup_k \max(\beta_{j,k}\Delta s_{j,k}, (1-\beta_{j,k})\Delta s_{j,k}) \\ &\le \left(\frac{1}{2} + \left|\beta_{j,k} - \frac{1}{2}\right|\right) \sup_k \Delta s_{j,k} \le \left(1 - \frac{q}{2}\right) \sup_k \Delta s_{j,k} \ .\end{aligned}$$

This shows that $\sup_k \Delta s_{j,k} \le \delta^j$ where $\delta = 1 - q/2$. Next, we use (5),

$$\begin{aligned}\sup_k \Delta s_{j+1,k} &= \sup_k \max(\beta_{j,k}\Delta s_{j,k}, (1-\beta_{j,k})\Delta s_{j,k}) \\ &\le \sup_k \left(\frac{1}{2}\Delta s_{j,k} + \frac{1}{2} c_\gamma (\Delta s_{j,k})^2\right) \ .\end{aligned}$$

Together with the exponential convergence, we get

$$\begin{aligned}2^j \sup_k \Delta s_{j,k} &\le 2^j \sup_k \Delta s_{j-1,k} \left(\frac{1}{2} + \frac{1}{2} c_\gamma \delta^{j-1}\right) \\ &\le 2^j \Delta s_0 \prod_{i=0}^{j-1} \left(\frac{1}{2} + \frac{1}{2} c_\gamma \delta^i\right) = \prod_{i=0}^{j-1} \left(1 + c_\gamma \delta^i\right) \\ &\le \prod_{i=0}^{j-1} \exp\left(c_\gamma \delta^i\right) = \exp\left(c_\gamma \sum_{i=0}^{j-1} \delta^i\right) \le \exp\left(\frac{c_\gamma}{1-\delta}\right) \ .\end{aligned}$$

The estimate (4) follows. □

We can now easily prove convergence.

Theorem 1. *Under the assumptions of Lemma* 1 *the normal approximation converges,*

$$\lim_{j\to\infty} \sup_{0\le s\le 1} ||\Gamma_j(s) - \Gamma(s)|| = 0 \ . \tag{8}$$

Proof. Consider first the interval $I_{j,k}$. Taylor expand and use (1, 2),

$$\begin{aligned}&\sup_{s\in I_{j,k}} ||\Gamma(s) - \Gamma_j(s)|| \\ &\quad = \sup_{s\in I_{j,k}} \left\| \frac{s - s_{j,k}}{\Delta s_{j,k}} [\Gamma(s_{j,k+1}) - \Gamma(s)] + \frac{s_{j,k+1} - s}{\Delta s_{j,k}} [\Gamma(s_{j,k}) - \Gamma(s)] \right\| \\ &\quad = \sup_{s\in I_{j,k}} \left\| \frac{(s - s_{j,k})(s - s_{j,k+1})^2}{2\Delta s_{j,k}} R(s, s_{j,k+1}) \right. \\ &\qquad\qquad \left. + \frac{(s_{j,k+1} - s)(s - s_{j,k})^2}{2\Delta s_{j,k}} R(s, s_{j,k}) \right\| \\ &\quad \le c_\gamma (\Delta s_{j,k})^2 \ .\end{aligned}$$

Consequently, by Lemma 1,

$$\sup_{0\le s\le 1} ||\Gamma(s) - \Gamma_j(s)|| \le c_\gamma \sup_k (\Delta s_{j,k})^2 \le c\, 2^{-2j} \ ,$$

and (8) follows. □

2.3 Decay of Wavelet Coefficients

One of the important features of a normal multiresolution is the decay of the offsets in each of the normal directions, defined as

$$w_{j,k} = ||v_{j+1,2k+1} - v^*_{j+1,2k+1}|| \ .$$

We will refer to these as wavelet coefficients. Fast decay of those coefficients together with stability means that efficient compression schemes can be devised. We show here that they decay as 2^{-2j}.

Theorem 2. *Under the assumptions of Lemma* 1 *then*

$$w_{j,k} \leq c_\gamma (\Delta s_{j,k})^2 \tag{9}$$

with c_γ as in (2)*. Moreover, there is a constant c independent of j such that*

$$\sup_k w_{j,k} \leq c\, 2^{-2j} \ . \tag{10}$$

Proof. As above, consider the interval $I_{j,k}$ and note that $s_{j+1,2k+1} \in I_{j,k}$ is the point where the constructed normal pierces the curve, $\Gamma(s_{j+1,2k+1}) = v_{j+1,2k+1}$. Denote the offset vector by u,

$$u = \Gamma(s_{j+1,2k+1}) - \frac{1}{2}(\Gamma(s_{j,k}) + \Gamma(s_{j,k+1})) \ ,$$

There exists $\eta \in I_{j,k}$ such that $\Gamma'(\eta)$ is parallel to $\Delta\Gamma_{j,k}$. We Taylor expand around η and using (1, 2) we get

$$\begin{aligned} u = \frac{1}{2}(2s_{j+1,2k+1} - s_{j,k} - s_{j,k+1})\Gamma'(\eta) + \frac{(s_{j+1,2k+1}-\eta)^2}{2} R(s_{j+1,2k+1},\eta) \\ - \frac{(s_{j,k}-\eta)^2}{4} R(s_{j,k},\eta) - \frac{(s_{j,k+1}-\eta)^2}{4} R(s_{j,k+1},\eta) \ . \end{aligned}$$

Let $\hat{n}$ be the vector normal to $\Gamma'(\eta)$ and to $\Delta\Gamma_{j,k}$. Then,

$$\begin{aligned} w_{j,k} = ||u|| = |\hat{n}\cdot u| &\leq \frac{2(s_{j+1,2k+1}-\eta)^2 + (s_{j,k}-\eta)^2 + (s_{j,k+1}-\eta)^2}{4} c_\gamma \\ &\leq c_\gamma (\Delta s_{j,k})^2 \ , \end{aligned}$$

which is (9). The decay rate for $\Delta s_{j,k}$ established in Lemma 1 finally gives (10). □

2.4 Normal Parameterization

Normal multiresolution induces a parameterization of the curve Γ, as exemplified in Fig. 2. Analytically, this parameterization is described as follows: We define, at every level j, $\mathtt{s}_j : [0,1] \mapsto \mathbb{R}$ to be the piecewise affine map with breakpoints at the $t_{j,k} = 2^{-j}k$, $k = 0, \ldots, 2^j$, and for which $\Gamma(\mathtt{s}_j(t_{j,k})) = v_{j,k}$, i.e. $\mathtt{s}_j(t_{j,k}) = s_{j,k}$,

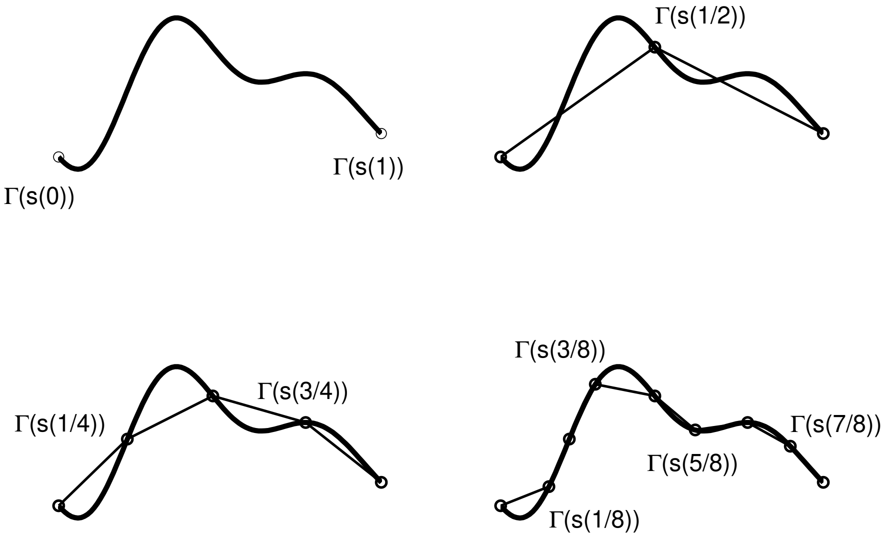

Fig. 2. Example of how the normal multiresolution induces a parameterization

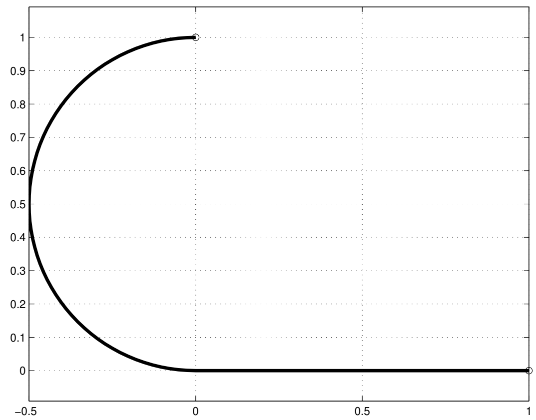

Fig. 3. Example of a curve with non smooth normal parameterization

see Fig. 2. If the node points on Γ_j approach each other when $j \to \infty$, i.e. if $\sup_k \Delta s_{j,k} \to 0$ then $\mathtt{s}_j(t)$ converges uniformly to a function $\mathtt{s}(t)$. The parameterization of the curve Γ induced by the normal multiresolution then maps $t \in [0,1]$ to $\Gamma(\mathtt{s}(t))$; we shall call this the *normal parameterization* of the curve Γ.

The regularity of the normal parameterization is related to the decay of the wavelet coefficients which directly determines the approximation quality. It can, however, also be significant independently, for instance when the normal scheme is used to generate meshes for computational purposes; the accuracy and stability of a numerical scheme is affected by the smoothness of the map from the computational to the physical domain. It should therefore be noted that the normal parameterization need *not* be smooth, even if Γ is. Consider the normal multiresolution as applied to the curve in Fig. 3, which consists of a $180°$ circle arc and a straight, tangent line segment with length equal to the diameter of the circle. At level zero, we have $v_{0,0} = (0,1) = \Gamma(\mathtt{s}(0))$ and $v_{0,1} = (1,0) = \Gamma(\mathtt{s}(1))$. Because of the special construction of Γ, the first inserted point $v_{1,1}$ coincides with the origin, $(0,0) = \Gamma(\mathtt{s}(1/2))$. After that, the normal multiresolution will induce a parameterization that corresponds to arc length for both the right and the left piece of the curve:

$$\Gamma(\mathtt{s}(t)) = \begin{cases} \frac{1}{2}(-\sin(2t\pi), 1+\cos(2t\pi)), & 0 \le t < \frac{1}{2}\ , \\ (2t-1, 0), & \frac{1}{2} \le t \le 1\ . \end{cases}$$

However, the two pieces have different lengths, so the parameterization must have a discontinuity in its gradient, indeed

$$\left\| \frac{\mathrm{d}\Gamma(\mathtt{s}(t))}{\mathrm{d}t} \right\| = \begin{cases} \pi, & 0 \le t < \frac{1}{2}\ , \\ 2, & \frac{1}{2} < t \le 1\ . \end{cases}$$

In this case the curve $\Gamma(s)$ is almost C^2 (its gradient $\Gamma'(s)$ is Lipschitz), yet its normal parameterization is only Lipschitz. This is because the regularity of the parameterization turns out to be limited not only by the smoothness of the curve, but also by the way the predicted point $v^*_{j,k}$ is computed as shown by the following argument. By definition,

$$\Big(\Gamma(\mathtt{s}(t+h)) - \Gamma(\mathtt{s}(t-h))\Big) \cdot \Big(\Gamma(\mathtt{s}(t+h)) - 2\Gamma(\mathtt{s}(t)) + \Gamma(\mathtt{s}(t-h))\Big) = 0\ ,$$

at odd dyadic points ($t = (2k+1)2^{-j}$, $h = 2^{-j}$), where ' $\cdot$ ' stands for the $\mathbb{R}^2$ inner product, $(u,v) \cdot (u',v') = uu' + vv'$. Now *if* the parameterization were $C^{4+\varepsilon}$, with $\varepsilon > 0$, then we could Taylor expand $\Gamma(\mathtt{s}(t \pm h))$ around t and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\mathrm{d}\Gamma(\mathtt{s}(t))}{\mathrm{d}t}\right\|^2 + \frac{h^2}{12}\left(\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left\|\frac{\mathrm{d}\Gamma(\mathtt{s}(t))}{\mathrm{d}t}\right\|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left\|\frac{\mathrm{d}^2\Gamma(\mathtt{s}(t))}{\mathrm{d}t^2}\right\|^2\right) = \mathrm{O}(h^4)\ ,$$

at odd dyadic points; if $\Gamma(\mathtt{s}(t)) \in C^{4+\varepsilon}$, then this equation extends to all t, h. By letting $h \to 0$ we see that we must have both $|\Gamma'|$ and $|\Gamma''|$ constant for this to hold, i.e. the curve Γ must be either a straight line or a circle segment, which is obviously not the case for general smooth curves Γ. In fact, typically the parameterization is not even piecewise smooth; the derivative is discontinuous at every dyadic point. We can, however, prove that it is always Lipschitz:

Theorem 3. *Under the assumptions of Lemma* 1 *there exists a Lipschitz continuous limit* $\mathtt{s}(t)$

$$\lim_{j\to\infty} \sup_{0\le t\le 1} |\mathtt{s}_j(t) - \mathtt{s}(t)| = 0\ . \tag{11}$$

Proof. Using Lemma 1 we get

$$|\mathtt{s}_{j+1}(t) - \mathtt{s}_j(t)| \le \sup_k \left|\beta_{j,k} - \frac{1}{2}\right| \Delta s_{j,k} \le c \sup_k (\Delta s_{j,k})^2 \le c' 4^{-j} \ .$$

Hence, if $m < n$

$$|\mathtt{s}_n(t) - \mathtt{s}_m(t)| \le \sum_{j=m}^{n-1} |\mathtt{s}_{j+1}(t) - \mathtt{s}_j(t)| \le c' \sum_{j=m}^{n-1} 4^{-j} \le \frac{c' 4^{-m+1}}{3} \ ,$$

which tends to zero with m. Therefore $\{\mathtt{s}_j(t)\}$ is Cauchy in sup norm and there exists a continuous limit $\mathtt{s}(t)$ satisfying (11). Finally, by Lemma 1,

$$|\mathtt{s}(t_1) - \mathtt{s}(t_2)| = \lim_{j\to\infty} |\mathtt{s}_j(t_1) - \mathtt{s}_j(t_2)| \le \lim_{j\to\infty} \sup_k 2^j |\Delta s_{j,k}||t_1 - t_2| \le c|t_1 - t_2| \ ,$$

which shows that $\mathtt{s}(t)$ is Lipschitz. □

2.5 Stability

Finally, we look at the stability of normal multiresolution. We assume that the initial points on the curve as well as the wavelet coefficients have some error or round-offs. The curve is then reconstructed as if they were exact. The result is a perturbed sequence of points $\tilde{v}_{j,k}$ and polylines $\tilde{\Gamma}_j$. We show here that this perturbation remains close to the exact curve Γ if the initial perturbations are small.

Let the errors in initial data be bounded as

$$\sup_k ||v_{0,k} - \tilde{v}_{0,k}|| = E_\gamma \ .$$

Since the wavelet coefficients rapidly decay with j we cannot expect stability unless the absolute error in the coefficients also decays. We therefore assume

$$\sup_k |w_{j,k} - \tilde{w}_{j,k}| = E_w 2^{-j\nu} \ , \qquad \nu > 0 \ .$$

We can then show

Theorem 4. *Under the assumptions of Lemma* 1 *and with the definition of the perturbations above there is a constant c depending on ν, but independent of j, E_γ and E_w such that*

$$\sup_k ||v_{j,k} - \tilde{v}_{j,k}|| \le c(E_\gamma + E_w) \ . \tag{12}$$

Proof. This proof consists of two points.

1. Estimate of error in one reconstruction step.
   We start from a fixed level j and consider how the error

$$\varepsilon_j := \sup_k ||v_{j,k} - \tilde{v}_{j,k}||$$

is amplified in level $j+1$. Since the even points in level $j+1$ are the same as the points in level j the error in those points does not change. We therefore only need to investigate the odd points. For simplicity we consider fixed indices j, k and drop all indices in the notation, so that $v = v_{j,k}$, $s = s_{j,k}$, $\Delta s = \Delta s_{j,k}$, etc. We also call the new odd points $v^* = v_{j+1,2k+1}$ and $\tilde{v}^* = \tilde{v}_{j+1,2k+1}$. Finally, we let $\hat{n}$ and $\tilde{\hat{n}}$ be the normal and perturbed normal respectively at the indices j, k. The error in the new value v^* can then be estimated,

$$\begin{aligned}||v^* - \tilde{v}^*|| &= \left\| \frac{\Gamma(s) + \Gamma(s+\Delta s)}{2} + w\hat{n} - \frac{\tilde{\Gamma}(s) + \tilde{\Gamma}(s+\Delta s)}{2} - \tilde{w}\tilde{\hat{n}} \right\| \\ &\leq \varepsilon_j + ||w(\tilde{\hat{n}} - \hat{n})|| + |w - \tilde{w}| \leq \varepsilon_j + E_w 2^{-j\nu} + w||\tilde{\hat{n}} - \hat{n}|| \ .\end{aligned}$$

Moreover, if $||\Delta\Gamma - \Delta\tilde{\Gamma}|| \geq ||\Delta\Gamma||$,

$$||\tilde{\hat{n}} - \hat{n}|| \leq 2 \leq 2\frac{||\Delta\Gamma - \Delta\tilde{\Gamma}||}{||\Delta\Gamma||} \leq \frac{4\varepsilon_j}{||\Delta\Gamma||} \ .$$

On the other hand, if $||\Delta\Gamma - \Delta\tilde{\Gamma}|| < ||\Delta\Gamma||$,

$$\begin{aligned}||\tilde{\hat{n}} - \hat{n}|| &= \frac{\Big\| \Delta\Gamma||\Delta\tilde{\Gamma}|| - ||\Delta\Gamma||\Delta\tilde{\Gamma} \Big\|}{|\Delta\tilde{\Gamma}|||\Delta\Gamma||} \leq \frac{\Big| ||\Delta\tilde{\Gamma}|| - ||\Delta\Gamma|| \Big| + ||\Delta\Gamma - \Delta\tilde{\Gamma}||}{||\Delta\Gamma||} \\ &\leq \frac{2||\Delta\Gamma - \Delta\tilde{\Gamma}||}{||\Delta\Gamma||} \leq \frac{4\varepsilon_j}{||\Delta\Gamma||}\end{aligned}$$

These estimates together with (6, 9) then gives

$$\begin{aligned}||\tilde{v}^* - v^*|| &\leq \varepsilon_j + E_w 2^{-j\nu} + 4\frac{w\varepsilon_j}{||\Delta\Gamma||} \leq \varepsilon_j + E_w 2^{-j\nu} + 4\varepsilon_j c_\gamma \frac{(\Delta s)^2}{||\Delta\Gamma||} \\ &\leq \varepsilon_j \left(1 + \frac{4c_\gamma}{q}\Delta s\right) + E_w 2^{-j\nu} \ ,\end{aligned}$$

which implies that

$$\varepsilon_{j+1} \leq \varepsilon_j \left(1 + \frac{4c_\gamma}{q} \sup_k \Delta s_{j,k}\right) + E_w 2^{-j\nu} \leq \varepsilon_j (1 + c2^{-j}) + E_w 2^{-j\nu}. \quad (13)$$

2. Stability.
   Let us define the sequence $\{a_j\}_{j=1}^{\infty}$ as,

$$a_1 = E_\gamma(1+c) + E_w, \qquad a_{j+1} = a_j(1 + c2^{-j}) + E_w 2^{-j\nu} \ , \qquad (14)$$

and set $b = \min(2, 2^\nu) > 1$. Clearly a_j is increasing. We then get

$$
\begin{aligned}
a_{j+1} &\leq a_j \left(1 + b^{-j}\left(c + \frac{E_w}{a_1}\right)\right) \leq a_j(1 + b^{-j}(c+1)) \\
&\leq a_1 \prod_{i=1}^{j}(1 + b^{-i}(c+1)) \leq a_1 \prod_{i=0}^{j} \exp(b^{-i}(c+1)) \\
&= a_1 \exp\left((c+1)\sum_{i=0}^{j} b^{-i}\right) \leq a_1 \exp\left(\frac{b(c+1)}{b-1}\right) .
\end{aligned}
$$

Moreover, since $\varepsilon_0 = E_\gamma$, induction on (13) and (14) shows that $\varepsilon_j \leq a_j$ for all j. Consequently,

$$
\varepsilon_j \leq a_j \leq c a_1 \leq c'(E_\gamma + E_w) .
$$

This shows (12). □

Remark 1. The result in Theorem 4 demonstrates that a compression scheme based on thresholding wavelet coefficients is stable. Suppose that we set

$$
\tilde{w}_{j,k} = \begin{cases} w_{j,k}, & |w_{j,k}| \geq \varepsilon , \\ 0, & |w_{j,k}| < \varepsilon . \end{cases}
$$

We then have $|w_{j,k} - \tilde{w}_{j,k}| \leq \varepsilon$ for all j, k, as well as $|w_{j,k} - \tilde{w}_{j,k}| \leq |w_{j,k}| \leq c2^{-2j}$. It follows that, for $0 \leq \kappa \leq 1$,

$$
|w_{j,k} - \tilde{w}_{j,k}| \leq \varepsilon^{1-\kappa} c^\kappa 2^{-2j\kappa} .
$$

If $\tilde{v}_{0,k} = v_{0,k}$ and $\kappa > 0$, then we obtain from Theorem 4 that

$$
\sup_k ||\tilde{v}_{j,k} - v_{j,k}|| \leq c' \, c^\kappa \varepsilon^{1-\kappa} , \qquad \kappa > 0 ,
$$

where c' depends on κ. To compare the perturbed polyline $\tilde{\Gamma}_j$ with Γ_j we need to consider their normal parameterizations,

$$
\Gamma_j(t) = 2^j(t - t_{j,k})v_{j,k+1} + 2^j(t_{j,k+1} - t)v_{j,k} , \qquad t_{j,k} \leq t \leq t_{j,k+1} ,
$$

and similarly for $\tilde{\Gamma}_j$. Then, clearly,

$$
\sup_{0\leq t\leq 1} ||\Gamma_j(t) - \tilde{\Gamma}_j(t)|| \leq c' \, c^\kappa \varepsilon^{1-\kappa} , \tag{15}
$$

We note that after a finite number of refinement levels the perturbed wavelet coefficients will all be zero by the decay estimate (10). This implies that there is a well defined limit $\tilde{\Gamma}_j \to \tilde{\Gamma}$ as $j \to \infty$. It then follows from (15) that

$$
\sup_{0\leq t\leq 1} ||\Gamma(\mathtt{s}(t)) - \tilde{\Gamma}(t)|| \leq c' \, c^\kappa \varepsilon^{1-\kappa} .
$$

A similiar, simpler, argument can be made when wavelet coefficients $\tilde{w}_{j,k}$ are set to zero beyond a certain level J. It leads to the estimate

$$
\sup_{0\leq t\leq 1} ||\Gamma(\mathtt{s}(t)) - \tilde{\Gamma}(t)|| \leq c2^{-2J} .
$$

Remark 2. When Γ is not in C^2 the proofs of the theorems above become more complicated, but the main results are still true. As long as $\Gamma \in C^r$ with $1 < r < 2$ Lemma 1 and Theorem 1 to Theorem 4 all hold, although the decay rates in (5), (9) and (10) go down to $(\Delta s_{j,k})^{r-1}$, $(\Delta s_{j,k})^r$ and 2^{-jr} respectively. In fact, Theorem 1 and (4) hold also for Lipschitz continuous Γ and Theorem 1 even for $\Gamma \in C^r$ with $0 < r < 1$. In the latter case the decay in (4) is just algebraic however, $\sup_k \Delta s_{j,k} \leq c/(1 + j^{r/(1-r)})$. There is no improvement in decay rates when Γ is smoother than C^2. See [3] for these proofs.

3 Higher Order Generalizations

3.1 Definitions and Notation

We shall be interested in using more general methods, which will lead to higher quality approximation for smooth curves. As illustrated in Fig. 4, the same basic plan is followed: we still set $v_{j+1,2k} = v_{j,k}$. However, the predicted point $v^*_{j+1,2k+1}$ is now defined via a *subdivision* scheme S (see below), and $v_{j+1,2k+1}$ is an intersection point between $v_{j,k}$ and $v_{j,k+1}$ of Γ with the normal through $v^*_{j+1,2k+1}$ to the segment $(v_{j,k}, v_{j,k+1})$. We still define the wavelet coefficient as $w_{j,k} = ||v_{j+1,2k+1} - v^*_{j+1,2k+1}||$. The results now also depend on S. In short, convergence requires additional assumptions but regularity and wavelet decay can be much improved.

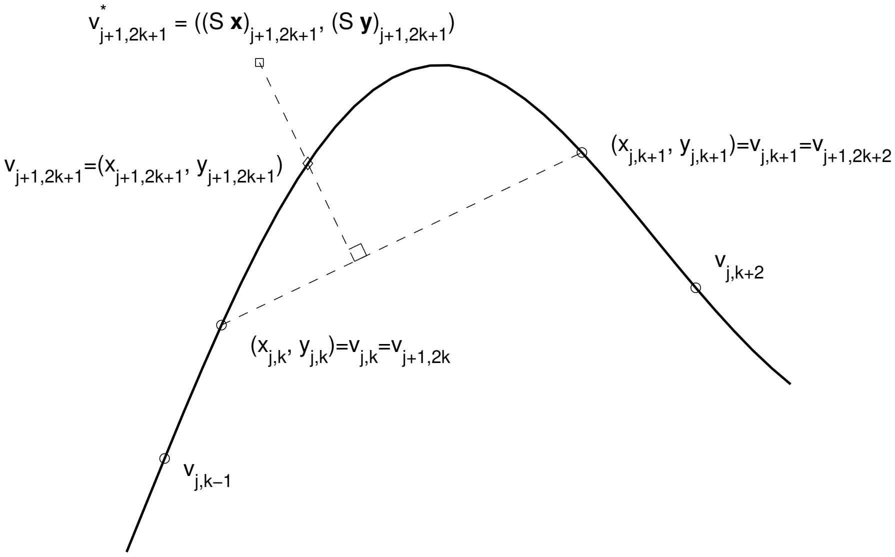

Fig. 4. Notation for the higher order normal scheme

To simplify notation we introduce infinite sequences. Sequences will be written in bold face, and elements of sequences in normal font, $\boldsymbol{x} := (x_k)$. We define the difference operator Δ as

$$(\Delta \boldsymbol{x})_k = x_{k+1} - x_k \ .$$

Often a sequence itself is indexed by the refinement level j; then we use the convention that $\boldsymbol{x}_j := (x_{j,k})$. We use the usual sup-norm for sequences, $|\boldsymbol{x}|_\infty = \sup_k |x_k|$.

A local, stationary subdivision scheme is characterized by a bounded linear operator S, defined by a sequence $\boldsymbol{a}$ as follows

$$(S\boldsymbol{x})_k = \sum_\ell a_{k-2\ell} x_\ell \ ,$$

where the number of non-zero elements in $\boldsymbol{a}$, and consequently also the sum, is finite. Given a starting sequence $\boldsymbol{x}_0$ we can apply S iteratively and define, for all $j \geq 0$,

$$\boldsymbol{x}_{j+1} = S\boldsymbol{x}_j \ . \tag{16}$$

The sequence $\boldsymbol{x}_0$ can be viewed as a coarse approximation of a function, on the integer grid; the sequences $\boldsymbol{x}_j$ then give successively finer approximations of the function on grids with spacing 2^{-j}. We are interested in the case when this process converges to a smooth limit function as j increases. A subdivision scheme is *interpolating* if $s_{2l} = \delta_{l,0}$, implying $x_{j+1,2k} = x_{j,k}$ for all j, k; in this case the $x_{j,k}$ can be interpreted as function values of the limit function $f(x)$ with $x_{j,k} = f(2^{-j}k)$.

Let $\boldsymbol{v}_j$ be a two-dimensional sequence, $\boldsymbol{v}_j = (\boldsymbol{x}_j, \boldsymbol{y}_j)$ and define $S\boldsymbol{v}_j = (S\boldsymbol{x}_j, S\boldsymbol{y}_j)$. The iteration corresponding to (16) for the normal scheme can then be written

$$\boldsymbol{v}_{j+1} = S\boldsymbol{v}_j + \boldsymbol{w}_j \cdot \hat{\boldsymbol{n}}_j \ ,$$

where $\hat{\boldsymbol{n}}_j$ is the two-dimensional sequence of normals and the last product is taken elementwise. Since the $\boldsymbol{w}_j$ sequences decay rapidly to zero we can therefore view the normal scheme as a non-linear perturbation of the underlying linear subdivision scheme S.

Examples of interpolating subdivision schemes are given by the so-called Lagrange interpolation subdivision schemes. For those the new odd-indexed points are defined as the values taken by a polynomial determined by several neighboring old points. We shall denote the 2ℓ-point scheme by $S_{2\ell}$. For instance, in the two-point scheme, $x_{j+1,2k+1}$ is given the value at $t = 1/2$ of the *linear* polynomial that takes the values $x_{j,k}$ at $t = 0$ and $x_{j,k+1}$ at $t = 1$; in other words,

$$(S_2\boldsymbol{x})_{2k+1} := x_{j+1,2k+1} = \frac{1}{2}(x_{j,k} + x_{j,k+1}) \ .$$

This is hence the scheme that was used in the basic normal approximation in Sect. 2. The construction strategy generalizes to higher order and in the four-point scheme, $x_{j+1,2k+1}$ is given the value at $t = 1/2$ of the *cubic* that takes the values $x_{j,k-1}$, $x_{j,k}$, $x_{j,k+1}$ and $x_{j,k+2}$ at $t = -1, 0, 1, 2$ respectively, leading to

$$(S_4\boldsymbol{x})_{2k+1} := x_{j+1,2k+1} = \frac{9}{16}(x_{j,k} + x_{j,k+1}) - \frac{1}{16}(x_{j,k-1} + x_{j,k+2}) \ .$$

In general, the 2ℓ-point scheme gives $x_{j+1,2k+1}$ the value at $t = 1/2$ of the $(2\ell - 1)$-degree polynomial that takes the values $x_{j,k+m}$ at $t = m$, where $m = -\ell + 1, \ldots, \ell$. Since these are all interpolating schemes, we have of course $(S_2\boldsymbol{x}_j)_{2k} = (S_4\boldsymbol{x}_j)_{2k} = (S_{2\ell}\boldsymbol{x}_j)_{2k} = x_{j,k}$.

We assume that the curve Γ is at least $C^2([0,1],\mathbb{R}^2)$. This means that it can always be broken up into adjacent finite length pieces, possibly overlapping, that can be well parameterized by the x-coordinate (with, say, $|dy/dx| \leq 2$) or by the y-coordinate (with, say, $|dx/dy| \leq 2$). For technical reasons we shall also assume that it can be broken up in this way in a *finite* number of pieces. Then the theorems below follow directly from the theorems in [3], since in each piece we would have either

$$|\Delta x_{j,k}| \leq |\Delta s_{j,k}| \leq \sqrt{5}|\Delta x_{j,k}| \quad \text{or} \quad |\Delta y_{j,k}| \leq |\Delta s_{j,k}| \leq \sqrt{5}|\Delta y_{j,k}| \ .$$

3.2 Convergence

When a general subdivision scheme is used to determine the predicted point there is no longer any guarantee that the normal cuts the curve in between the previous two points. This makes the convergence analysis more complicated. To have a proper parameterization, we need to ensure that all $\boldsymbol{s}_j$ sequences are increasing, i.e., $\Delta\boldsymbol{s}_j > 0$, given that the initial sequence $\boldsymbol{s}_0$ is increasing. In general there are very few subdivision schemes that preserve increasing sequences. In our case, the $\boldsymbol{s}_j$ sequences are obtained by a nonlinear perturbation of subdivision so the situation is even more complex. Fortunately, there are conditions on both the subdivision scheme and the initial sequence that guarantee that the $\boldsymbol{s}_j$ will be increasing. The following theorem introduces a non-uniformity measure $\mathcal{N}$ of a sequence which is the maximal ratio of the length of two neighboring intervals; it states that if the non-uniformity of the initial sequence is bounded and the subdivision scheme preserves this bound, the sequences $\boldsymbol{s}_j$ generated by the normal scheme will be increasing and converge exponentially, provided the initial sequence also resolves the curve well enough.

Theorem 5. *Let S be an interpolating subdivision scheme. Let the non-uniformity $\mathcal{N}(\boldsymbol{x})$ be defined by*

$$\mathcal{N}(\boldsymbol{x}) := \sup_k \max\left(\frac{|(\Delta\boldsymbol{x})_k|}{|(\Delta\boldsymbol{x})_{k+1}|}, \frac{|(\Delta\boldsymbol{x})_{k+1}|}{|(\Delta\boldsymbol{x})_k|}\right) . \tag{17}$$

Suppose there is an R such that for every strictly increasing $\boldsymbol{x}$ with $\mathcal{N}(\boldsymbol{x}) \leq R$, $S\boldsymbol{x}$ is strictly increasing as well, and satisfies $\mathcal{N}(S\boldsymbol{x}) \leq \mathcal{N}(\boldsymbol{x})$. Suppose $\boldsymbol{s}_0$ is strictly increasing, $\mathcal{N}(\boldsymbol{s}_0) < R$ and that $|\Delta\boldsymbol{s}_0|_\infty$ is sufficiently small. If $\Gamma \in C^2([0,1];\mathbb{R}^2)$, then the normal approximation converges,

$$\lim_{j\to\infty} \sup_{0\leq s\leq 1} ||\Gamma_j(s) - \Gamma(s)|| = 0 \ ,$$

where Γ_j is defined in (3)*. Moreover, $\boldsymbol{s}_j$ is strictly increasing for all j, with $\mathcal{N}(\boldsymbol{s}_j) \leq R$ for all j, and the $\boldsymbol{s}_j$ converge exponentially, i.e., there is a $\delta < 1$ so that*

$$|\Delta \boldsymbol{s}_j|_\infty \leq c\, \delta^j\, |\Delta \boldsymbol{s}_0|_\infty\,, \qquad \forall j\ .$$

Examples of subdivision schemes that meet the requirements in the theorem are, for instance, the first Lagrange interpolation schemes introduced above.

Remark 3. As was seen in Sect. 2, when $S = S_2$ the smallness assumptions on $|\Delta \boldsymbol{s}_0|_\infty$ and $\mathcal{N}(\boldsymbol{s}_0)$ are not necessary and exponential convergence follows when Γ is merely Lipschitz continuous.

3.3 Wavelet Decay, Regularity and Stability

To characterize the approximation quality and stability of the higher order normal scheme we need to introduce two additional subdivision concepts: the *order* of the operator and the *derived* operators.

The *order* of a subdivision scheme S is the largest degree for which it leaves the corresponding space of monic polynomials invariant. More precisely, let $\boldsymbol{k}$ be the sequence for which the k-th entry is k itself and denote the order of S by $\mathcal{P}$. Then $\mathcal{P}$ is the largest integer such that for all p-degree monic polynomials P with $0 \leq p < \mathcal{P}$, a p-degree monic polynomial Q exists so that $SP(\boldsymbol{k}) = Q(\boldsymbol{k}/2)$, where the polynomials are applied elementwise to generate new sequences. If S is interpolating, then $SP(\boldsymbol{k}) = P(\boldsymbol{k}/2)$. For example, S_2 is of order two and $S_2\boldsymbol{k} = \boldsymbol{k}/2$. In general the 2ℓ-point Lagrange interpolation scheme is of order 2ℓ. We always assume that $\mathcal{P}$ is at least one so that for a constant sequence $\mathbf{1}$ we have $S\mathbf{1} = \mathbf{1}$.

The *derived* subdivision schemes are defined as

$$S^{[0]} = S, \qquad S^{[p]} = 2\Delta S^{[p-1]}\Delta^{-1}, \qquad p > 0\ .$$

The significance of those schemes is that if the sequences $\{\boldsymbol{x}_j\}$ are generated by (16) then the divided differences of those sequences are generated by the corresponding derived schemes,

$$\boldsymbol{x}_{j+1}^{[p]} = S^{[p]}\boldsymbol{x}_j^{[p]},$$

where

$$\boldsymbol{x}_j^{[p]} = D_j^p \boldsymbol{x}_j\ , \qquad (D_j\boldsymbol{x})_k = 2^j(x_{k+1} - x_k)\ .$$

Note that $S^{[p]}$ is well-defined as long as $S^{[p-1]}$ has at least order one, and that the order of $S^{[p]}$ is one less than the order of $S^{[p-1]}$. Thus $S^{[p]}$ is defined for $p \leq \mathcal{P}$. The derived schemes can easily be written down explicitly for a given subdivision scheme. For example, the first derived scheme for the four-point scheme is

$$(S_4^{[1]}\boldsymbol{x})_{2k} = \frac{1}{8}(x_{k-1}+8x_k-x_{k+1})\ , \qquad (S_4^{[1]}\boldsymbol{x})_{2k+1} = \frac{1}{8}(-x_{k-1}+8x_k+x_{k+1})\ .$$

Note that derived schemes are typically not interpolating even if the base scheme is.

We are now ready to state a theorem that corresponds to Theorem 3, Theorem 2 and Theorem 4 in Sect. 2. We use the same definitions and notation. Furthermore, we introduce in (18) a growth rate μ that characterizes a derived operator $S^{[p]}$. The value of μ can in particular be chosen as $\log_2 |S^{[p]}|_\infty$. If the ℓ_∞-spectral radius σ_p of $S^{[p]}$ is strictly smaller, a more precise choice can be made: $\mu = \log_2 \sigma_p + \varepsilon$ for any $\varepsilon > 0$. (This follows from the well-known identity $\sigma_p = \lim_{j\to\infty} |S^{[p]^j}|_\infty^{1/j}$.)

The results in this higher order case depend on the smoothness of S, determined via $\mathcal{P}$, p and μ, and on the regularity r of the curve Γ. For sufficiently regular Γ and high order S, i.e. for r and $\mathcal{P}$ large enough, the regularity of $\mathtt{s}(t)$ is almost $C^{p-\mu}$ while the wavelet decay is one order higher, $2^{-j(p-\mu+1)}$. The best bound one can get from this theorem is thus obtained for that combination of p and μ where $p-\mu$ is maximal. This maximum need not be reached at $p = \mathcal{P}$.

Note that the exponential decay rate of $|\Delta \boldsymbol{s}_j|_\infty$ in (19) is, for example, established by Theorem 5.

Theorem 6. *Let S be the $\mathcal{P}$-th order interpolating subdivision scheme used in the normal scheme. Assume that there are positive real numbers C, μ and integer $p \le \mathcal{P}$ such that*

$$\left| S^{[p]^j} \right|_\infty \le C 2^{\mu j} , \qquad \forall j \ge 0 , \qquad \mu \le p-1 , \tag{18}$$

that there is a $\delta < 1$ so that

$$|\Delta \boldsymbol{s}_j|_\infty \le C\delta^j \tag{19}$$

and that $\Gamma \in C^r([0,1];\mathbb{R}^2)$ with $r \ge 2$ satisfies the hypothesis given in Sect. 3.1. *Then*

1. *Decay of wavelet coefficients.*
   For all $\varepsilon > 0$ there is a constant C_ε such that
   $$|\boldsymbol{w}_j|_\infty \le C_\varepsilon 2^{-j(Q-\varepsilon)} , \qquad Q := \min(p-\mu+1, r, \mathcal{P}) .$$
2. *Regularity of normal parameterization.*
   There is a limit function $\mathtt{s}(t)$ such that
   $$\lim_{j\to\infty} \sup_{0\le t\le 1} |\mathtt{s}_j(t) - \mathtt{s}(t)| = 0 .$$
   The limit $\mathtt{s}(t)$ belongs to $C^{Q'-\varepsilon}([0,1])$, for all $\varepsilon > 0$, and $Q' := \min(p-\mu, r)$.
3. *Stability.*
   If $p-\mu > 1$ when $p > 1$ there is a constant c depending on ν, but independent of j, E_γ and E_w such that
   $$\sup_k ||v_{j,k} - \tilde{v}_{j,k}|| \le c(E_\gamma + E_w) .$$

Remark 4. The regularity of the normal parameterization is the same regularity that we get for the limit function of the pure predictor subdivision scheme S when we use the same method of proof. If we take the very special case $\Gamma(s) = (s, 0)$ for all s, then the normal multiresolution scheme gives $\boldsymbol{s}_{j+1} = S\boldsymbol{s}_j$. In this case $\boldsymbol{w}_j = 0$, and we no longer have a curve approximation problem. However, we can define $\mathtt{s}_j(t)$ as before, and the convergence of $\mathtt{s}_j(t)$ still holds, as a special case of the theorem. Theorem 6 can thus be viewed as an extension, without loss in the strength of the estimates, of standard convergence results for linear subdivision, see e.g. [1, 2].

We conclude with some examples where we apply Theorem 6 to the first Lagrangian interpolating subdivision predictors S_{2l}. A numerical illustration is presented in Fig. 5. In the two-point case we simply have

$$\left| S_2^{[1]} \boldsymbol{x}_j^{[1]} \right|_\infty = \left| \boldsymbol{x}_j^{[1]} \right|_\infty .$$

Hence we can take $\mu = 0$ and $p = 1$. Since $\mathcal{P} = 2$ we get $Q = 2$ and $Q' = 1$ if $\Gamma \in C^2$. Then $\mathtt{s} \in C^{1-\varepsilon}$ and $|\boldsymbol{w}_j| \leq c2^{-(2-\varepsilon)j}$, agreeing with the results in Sect. 2. For the four-point operator we start from the estimate

$$\left| S_4^{[3]} \boldsymbol{x}_j^{[3]} \right|_\infty \leq 2 \left| \boldsymbol{x}_j^{[3]} \right|_\infty .$$

Here we thus take $\mu = 1$, $p = 3$ and with $\mathcal{P} = 4$ we get $Q = 3$, $Q' = 2$ when $\Gamma \in C^3$. Thus, $\mathtt{s}(t)$ is almost C^2 and $\boldsymbol{w}_j$ decay, almost, as 2^{-3j}. For S_6 and S_8 it is more difficult to get estimates of the optimal pair p, μ. Empirically the wavelet decay is consistent with what is obtained for pure subdivision, where the limit functions of the S_6-scheme belong to $C^{2.83}$ and S_8 generates $C^{3.55}$ functions; the corresponding wavelet decay in the normal scheme is 3.83 and 4.55 respectively.

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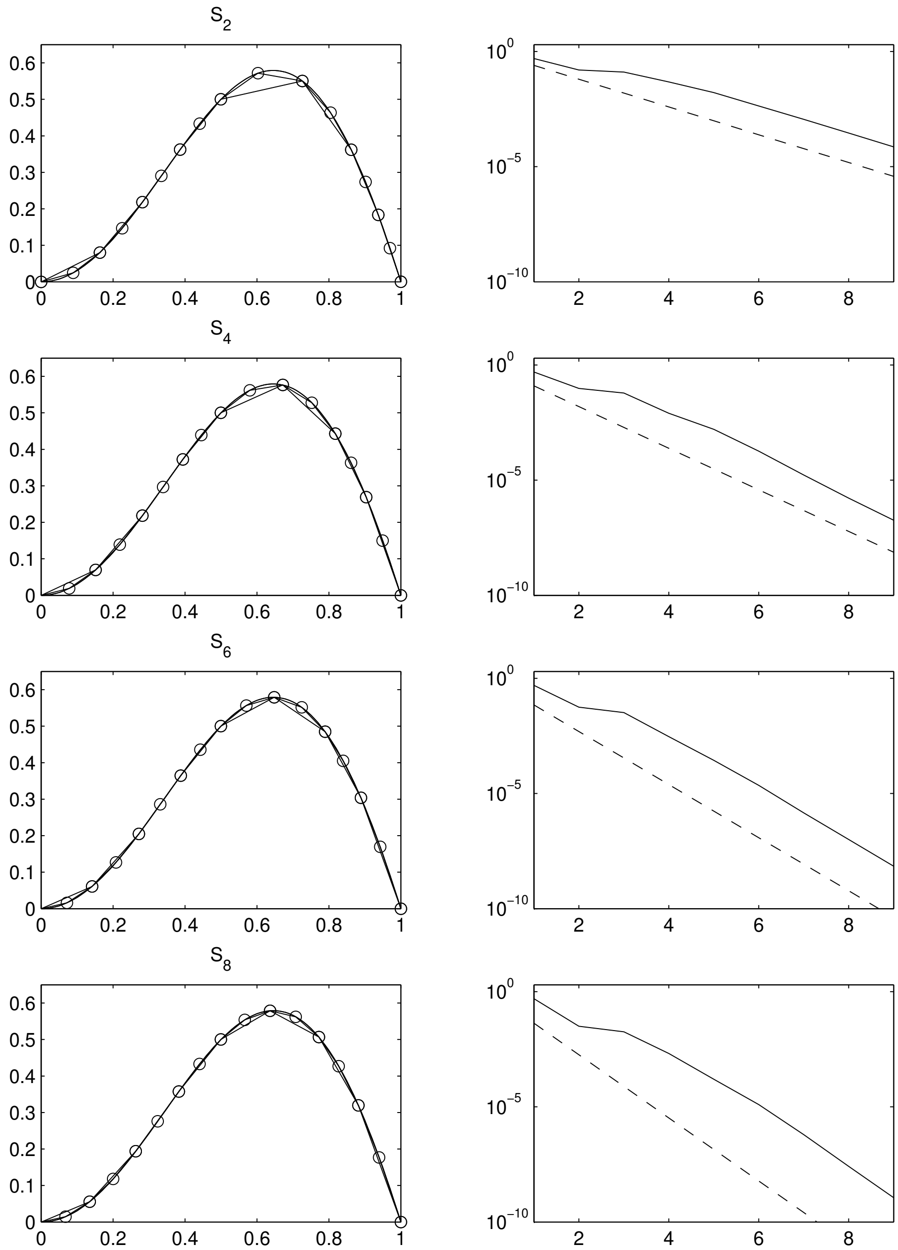

Fig. 5. Numerical study of the first Lagrange interpolation subdivision schemes, S_2 to S_8. Left column shows the normal multiresolution approximation at levels $j = 3, 4$. Right column shows the decay of wavelet coefficients as a function of level j (*solid line*) compared with the function 2^{-jQ} with $Q = 2, 3, 3.83, 4.55$ (*dashed line*)

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