Projection Generated Homogenization

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Abstract. Homogenization is a class of analytical techniques for approximating multiscale differential equations. The number of scales are reduced and the resulting equations are easier to analyze or numerically approximate. The classical homogenization technique is based on asymptotic expansions. We shall introduce a homogenization procedure based on projections of differential or discrete approximation operators onto coarser subspaces. The projection procedure is quite general and we give a presentation of a framework in Hilbert spaces. We apply the framework to a simple differential equation with oscillatory coefficients and to two discrete problems. One is wavelet based numerical homogenization and the other a multiscale finite element method.

1 Introduction

The existence of subgrid scale phenomena in the numerical simulation of differential equations poses considerable difficulties. With subgrid scale phenomena we mean those processes which could influence the solution on the computational grid, but which have length scales shorter than the grid size. One way to deal with these difficulties is to replace the multiscale differential equation by homogenized or effective equations, see e.g. [3]. The homogenized equations are easier to approximate numerically since they do not explicitly contain the smallest scales. Another possibility is to derive numerical methods that mainly operate on the coarser scales. These methods typically originate from a fine grid representation of the original problem. We shall consider two such methods, wavelet based numerical homogenization [1] and a multiscale finite element method, [14,15]. There are also other similar methods based on coarsening techniques from algebraic multigrid, see for example [18,20,21].

In this paper we present a framework for the derivation of both analytical and numerical homogenization. The objective is to find simplified operators that accurately reproduce the effect of subgrid scales. Let L_ε be a linear differential operator for which ε indicates small scales in the coefficients. The solution u_ε of the differential equation

$$L_\varepsilon u_\varepsilon = f_\varepsilon, \tag{1}$$

will typically inherit the small scales from the operator L_ε or the data f_ε. Consider for example the simple model problem

$$L_\varepsilon u_\varepsilon = -\frac{d}{dx}\left(g^\varepsilon(x)\frac{d}{dx}\right)u_\varepsilon(x) = f(x), \qquad 0 < x < 1, \tag{2}$$

$$u_\varepsilon(0) = u_\varepsilon(1) = 0. \tag{3}$$

The coefficient $g^\varepsilon(x)$ can for instance be oscillatory,

$$g^\varepsilon(x) = g(x/\varepsilon), \qquad g(y) \text{ 1-periodic.} \tag{4}$$

Our goal is to find an accurate and computationally efficient finite dimensional approximation of (1),

$$\bar{L}_{\varepsilon\Delta}\bar{u}_{\varepsilon\Delta} = \bar{f}_{\varepsilon\Delta}. \tag{5}$$

Difficulties originate from the small scales or high frequencies in (1). If ε denotes a typical wave length in u_ε and h a typical element size or step size in (5), then h must be substantially smaller than ε in order to resolve the ε-scale in the numerical approximation. This can be costly if ε is small compared to the overall size of the computational domain. Ideally the discretized operator in (5) should be designed such that $\bar{u}_{\varepsilon\Delta}$ is a good approximation of u_ε even if h is not small compared to ε. This goal resembles that of classical analytical homogenization.

Homogenization based on asymptotic derivation is a well established analytical technique. Let us here summarize a few simple results. Consider a set of operators L_ε in (1), indexed by the small parameter ε. The homogenization problem is to find the limit solution $\bar{u}$ and the *homogenized operator* $\bar{L}$ defined by

$$L_\varepsilon u_\varepsilon = f, \qquad \lim_{\varepsilon\to 0} u_\varepsilon = \bar{u}, \qquad \bar{L}\bar{u} = f. \tag{6}$$

In certain cases the convergence above and existence of the homogenized operator can be proved, [3].

For simple model problems, with coefficients that are periodic on the fine scale, exact closed form solutions can be obtained. For instance, with $g(y)$ positive, 1-periodic and bounded away from zero, we have for the one-dimensional example (2),

$$L_\varepsilon = -\frac{d}{dx}\left(g(x/\varepsilon)\frac{d}{dx}\right), \qquad \bar{L} = -\tilde{g}\frac{d^2}{dx^2}, \qquad \tilde{g} = \left(\int_0^1 \frac{dy}{g(y)}\right)^{-1}. \tag{7}$$

With the same $\tilde{g}$ we get for the hyperbolic operators,

$$L_\varepsilon = \frac{\partial}{\partial t} + g(x/\varepsilon)\frac{\partial}{\partial x}, \qquad \bar{L} = \frac{\partial}{\partial t} + \tilde{g}\frac{\partial}{\partial x}. \tag{8}$$

In higher dimensions, the solution to (6) is a little more complicated, although for some model problems the structure of the homogenized operator can still be written down, as long as the coefficients are periodic or stochastic. In the d-dimensional elliptic case, let $G(y) : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ be uniformly elliptic and 1-periodic in each of its arguments. Let I_d denote the d-dimensional unit square. It can then be shown, [3], that

$$L_\varepsilon = -\nabla\cdot\left(G\left(\frac{x}{\varepsilon}\right)\nabla\right), \quad \bar{L} = -\nabla\cdot(\tilde{g}\nabla), \quad \tilde{g} = \int_{I_d} G(y) - G(y)\frac{d\chi(y)}{dy}dy, \tag{9}$$

where $d\chi/dy$ is the Jacobian of the function $\chi(y) : \mathbb{R}^d \mapsto \mathbb{R}^d$, given by solving the so called cell problem,

$$\nabla\cdot G(y)\frac{d\chi(y)}{dy} = \nabla\cdot G(y), \qquad y \in I_d, \tag{10}$$

with periodic boundary conditions for χ. Note that $\tilde{g}$ is a constant matrix.

The solution u_ε will be of the form

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \mathcal{O}(\varepsilon^3). \tag{11}$$

This is also the asymptotic ansatz, which is often used in the derivation of $\bar{L}$.

In the following section we introduce the projection technique for homogenzation in a Hilbert space setting. We show, in Sect. 3, that the zero ε-limit of the projection generated operator is the same as the effective operator $\bar{L}$ in classical homogenization for a model differential equation. In Sect. 4 and Sect. 5 we apply the technique to numerical finite dimensional operators.

2 Projection Framework

In this section we describe the following approach to homogenization. Consider an equation $Lu = f$ where L is a linear operator, f a right hand side and u a solution that contains fine scales. Let P be a projection operator onto a subspace where the fine scales in the original solution do not exist. Our objective is to find the (projection generated) *homogenized* operator $\bar{L}$ such that $\bar{L}Pu = f$ for all f such that $Pf = f$. (When $Pf \neq f$ we also need to find the homogenized right hand side $\bar{f}$.) We confine ourselves to the case of Hilbert spaces.

Let X be a Hilbert space of functions, typically a Sobolev space. Let $X_0 \subset X$ be a closed subspace representing the coarse part of the functions, and denote by P_x the orthogonal (and symmetric) projection operator in X onto X_0. Let the spaces X_0 and $X_0^\perp$ inherit the innerproduct and norm of X, so that $\|u\|_X = \|u\|_{X_0}$ and $(u,v)_X = (u,v)_{X_0}$ when $u, v \in X_0$, and similar for $X_0^\perp$. In addition, set $Q_x = I_x - P_x$ where I_x is the identity operator in X, and introduce the unitary operator $\mathcal{W}_x$ on X defined by

$$\mathcal{W}_x : X \mapsto X_0^\perp \times X_0, \qquad \mathcal{W}_x u = \begin{pmatrix} Q_x u \\ P_x u \end{pmatrix}. \tag{12}$$

In the same way, define the corresponding operators P_y, Q_y and $\mathcal{W}_y$ for another Hilbert space Y with subspace Y_0. Let $\mathcal{L}(X,Y)$ be the set of bounded linear maps from X to Y. For an operator $L \in \mathcal{L}(X,Y)$, we have

$$\mathcal{W}_y L \mathcal{W}_x^* \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{W}_y L (P_x + Q_x)(u+v) = \begin{pmatrix} Q_y L(P_x + Q_x)(u+v) \\ P_y L(P_x+Q_x)(u+v) \end{pmatrix}$$
$$= \begin{pmatrix} Q_y L(P_x v + Q_x u) \\ P_y L(P_x v + Q_x u) \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{13}$$

where

$$\begin{aligned} A &= Q_y L Q_x \in \mathcal{L}(X_0^\perp, Y_0^\perp), & B &= Q_y L P_x \in \mathcal{L}(X_0, Y_0^\perp), \\ C &= P_y L Q_x \in \mathcal{L}(X_0^\perp, Y_0), & D &= P_y L P_x \in \mathcal{L}(X_0, Y_0). \end{aligned} \tag{14}$$

When A is invertible the following definition can be stated:

Definition 1. Suppose $L \in \mathcal{L}(X,Y)$ and $f \in Y$. When A in (13, 14) is invertible (one-to-one and onto), we define the *homogenized* operator $\bar{L} : X_0 \mapsto Y_0$ as the Schur complement with respect to the decomposition in (13),

$$\bar{L} = D - CA^{-1}B, \tag{15}$$

and the homogenized right hand side as

$$\bar{f} = P_y f - CA^{-1} Q_y f. \tag{16}$$

We will write $\bar{L}_{X,X_0}$ and $\bar{f}_{X,X_0}$ when there is a need to show explicitly between which spaces the homogenization step is made. From Definition 1 we immediately have

Lemma 1. *Suppose $Lu = f$, where $L \in \mathcal{L}(X,Y)$, $u \in X$ and $f \in Y$. If A^{-1} exists,*

$$\bar{L} P_x u = \bar{f}. \tag{17}$$

Proof. Since $Lu = f$ we get

$$\mathcal{W}_y L \mathcal{W}_x^* \mathcal{W}_x u = \mathcal{W}_y f \qquad \Rightarrow \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Q_x u \\ P_x u \end{pmatrix} = \begin{pmatrix} Q_y f \\ P_y f \end{pmatrix}. \tag{18}$$

Moreover, since A is invertible, this system can be reduced with Gaussian elimination. It yields (17).

Note that $\bar{L}$ and $\bar{f}$ do not exist for all elements of $\mathcal{L}(X,Y)$ since A^{-1} may not exist. In particular, $X_0^\perp$ and $Y_0^\perp$ must have the same cardinality. For some classes of operators A is however indeed invertible. Before we show one such case, let us comment on how to define projections on the dual of X. We denote the duality product between X^* and X by $\langle \cdot, \cdot \rangle$, so that e.g. $\langle u, \cdot \rangle$ represents the bounded linear functional $u \in X^*$. It is clear that a bounded linear functional $\langle f, \cdot \rangle$ on $X_0 \subset X$, can be extended to a bounded linear functional on all of X, simply by adding the projection operator, $\langle f, P_x \cdot \rangle$. In fact, we shall identify X_0^* with a subspace of X^*, such that the projection P_{x^*} in X^* on X_0^* satisfies

$$\langle P_{x^*} f, v \rangle \equiv \langle f, P_x v \rangle, \qquad \forall v \in X. \tag{19}$$

Another way of putting this, is to say that $X_0^* = \mathcal{A}^{-1} X_0$ and $P_{x^*} = \mathcal{A}^{-1} P_x \mathcal{A}$, where $\mathcal{A} : X^* \mapsto X$ is the isomorphic Riesz mapping defined by $\langle u, v \rangle = (\mathcal{A}u, v)_X$ for all $v \in X$.

Proposition 1. *Suppose the operator $L \in \mathcal{L}(X, X^*)$ is given by the weak form*

$$Lu = f \qquad \Leftrightarrow \qquad a[u, v] = \langle f, v \rangle, \qquad \forall v \in X, \tag{20}$$

where $a[\cdot,\cdot] : X \times X \mapsto \mathbb{R}$ is a bilinear form. Let P_{x^*} be defined as in (19) and suppose that for some positive constants c_1 and c_2,

$$\begin{aligned} a[u, v] \qquad &\le c_1 \, \|u\|_X \, \|v\|_X \,, \qquad &\forall u, v \in X, \\ a[Q_x u, Q_x u] &\ge c_2 \, \|Q_x u\|_X^2 \,, \qquad &\forall u \in X, \end{aligned} \tag{21}$$

Then L is well-defined and $A : X_0^\perp \mapsto (X_0^\perp)^*$ is one-to-one and onto.

Proof. By the first inequality in (21), the bilinear form $a[u, v]$ is a bounded linear functional on X for each fixed u, so L is well-defined. To show that A is invertible, we need to show that for every $f \in (X_0^\perp)^*$ there is a unique $u \in X_0^\perp$ such that

$$a[u, v] = \langle f, v \rangle, \qquad \forall v \in X_0^\perp. \tag{22}$$

This follows from the Lax–Milgram lemma applied to $X_0^\perp$ if (21) holds.

Some properties kept by the homogenized operator are surjectivity, symmetry and ellipticity, as seen by this proposition. We show it in the case when $X = Y$ and $X_0 = Y_0$.

Proposition 2. *Let $L \in \mathcal{L}(X, X)$ be such that $\bar{L}$ exists. If L is onto, $\bar{L}$ is onto. If L is self-adjoint, $\bar{L}$ is self-adjoint. If L is self-adjoint and also*

$$c_1 ||u||_X^2 \le (Lu, u) \le c_2 ||u||_X^2, \qquad \forall u \in X. \tag{23}$$

for some positive constants c_1, c_2. Then, for the same constants,

$$c_1 ||v||_{X_0}^2 \le \left(\bar{L}v, v\right) \le (Dv, v) \le c_2 ||v||_{X_0}^2, \qquad \forall v \in X_0. \tag{24}$$

Proof. Suppose L is onto. For any $f \in X_0 \subset X$ there exists a $u \in X$ such that $Lu = f$. By Lemma 1 then $\bar{L}Pu = \bar{f}$. From (16) we see that $\bar{f} = f$, so $\bar{L}$ is also onto. Let L be self-adjoint. Then $\mathcal{W}_x L \mathcal{W}_x^*$ is self-adjoint and therefore $A = A^*$, $D = D^*$ and $C = B^*$. When A is invertible this implies that $(A^{-1})^* = A^{-1}$. Hence, $\bar{L}^* = D^* - B^*(A^{-1})^* C^* = D - CA^{-1}B = \bar{L}$.

Next, assume (23). Take $v \in X_0$ and set $u = -A^{-1}Bv \in X_0^{\perp}$. Since $\mathcal{W}_x$ is unitary,

$$c_1||v||_{X_0}^2 \le c_1||v+u||_X^2 \le (L(u+v),(u+v)) = \left(\begin{pmatrix} A & B \\ C & D\end{pmatrix}\begin{pmatrix} u \\ v\end{pmatrix}, \begin{pmatrix} u \\ v\end{pmatrix}\right)$$
$$= \left(\begin{pmatrix} 0 \\ (-CA^{-1}B+D)v\end{pmatrix}, \begin{pmatrix} -A^{-1}Bv \\ v\end{pmatrix}\right) = \left(\bar{L}v, v\right). \tag{25}$$

This proves the left inequality in (24). Since $C = B^*$ and A^{-1} is positive by the positivity of L, we have for the right inequality

$$\begin{aligned}\left(\bar{L}v, v\right) &= \left((D - CA^{-1}B)v, v\right) = (Dv, v) - \left(A^{-1}Bv, Bv\right) \\ &\le (Dv, v) = (Lv, v) \le c_2||v||_X^2 = c_2||v||_{X_0}^2.\end{aligned} \tag{26}$$

We conclude this section by showing that the homogenization is "independent of the path," again for $X = Y$ and $X_0 = Y_0$.

Proposition 3. *Let X_0 and X_1 be two closed subspaces of X, nested such that $X_0 \subset X_1 \subset X$. Suppose $L \in \mathcal{L}(X,X)$ is onto and that $\bar{L}_{X,X_0}$, $\bar{L}_{X,X_1}$ and $\overline{\bar{L}_{X,X_1}}_{X_1,X_0}$ all exist. If $\bar{L}_{X,X_0}$ is one-to-one, then*

$$\bar{L}_{X,X_0} = \overline{\bar{L}_{X,X_1}}_{X_1,X_0}, \tag{27}$$
$$\bar{f}_{X,X_0} = \overline{\bar{f}_{X,X_1}}_{X_1,X_0}. \tag{28}$$

Proof. Let P_j be the projection onto X_j. Take $u \in X_0$ and let $f = \bar{L}_{X,X_0}u \in X_0$. Since L is onto, there is a $w \in X$ such that $Lw = f$. By Lemma 1, we have

$$\bar{L}_{X,X_0}P_0w = \bar{f}_{X,X_0}, \qquad \bar{L}_{X,X_1}P_1w = \bar{f}_{X,X_1}. \tag{29}$$

Since $f \in X_0$ we must have $\bar{f}_{X,X_0} = \bar{f}_{X,X_1} = f$, by (16). Another application of Lemma 1, then gives

$$\overline{\bar{L}_{X,X_1}}_{X_1,X_0}P_0P_1w = \bar{f}_{X_1,X_0}. \tag{30}$$

Again, $\bar{f}_{X_1,X_0} = f$ and $P_0P_1w = P_0w$ since $X_0 \subset X_1$. Hence,

$$\bar{L}_{X,X_0}P_0w = \overline{\bar{L}_{X,X_1}}_{X_1,X_0}P_0w = f = \bar{L}_{X,X_0}u. \tag{31}$$

But, $\bar{L}_{X,X_0}$ is one-to-one so $P_0w = u$ and since u was arbitrary, (27) follows. Finally, letting f be an arbitrary element of X, there exists a u such that $Lu = f$ since L is onto and by Lemma 1

$$\bar{f}_{X,X_0} = \bar{L}_{X,X_0}Pu = \overline{\bar{L}_{X,X_1}}_{X_1,X_0}Pu = \overline{\bar{f}_{X,X_1}}_{X_1,X_0}. \tag{32}$$

This shows (28).

3 Relationship Between the Homogenization Approaches

In this section we analyze how the projection generated homogenized operator relates to the classical homogenized operator. In the elliptic case, there is a striking similarity between the Schur complement in Definition 1 and the classical homogenized operator in (9), repeated here for convenience,

$$PLP - PLQ(QLQ)^{-1}QLP, \tag{33}$$

$$\nabla \left(\int_{I_d} G(y)dy \right) \nabla - \nabla \left(\int_{I_d} G(y) \frac{d\chi(y)}{dy} dy \right) \nabla. \tag{34}$$

Both are written as the average of the original operator minus a correction term, which is computed in a similar way for both operators. For the analytical case, a local elliptic cell problem is solved to get $G\partial_y \chi$, while in the projection case, a positive operator $A = QLQ$ defined on a subspace is inverted to obtain $LQA^{-1}B$. The average over the terms is obtained by integration in the analytical case, and by applying P in the projection case.

The relationship can be made more precise, and we will illustrate it by considering the case set in the one-dimensional torus $\mathbb{T}$, which means we are considering 1-periodic functions. The coarse space will be given by the lowest Fourier modes. Similar results in one dimension have been shown for wavelet bases, see [13,22].

Consider (2) with $g^\varepsilon = g(x/\varepsilon)$ where $g \in L^\infty(\mathbb{T})$ is 1-periodic. Let $X = H^1(\mathbb{T})$ with the dual $X^* = H^{-1}(\mathbb{T})$. The duality product between H^{-1} and H^1 will be written $\langle \cdot, \cdot \rangle$ and the L^2 innerproduct will be denoted by $(\cdot, \cdot)$. Define the operator $L_\varepsilon \in \mathcal{L}(X, X^*)$ by the weak form of the equations, such that $L_\varepsilon u_\varepsilon = f$ is equivalent to

$$a[u, v] := (g(x/\varepsilon)\partial_x u_\varepsilon, \partial_x v) = \langle f, v \rangle, \qquad \forall v \in H^1(\mathbb{T}). \tag{35}$$

We want to compare this with the classical homogenized operator,

$$\bar{L}\bar{u} = f \qquad \Leftrightarrow \qquad \tilde{g}\,(\partial_x \bar{u}, \partial_x v) = \langle f, v \rangle, \qquad \forall v \in H^1(\mathbb{T}), \tag{36}$$

where $\tilde{g}$ was given in (7) and (9),

$$\tilde{g} = \left(\int_0^1 \frac{dy}{g(y)} \right)^{-1} = \int_0^1 g(y) - g(y)\chi_y(y)dy \tag{37}$$

with χ being the solution to (10) in one dimension.

We start by introducing a family of ε-dependent spaces $\{X_0^\varepsilon\}$, which are assumed to be closed subspaces of $H^1(\mathbb{T})$. We denote the orthogonal projection in $H^1(\mathbb{T})$ onto X_0^ε by P_ε. On the dual side we use the subspaces $(X_0^\varepsilon)^*$ as discussed in (19). Their corresponding projections will be denoted by P_{ε^*}. We index the remaining operators by ε in the obvious way, setting $Q_\varepsilon = I - P_\varepsilon$ and

$$\bar{L}_\varepsilon = D_\varepsilon - C_\varepsilon A_\varepsilon^{-1} B_\varepsilon. \tag{38}$$

Our strategy is now to explicitly construct (38) corresponding to (35) for fixed ε and to prove that

$$\lim_{\varepsilon\to 0} \bar{L}_\varepsilon P_\varepsilon u = \bar{L}u \qquad \forall u \in H^1(\mathbb{T}), \tag{39}$$

in H^{-1}. If the spaces $\{X_0^\varepsilon\}$ have some additional structural properties, (40), and if $\bar{L}_\varepsilon$ is elliptic uniformly in ε, (43), the following theorem shows that the solution we get from solving $\bar{L}_\varepsilon \bar{u}_\varepsilon = f$ tends to the homogenized solution $\bar{u}$ strongly in H^1.

Theorem 1. *Let $\{X_0^\varepsilon\}$ be any ε-dependent family of closed subspaces of $H^1(\mathbb{T})$ such that, for all $u \in H_0^1(0,1)$, the corresponding projection operators $\{P_\varepsilon\}$ satisfy*

$$\lim_{\varepsilon\to 0} ||P_\varepsilon u - u||_{H^1} = 0, \qquad P_\varepsilon \mathbf{1} = \mathbf{1}, \tag{40}$$

where $\mathbf{1}$ is the constant function. Let $\bar{L}_\varepsilon$ be the operator in (38) corresponding to (35) and $\{X_0^\varepsilon\}$. Finally, let $\bar{L}$ be the classical homogenized operator (36). If

$$\bar{L}\bar{u} = f = \bar{L}_\varepsilon \bar{u}_\varepsilon, \qquad \bar{u} \in H_0^1(0,1), \quad \bar{u}_\varepsilon \in H_0^1(0,1) \cap X_0^\varepsilon, \tag{41}$$

$$\lim_{\varepsilon\to 0} \left\| (\bar{L}_\varepsilon P_\varepsilon - \bar{L})\bar{u} \right\|_{H^{-1}} = 0, \tag{42}$$

$$\inf_\varepsilon \left\langle \bar{L}_\varepsilon u, u \right\rangle \geq \delta \, ||u||_{H^1}^2, \qquad \forall u \in H_0^1 \cap X_0^\varepsilon, \tag{43}$$

then

$$\lim_{\varepsilon\to 0} ||\bar{u} - \bar{u}_\varepsilon||_{H^1} = 0. \tag{44}$$

Proof. By (43), for fixed $\varepsilon > 0$,

$$||u||_{H^1}^2 \leq \frac{1}{\delta} \left\langle \bar{L}_\varepsilon u, u \right\rangle \leq \frac{1}{\delta} ||\bar{L}_\varepsilon u||_{H^{-1}} ||u||_{H^1}, \qquad \forall u \in H_0^1 \cap X_0^\varepsilon. \tag{45}$$

So,

$$||u||_{H^1} \leq \frac{1}{\delta} ||\bar{L}_\varepsilon u||_{H^{-1}}, \qquad \forall u \in H_0^1 \cap X_0^\varepsilon. \tag{46}$$

Let P'_ε be the orthogonal projection onto $H_0^1(0,1) \cap X_0^\varepsilon$, defined by $P'_\varepsilon u = P_\varepsilon u - (P_\varepsilon u)(0)$. Then for all $u \in H^1(\mathbb{T})$, by (40),

$$\bar{L}_\varepsilon (P_\varepsilon - P'_\varepsilon) u = (P_\varepsilon u)(0) \bar{L}_\varepsilon \mathbf{1} = (P_\varepsilon u)(0) (P_\varepsilon L_\varepsilon - C_\varepsilon A_\varepsilon^{-1} P_\varepsilon L_\varepsilon) \mathbf{1} = 0, \tag{47}$$

since $L_\varepsilon \mathbf{1} = 0$. Using (46, 47, 41) and the fact that $\bar{u}_\varepsilon \in H_0^1 \cap X_0^\varepsilon$, we obtain

$$\begin{aligned}
||\bar{u} - \bar{u}_\varepsilon||_{H^1} &\le ||P'_\varepsilon \bar{u} - \bar{u}||_{H^1} + ||P'_\varepsilon(\bar{u} - \bar{u}_\varepsilon)||_{H^1} \\
&\le ||P'_\varepsilon \bar{u} - \bar{u}||_{H^1} + \frac{1}{\delta}||\bar{L}_\varepsilon P'_\varepsilon(\bar{u} - \bar{u}_\varepsilon)||_{H^{-1}} \\
&= ||P'_\varepsilon \bar{u} - \bar{u}||_{H^1} + \frac{1}{\delta}||(\bar{L}_\varepsilon P_\varepsilon - \bar{L})\bar{u}||_{H^{-1}} \\
&\le ||P_\varepsilon \bar{u} - \bar{u}||_{H^1} + |(P_\varepsilon \bar{u})(0)| + \frac{1}{\delta}||(\bar{L}_\varepsilon P_\varepsilon - \bar{L})\bar{u}||_{H^{-1}}. \quad (48)
\end{aligned}$$

By (40, 42) and the Sobolev inequality $|u|_\infty \le ||u||_{H^1}$, the remaining terms vanish when $\varepsilon \to 0$.

In the remainder of the section we let $\{X_0^\varepsilon\}$ be the spaces of band limited functions, defined by

$$X_0^\varepsilon = \left\{ u = \sum_{|k|<1/2\varepsilon} a_k e^{2\pi i k x},\ a_k \in \mathbb{C} \right\}. \quad (49)$$

Those finite-dimensional spaces are closed subspaces of both $L^2(\mathbb{T})$ and $H^1(\mathbb{T})$. The corresponding orthogonal projections are

$$P_\varepsilon u = \sum_{|k|<1/2\varepsilon} \hat{u}_k e^{2\pi i k x}, \qquad \hat{u}_k = \int_0^1 u(x) e^{-2\pi i k x} dx, \quad (50)$$

where $\hat{u}_k$ is the k^{th} Fourier coefficient of $u(x)$. For this choice of spaces we will actually have that $\bar{L}_\varepsilon = P_{\varepsilon^*} \bar{L} P_\varepsilon$.

Before continuing, we introduce some additional notation. We let $y = x/\varepsilon$. A y-dependent function seen as a function of x we will indicate by a superscripted epsilon, e.g. $g(x/\varepsilon) = g^\varepsilon(x)$ as before. A bar above the function denotes its mean value,

$$\bar{g} = \int_0^1 g(x)dx = \hat{g}_0, \quad (51)$$

where $\hat{g}_0$ is the zeroth Fourier coefficient of g.

We now begin by showing some properties of the projections.

Lemma 2. *The projections P_ε and Q_ε commutes with differentiation,*

$$\partial_x P_\varepsilon u = P_\varepsilon \partial_x u, \qquad \partial_x Q_\varepsilon u = Q_\varepsilon \partial_x u, \qquad \forall u \in H^1(\mathbb{T}). \quad (52)$$

For $g \in L^2(\mathbb{T})$ and $\varepsilon = 1/n$ with $n \in \mathbb{Z}^+$,

$$P_\varepsilon g^\varepsilon P_\varepsilon u = \bar{g} P_\varepsilon u, \qquad \forall u \in L^2(\mathbb{T}). \quad (53)$$

For $u \in L^2(\mathbb{T})$,

$$\overline{P_\varepsilon u} = \overline{u}. \quad (54)$$

Finally,

$$||\partial_x Q_\varepsilon u||_{L^2} \geq \frac{\pi}{\varepsilon}||Q_\varepsilon u||_{L^2}. \tag{55}$$

Proof. Take $u \in H^1(\mathbb{T})$ and let $\{\hat{u}_k\}$ be its Fourier coefficients. Then

$$\begin{aligned}\partial_x P_\varepsilon u &= \partial_x \sum_{|k|<1/2\varepsilon} \hat{u}_k e^{2\pi ikx} = \sum_{|k|<1/2\varepsilon} 2\pi ik\hat{u}_k e^{2\pi ikx} \\ &= P_\varepsilon \sum_k 2\pi ik\hat{u}_k e^{2\pi ikx} = P_\varepsilon \partial_x u. \end{aligned} \tag{56}$$

Since $Q_\varepsilon = I - P_\varepsilon$, we have shown (52). Moreover,

$$P_\varepsilon g^\varepsilon P_\varepsilon u = P_\varepsilon \sum_\ell \sum_{|k|<n/2} \hat{g}_\ell \hat{u}_k e^{2\pi i(k+n\ell)x} = \hat{g}_0 \sum_{|k|<n/2} \hat{u}_k e^{2\pi ikx} = \bar{g} P_\varepsilon u, \tag{57}$$

since $|k+n\ell| \geq n/2$ when $\ell \neq 0$ and $|k| < n/2$. This proves (53). Next, for $u \in L^2$,

$$\overline{P_\varepsilon u} = (P_\varepsilon u, 1)_{L^2} = \sum_{|k|<1/2\varepsilon} \hat{u}_k \left(e^{2\pi ikx}, 1\right) = \hat{u}_0 = \overline{u}, \tag{58}$$

showing (54). Finally, (55) follows from

$$\|\partial_x Q_\varepsilon u\|_{L^2}^2 = \sum_{|k|\geq 1/2\varepsilon} 4\pi^2 k^2 \hat{u}_k^2 \geq \frac{\pi^2}{\varepsilon^2} \sum_{|k|\geq 1/2\varepsilon} \hat{u}_k^2 = \frac{\pi^2}{\varepsilon^2} \|Q_\varepsilon u\|_{L^2}^2. \tag{59}$$

By (55), we now have

$$a[u,v] \leq \|g\|_{L^\infty} \|u\|_{H^1} \|v\|_{H^1}, \tag{60}$$

$$a[Q_\varepsilon u, Q_\varepsilon u] \geq \operatorname{ess\,inf}_{x\in\mathbb{T}} |g(x)| \ \ \|\partial_x Q_\varepsilon u\|_{L^2}^2 \geq C \operatorname{ess\,inf}_{x\in\mathbb{T}} |g(x)| \, \|Q_\varepsilon u\|_{H^1}^2, \tag{61}$$

so Proposition 1 shows that A_ε is invertible and $\bar{L}_\varepsilon$ exists for all ε. We can then show

Theorem 2. *The projection generated homogenized operator $\bar{L}_\varepsilon$ corresponding to (35) and $\{X_0^\varepsilon\}$ as in (49), satisfies*

$$\bar{L}_\varepsilon = D_\varepsilon - C_\varepsilon A_\varepsilon^{-1} B_\varepsilon = P_{\varepsilon^*} \bar{L} P_\varepsilon, \tag{62}$$

if $\varepsilon = 1/n$ and $n \in \mathbb{Z}^+$. Then we also have the convergence,

$$\lim_{\varepsilon\to 0} \left\|(\bar{L}_\varepsilon P_\varepsilon - \bar{L})u\right\|_{H^{-1}} = 0, \qquad \forall u \in H^1(\mathbb{T}). \tag{63}$$

Proof. By (52, 53) and (19) we have for D_ε,

$$\begin{aligned}\langle D_\varepsilon u, v\rangle = \langle P_{\varepsilon^*} L_\varepsilon P_\varepsilon u, v\rangle &= \langle L_\varepsilon P_\varepsilon u, P_\varepsilon v\rangle = (g^\varepsilon \partial_x P_\varepsilon u, \partial_x P_\varepsilon v) \\ &= (g^\varepsilon P_\varepsilon \partial_x u, P_\varepsilon \partial_x v) = (P_\varepsilon g^\varepsilon P_\varepsilon \partial_x u, \partial_x P_\varepsilon v) \\ &= \bar{g}\left(\partial_x P_\varepsilon u, \partial_x P_\varepsilon v\right), \end{aligned} \tag{64}$$

when $u, v \in H^1(\mathbb{T})$. For the second part of the operator, the equation $A_\varepsilon w = B_\varepsilon u$ needs to be solved. Then $C_\varepsilon A_\varepsilon^{-1} B_\varepsilon u$ is given by $C_\varepsilon w$. In this case we can directly write down the expression for w, namely

$$w(x) = \int_0^x \chi_y \left(\frac{x'}{\varepsilon}\right) \partial_x P_\varepsilon u(x') dx'. \tag{65}$$

where χ is the solution to the cell problem (10). It is well-known that $\chi \in H^1(\mathbb{T})$. By also using (52, 53, 54) and the fact that $P_\varepsilon u \in L^\infty(\mathbb{T})$, we have

$$\|w\|_{L^2} \le |w|_\infty \le \left(|\chi_y^\varepsilon|, |\partial_x P_\varepsilon u|\right) \le C \|\chi\|_{H^1} \|P_\varepsilon u\|_{H^1}, \tag{66}$$

$$\|\partial_x w\|_{L^2} \le |\partial_x P_\varepsilon u|_\infty \left\|\chi_y^\varepsilon\right\|_{L^2} \le C |\partial_x P_\varepsilon u|_\infty \|\chi\|_{H^1}, \tag{67}$$

$$w(1) = \overline{\chi_y^\varepsilon \partial_x P_\varepsilon u} = \overline{P_\varepsilon \chi_y^\varepsilon P_\varepsilon \partial_x u} = 0 = w(0). \tag{68}$$

Therefore $w \in H^1(\mathbb{T})$. Moreover, by (52) and (53)

$$\partial_x Q_\varepsilon w = \partial_x w - P_\varepsilon \partial_x w = \partial_x w - P_\varepsilon \chi_y^\varepsilon P_\varepsilon \partial_x u = \partial_x w. \tag{69}$$

Now, from (10) it follows that $g^\varepsilon \chi_y^\varepsilon = g^\varepsilon + c_0$ a.e., where c_0 is a constant. Together with (69) this shows that w indeed satisfies $A_\varepsilon w = B_\varepsilon u$,

$$\begin{aligned} \langle A_\varepsilon w, v\rangle &= \langle L_\varepsilon Q_\varepsilon w, Q_\varepsilon v\rangle = (g^\varepsilon \partial_x Q_\varepsilon w, \partial_x Q_\varepsilon v) = \left(g^\varepsilon \chi_y^\varepsilon \partial_x P_\varepsilon u, \partial_x Q_\varepsilon v\right) \\ &= (g^\varepsilon \partial_x P_\varepsilon u, \partial_x Q_\varepsilon v) + c_0 (\partial_x P_\varepsilon u, \partial_x Q_\varepsilon v) \\ &= \langle L_\varepsilon P_\varepsilon u, Q_\varepsilon v\rangle + c_0 (P_\varepsilon \partial_x u, Q_\varepsilon \partial_x v) = \langle B_\varepsilon u, v\rangle . \end{aligned} \tag{70}$$

Then, finally,

$$\begin{aligned} \left\langle C_\varepsilon A_\varepsilon^{-1} B_\varepsilon u, v\right\rangle &= \langle C_\varepsilon w, v\rangle = \langle L_\varepsilon Q_\varepsilon w, P_\varepsilon v\rangle = (g^\varepsilon \partial_x Q_\varepsilon w, \partial_x P_\varepsilon v) \\ &= \left(g^\varepsilon \chi_y^\varepsilon \partial_x P_\varepsilon u, \partial_x P_\varepsilon v\right) = \left(P_\varepsilon g^\varepsilon \chi_y^\varepsilon P_\varepsilon \partial_x u, P_\varepsilon \partial_x v\right) \\ &= \overline{g\chi_y} (\partial_x P_\varepsilon u, \partial_x P_\varepsilon v) , \end{aligned} \tag{71}$$

which gives

$$\begin{aligned} \left\langle D_\varepsilon - C_\varepsilon A_\varepsilon^{-1} B_\varepsilon u, v\right\rangle &= (\bar{g} - \overline{g\chi_y}) (\partial_x P_\varepsilon u, \partial_x P_\varepsilon v) = \left\langle \bar{L} P_\varepsilon u, P_\varepsilon v\right\rangle \\ &= \left\langle P_{\varepsilon^*} \bar{L} P_\varepsilon u, v\right\rangle . \end{aligned} \tag{72}$$

The convergence (63) follows easily. For $u \in H^1(\mathbb{T})$,

$$\begin{aligned} \left\|(\bar{L}_\varepsilon P_\varepsilon - \bar{L})u\right\|_{H^{-1}} &= \left\|(P_{\varepsilon^*} \bar{L} P_\varepsilon - \bar{L})u\right\|_{H^{-1}} = \sup_{\|v\|_{H^1}=1} \left|\left\langle (P_{\varepsilon^*} \bar{L} P_\varepsilon - \bar{L})u, v\right\rangle\right| \\ &= \tilde{g} \sup_{\|v\|_{H^1}=1} |(P_\varepsilon \partial_x u, P_\varepsilon \partial_x v) - (\partial_x u, \partial_x v)| \\ &\le \tilde{g} \|P_\varepsilon u - u\|_{H^1} , \end{aligned} \tag{73}$$

which vanishes with ε.

The result in Theorem 2 together with Theorem 1 now gives the strong convergence in H^1 of $\bar{u}_\varepsilon$ to $\bar{u}$.

Corollary 1. *Let $\bar{L}_\varepsilon$ be the operator in (38) corresponding to (35) and $\{X_0^\varepsilon\}$ as in (49). Let $\bar{L}$ be the classical homogenized operator (36). If*

$$\bar{L}\bar{u} = f = \bar{L}_\varepsilon \bar{u}_\varepsilon, \qquad \bar{u} \in H_0^1(0,1), \quad \bar{u}_\varepsilon \in H_0^1(0,1) \cap X_0^\varepsilon, \tag{74}$$

then

$$\lim_{\varepsilon \to 0} ||\bar{u} - \bar{u}_\varepsilon||_{H^1} = 0. \tag{75}$$

Proof. Since (42) follows from Theorem 2 we only need to show (40, 43) to obtain (75) from Theorem 1. Standard results in harmonic analysis give (40). By Poincaré's inequality we have $\langle \bar{L}u, u \rangle \geq \delta \, ||u||_{H^1}^2$ for all $u \in H_0^1(0,1)$. Together with (62) in Theorem 2 we then get, for all $u \in H_0^1 \cap X_0^\varepsilon$,

$$\langle \bar{L}_\varepsilon u, u \rangle = \langle \bar{L} P_\varepsilon u, P_\varepsilon u \rangle = \langle \bar{L}u, u \rangle \geq \delta \, ||u||_{H^1}^2 . \tag{76}$$

4 Wavelet Based Homogenization

A wavelet representation lends itself naturally to analyzing the fine and coarse scales of a function. In this section we show how the projection generated homogenization works when we choose wavelet spaces as our coarse and fine space decomposition. We assume a basic knowledge of multiresolution analysis and its related wavelet spaces. For a detailed description we refer the reader to e.g. the book by Daubechies, [10]. In this setting, we let $Lu = f$ be a finite-dimensional approximation of a partial differential equation and apply the machinery in Sect. 2. We also demonstrate how the resulting $\bar{L}$ can be approximated by a sparse matrix and used for computing an approximate solution to the differential equation on a coarse grid.

Let us consider the equation

$$L_{j+1}U = F, \qquad U, F \in V_{j+1}, \qquad L_{j+1} \in \mathcal{L}(V_{j+1}, V_{j+1}), \tag{77}$$

set in the wavelet space V_{j+1}. This equation may originate from a finite difference, finite element or finite volume discretization of a given differential equation. In the simple case of Haar wavelets, U can be identified as a piecewise constant approximation of $u(x)$, the solution to the continuous problem.

For a function in V_{j+1}, the coarse scale is represented by V_j. The fine scale part lies in the wavelet space W_j and, assuming the wavelet system is orthogonal, $V_j \perp W_j$. In the first step we are therefore interested in the homogenized operator given by $\bar{L}_{V_{j+1},V_j}$. Let $\mathcal{W}_j : V_{j+1} \mapsto V_j \times W_j$ be the unitary operator corresponding to (12). As in the infinite dimensional case (18), we apply the transformation $\mathcal{W}_j$ to L_{j+1} from the left and get

$$\mathcal{W}_j L_{j+1} \mathcal{W}_j^* (\mathcal{W}_j U) = \mathcal{W}_j F, \tag{78}$$

or if we decompose $\mathcal{W}_j L_{j+1} \mathcal{W}_j^*$ according to (13),

$$\begin{pmatrix} A_j & B_j \\ C_j & L_j \end{pmatrix} \begin{pmatrix} U_{\mathrm{f}} \\ U_{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} F_{\mathrm{f}} \\ F_{\mathrm{c}} \end{pmatrix}, \qquad U_{\mathrm{f}}, F_{\mathrm{f}} \in W_j, \qquad U_{\mathrm{c}}, F_{\mathrm{c}} \in V_j. \tag{79}$$

Here the subindex "f" means projection onto the fine scale subspace, W_j in this case, and subindex "c" stands for projection onto the coarse scale subspace, V_j here. This is a convention we will use frequently. Also note that we changed the notation of the block previously called D_j to L_j, to indicate that $L_j = PL_{j+1}P$ actually corresponds to one type of direct discretization on the coarse scale.

The homogenized "coarse grid operator" is now given by block Gaussian elimination of (79),

$$(L_j - C_j A_j^{-1} B_j) U_{\mathrm{c}} = F_{\mathrm{c}} - C_j A_j^{-1} F_{\mathrm{f}}. \tag{80}$$

For simplicity we will use the notation $\bar{L}_j \equiv \overline{L_{j+1}}_{V_{j+1}, V_j}$ and $\bar{F}_j \equiv \bar{F}_{V_{j+1}, V_j}$. Hence,

$$\bar{L}_j = L_j - C_j A_j^{-1} B_j, \qquad \bar{F}_j = F_{\mathrm{c}} - C_j A_j^{-1} F_{\mathrm{f}}. \tag{81}$$

Since L_j corresponds to a direct coarse discretization, we can interpret $C_j A_j^{-1} B_j$ as a correction term, which includes subgrid phenomena in $\bar{L}_j$.

In the finite dimensional case, we represent the operators using their matrix representations with respect to the wavelet bases. For instance in the one dimensional Haar basis, $\mathcal{W}_j$ is represented by the matrix

$$\mathcal{W}_j = \begin{pmatrix} \tilde{Q}_j \\ \tilde{P}_j \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & \cdots & & \\ 0 & 0 & 1 & -1 & 0 & \cdots \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 1 & 1 & 0 & \cdots & & \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}. \tag{82}$$

Note, in this representation $\tilde{P}_j^*$ and $\tilde{Q}_j^*$ are bases in V_{j+1} for V_j and W_j, respectively. The corresponding projections in V_{j+1} are $\tilde{P}_j^* \tilde{P}_j$ and $\tilde{Q}_j^* \tilde{Q}_j$. The matrix representation of (79) is a block matrix decomposition of the same form as the operator decomposition.

Before we go on we should note that in general $\bar{L}_j$ will not be represented by a sparse matrix even if L_{j+1} is, because A_j^{-1} will typically be dense. For the homogenization method to be efficient we must be able to approximate $\bar{L}_j$ with a sparse matrix. If this sparse matrix is on banded form, it can be seen as a discretization of a local differential operator acting on the coarse space. In many applications this is indeed true and we will get back to the issue below.

The procedure to obtain $\bar{L}_j$ described above can be applied recursively on $\bar{L}_j$ itself to get $\bar{L}_{j-1}$ and so on. This can easily be verified when L_{j+1} is symmetric positive definite. Then A_j is always invertible by Proposition 1 (with $a[u,v] = (Lu, v)$) and Proposition 2 shows that also $\bar{L}_j$ is symmetric positive definite. By induction $\bar{L}_k$ exists for $k \le j$. Also note that Proposition 3 gives $\bar{L}_k = \overline{L_{j+1}}_{V_{j+1},V_k}$ and $\bar{F}_k = \bar{F}_{V_{j+1},V_k}$ for $k \le j$. Moreover, an improvement in the condition number can often be estimated. Typically for standard discretizations

$$\max_{||v||=1} (L_k v, v) < \max_{||u||=1} (L_{j+1} u, u), \tag{83}$$

when $k \le j$. Then, from Proposition 2,

$$\begin{aligned} \kappa(\bar{L}_k) &= \frac{\max_{||v||=1} \left(\bar{L}_k v, v\right)}{\min_{||v||=1} \left(\bar{L}_k v, v\right)} \le \frac{\max_{||v||=1} (L_k v, v)}{\min_{||v||=1} (L_{j+1} v, v)} \\ &< \frac{\max_{||v||=1} (L_{j+1} v, v)}{\min_{||v||=1} (L_{j+1} v, v)} = \kappa(L_{j+1}). \end{aligned} \tag{84}$$

The two-dimensional wavelet transform

$$\boldsymbol{\mathcal{W}}_j : \boldsymbol{V}_{j+1} \mapsto \boldsymbol{W}_j \times \boldsymbol{V}_j \tag{85}$$

can be written as a tensor product of one-dimensional transforms,

$$\boldsymbol{\mathcal{W}}_j = \mathcal{W}_j \otimes \mathcal{W}_j. \tag{86}$$

A linear operator L_{j+1} that acts on the space $\boldsymbol{V}_{j+1}$ can be decomposed in a way similar to the one-dimensional case. The equation

$$L_{j+1} U = F, \qquad U, F \in \boldsymbol{V}_{j+1}, \qquad L_{j+1} \in \mathcal{L}(\boldsymbol{V}_{j+1}, \boldsymbol{V}_{j+1}) \tag{87}$$

can be transformed to

$$\begin{pmatrix} A_j & B_j \\ C_j & L_j \end{pmatrix} \begin{pmatrix} U_{\mathrm{f}} \\ U_{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} F_{\mathrm{f}} \\ F_{\mathrm{c}} \end{pmatrix}, \qquad U_{\mathrm{f}}, F_{\mathrm{f}} \in \boldsymbol{W}_j, \qquad U_{\mathrm{c}}, F_{\mathrm{c}} \in \boldsymbol{V}_j, \tag{88}$$

and the coarse grid operator is again the Schur complement,

$$\overline{L}_{\boldsymbol{V}_{j+1},\boldsymbol{V}_j} \equiv \bar{L}_j = L_j - C_j A_j^{-1} B_j. \tag{89}$$

To get the matrix representation of (88), we take the one-dimensional matrix representations of $\mathcal{W}_j$ and, following (86), compute its Kronecker tensor product with itself. We must thereafter also apply a suitable permutation matrix to the product in order to get the same block structure of the matrix as in (88).

Note that the fine scale part of U can be decomposed as

$$U_{\mathrm{f}} = \begin{pmatrix} U_{\mathrm{ff}} \\ U_{\mathrm{cf}} \\ U_{\mathrm{fc}} \end{pmatrix}, \qquad U_{\mathrm{ff}} \in W_j \otimes W_j, \qquad U_{\mathrm{cf}} \in V_j \otimes W_j, \qquad U_{\mathrm{fc}} \in W_j \otimes V_j. \tag{90}$$

In some cases, the homogenized operator keeps important properties of the original operator. Let the forward and backward undivided differences be defined as

$$\Delta_+ u_i = u_{i+1} - u_i, \qquad \Delta_- u_i = u_i - u_{i-1}. \tag{91}$$

In [11] it was shown that the one-dimensional elliptic model equation $-(gu')' = f$ discretized with

$$-\Delta_+ \mathrm{diag}(g) \Delta_- U = h^2 F \tag{92}$$

will preserve its divergence form during homogenization. That is, we will get

$$\bar{L}_j = \frac{1}{h^2} \Delta_+ H_j \Delta_-, \tag{93}$$

where H_j is a strongly diagonal dominant matrix which can be interpreted as the effective material coefficient related to g. Analogously, for the first order differential operator $g(x)\frac{\partial}{\partial x}$ the discretized form $\mathrm{diag}(g)\Delta_-/h$ is preserved during homogenization,

$$\bar{L}_j = \frac{1}{h} H_j \Delta_-. \tag{94}$$

In two dimensions, the elliptic model equation $-\nabla(g(x,y)\nabla u) = f$ can be discretized as

$$L_{j+1} = -\frac{1}{h^2} \Delta_+^x G \Delta_-^x - \frac{1}{h^2} \Delta_+^y G \Delta_-^y, \qquad L_{j+1} U = F. \tag{95}$$

Then $\bar{L}_j$ is not any longer of exactly the same form as L_{j+1}. The cross-derivatives must also be included. We get

$$\bar{L}_j = -\frac{1}{h^2} \Delta_+^x H^{xx} \Delta_-^x - \frac{1}{h^2} \Delta_+^y H^{yx} \Delta_-^x - \frac{1}{h^2} \Delta_+^x H^{xy} \Delta_-^y - \frac{1}{h^2} \Delta_+^y H^{yy} \Delta_-^y. \tag{96}$$

When the operator L_{j+1} is derived from a finite difference, finite element or finite volume discretization, it is sparse and of a certain structure. In one dimension it might, for instance, be tridiagonal. However, as remarked above the matrix $\bar{L}_j$ is in general not sparse since A_j^{-1} is usually dense. Computing all components of $\bar{L}_j$ would be inefficient. Fortunately, $\bar{L}_j$ will be diagonal dominant in many important cases and we should then be able to find a sparse matrix that is a close approximation of $\bar{L}_j$.

In some simple cases the matrix A_j is in fact diagonal. Examples include operators L of the form

$$Lu = \int_0^x a(t)u(t)dt + b(x)u(x). \tag{97}$$

Let P_j be the orthogonal projection on V_j. Suppose L is discretized in V_{j+1} as $L_{j+1} = P_{j+1} L P_{j+1}$ with a in (97) replaced by $P_{j+1} a P_{j+1}$, and similar for b. When the Haar system is used, A_j is diagonal and $\bar{L}_j$ is of the same form as L_{j+1}. Let $\bar{A}_k$ be related to $\bar{L}_{k+1}$ via (78, 79) in the same way as A_j

relates to L_{j+1}. By induction, $\bar{A}_k$ is also diagonal for $k < j$. The operators in (97) turn up for instance in problems with systems of ordinary differential equations and one-dimensional elliptic equations, see [4,13]. In these cases, an explicit recurrence relation between scale levels can be established, which permits the computation of $\bar{L}_k$ on any fixed level k as the starting level, $j+1$, tends to infinity.

For more general problems one must instead rely on the rapid decay of elements in A_j and $\bar{L}_j$ off the diagonal, which is a consequence of the wavelet spaces' good approximation properties. The decay rate of A_j for Calderon–Zygmund and pseudo-differential operators was given by Beylkin, Coifman and Rokhlin in [5]. Letting $L_{j+1} = P_{j+1}LP_{j+1}$, $A_j = \{a^j_{k\ell}\}$, $B_j = \{b^j_{k\ell}\}$ and $C_j = \{c^j_{k\ell}\}$, they show that

$$|a^j_{k\ell}| + |b^j_{k\ell}| + |c^j_{k\ell}| \leq \frac{2^{-\lambda j} C_M}{1 + |k-\ell|^{M+1}}, \qquad |k-\ell| \geq \nu, \tag{98}$$

when the wavelet system has M vanishing moments. For Calderon–Zygmund operators $\lambda = 0$ and $\nu = 2M$. For a pseudo-differential operator $\nu = 0$ and if its symbol is $\sigma(x,\xi)$, the value of λ should be taken such that the estimates

$$|\partial^\alpha_\xi \partial^\beta_x \sigma(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{\lambda-\alpha+\beta}, \tag{99}$$

$$|\partial^\alpha_\xi \partial^\beta_x \sigma^*(x,\xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{\lambda-\alpha+\beta}, \tag{100}$$

are satisfied for some constants $C_{\alpha,\beta}$. For instance, in the second order elliptic case $\lambda = 2$. Moreover, Beylkin and Coult, [6], showed that if (98) holds with $\lambda = 0$ for A_j, B_j and C_j given by L_{j+1} in (78, 79), then the same estimate also holds for $\bar{A}_{j'}$, $\bar{B}_{j'}$ and $\bar{C}_{j'}$, here given by $P_{j'}\bar{L}_{k+1}P_{j'}$ for $j' \leq k < j$ with $j+1$ being the starting homogenization level. Hence, the decay rate is preserved after homogenization.

The decay estimate in [6] for $\bar{A}_{j'}$ is uniform in k and may not be sharp for a fixed k. There is, for example, a general result by Concus, Golub and Meurant, [9], for diagonal dominant, symmetric and tridiagonal matrices. For those cases, which include A_j corresponding to the discretization in (92) of the one-dimensional elliptic operator, the inverse has exponential decay,

$$\left|\left(A_j^{-1}\right)_{k\ell}\right| \leq C\varrho^{|k-\ell|}, \qquad 0 < \varrho < 1. \tag{101}$$

This holds also when the elliptic operator has a lower order term of type $b(x)\partial_x$ discretized with upwinding, [19].

One simple way to approximate $\bar{L}_j$ is to set all components outside a prescribed bandwidth ν equal to zero. Let us define

$$\operatorname{trunc}(M,\nu)_{ij} = \begin{cases} M_{ij}, & \text{if } 2\,|i-j| \leq \nu - 1 \\ 0, & \text{otherwise.} \end{cases} \tag{102}$$

For $\nu = 1$ the matrix is diagonal. For $\nu = 3$ it is a tridiagonal and so on.

Another possible strategy is the basic truncation method used in [5],

$$\operatorname{trunc}_{\mathrm{BCR}}(M,\varepsilon)_{ij} = \begin{cases} M_{ij}, & \text{if } |M_{ij}| > \varepsilon, \\ 0, & \text{otherwise} \end{cases} \tag{103}$$

but it is not practical here since the location of the non-zero elements cannot be controlled.

The matrix $\bar{L}_j$ can also be projected onto banded form in a more effective manner. Let $\{\boldsymbol{v}_j\}_{j=1}^{\nu}$ be a set of linearly independent vectors in $\mathbb{R}^N$. We define the *band projection*, $\operatorname{band}(M,\nu)$, of a matrix $M \in \mathbb{R}^{N\times N}$ as the projection of M onto the subspace of matrices with bandwidth ν such that

$$M\boldsymbol{x} = \operatorname{band}(M,\nu)\boldsymbol{x}, \qquad \forall \boldsymbol{x} \in \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_\nu\}. \tag{104}$$

In our setting M will usually operate on vectors representing smooth functions, for instance solutions to elliptic equations, and a natural choice for $\boldsymbol{v}_j$ vectors are thus the first ν polynomials,

$$\boldsymbol{v}_j = \{1^{j-1}, 2^{j-1}, \dots, N^{j-1}\}^T, \qquad j = 1, \dots, \nu. \tag{105}$$

Smooth solutions to the homogenized problem should be well approximated by these vectors. For the case $\nu = 1$ we get the standard "masslumping" of a matrix, often used in the context of finite element methods.

This technique is similar to the probing technique used by Chan et al., [8]. In that case the vectors $\boldsymbol{v}_j$ are sums of unit vectors. Other probing techniques have been suggested by Axelsson, Pohlman and Wittum, see Chapter 8 in [2]. The choice of $\boldsymbol{v}_j$ vectors could be optimized if there is some a priori knowledge of the homogenized solution. See [1] for further discussion.

It would be expensive to compute $\operatorname{trunc}(\bar{L}_j,\nu)$ using a naive method, such as calculating the full inverse of A_j. By capitalizing on the nearly sparse structure of the matrices involved, it was, however, shown in [6] that $\operatorname{trunc}(\bar{L}_j,\nu)$ can be computed with a cost of only $\mathcal{O}(N)$ operations for N unknowns and fixed accuracy. This method uses a multiresolution based LU decomposition procedure described in [7]. The computation of $\operatorname{band}(\bar{L}_j,\nu)$ can be based on $\operatorname{trunc}(\bar{L}_j,\mu)$, $\mu > \nu$. The additional computational cost is proportional to $(\nu^3 + \mu\nu)N$. The $\nu^3 N$ term corresponds to solving N $\nu\times\nu$ systems and $\mu\nu N$ to computing the right hand sides, see also [2]. Finally, note that the effective cost of using $\operatorname{trunc}(\bar{L}_j,\nu)$ is typically smaller, since the same homogenized operator will often be reused multiple times, for instance with different right hand sides, or in different places of the geometry as a subgrid model.

The truncation methods described above are even more efficient when applied to H_j instead of directly to the homogenized operator. The following proposition shows that when the solution to the homogenized problem belongs to the Sobolev space H^1, the accuracy is one order higher (see [1] for a proof).

Proposition 4. *Suppose $L = \Delta_+ H \Delta_-$ and $LU = h^2 f$. Let $\kappa = ||L^{-1}|| \cdot ||H||$ and*

$$r = \frac{||\delta L||}{||L||} = \frac{||\delta H||}{||H||} < \frac{1}{4\kappa}, \tag{106}$$

where $||\cdot||$ denotes the l_2-norm. Furthermore, let u be any H^1-function such that $U_j = u(jh)$. Then, if

$$(L + \delta L)(U + \delta U_L) = h^2 f, \tag{107}$$

$$\Delta_+(H + \delta H)\Delta_-(U + \delta U_H) = h^2 f \tag{108}$$

the following bounds on the perturbed solutions hold $\forall f$

$$||\delta U_L|| \le \frac{4r\kappa}{1 - 4r\kappa}||U||, \tag{109}$$

$$||\delta U_H|| \le \frac{1}{2} \cdot \frac{4r\kappa}{1 - 4r\kappa}||\Delta_- U|| \le \frac{h}{2} \cdot \frac{4r\kappa}{1 - 4r\kappa}||u||_{H^1}. \tag{110}$$

The bound (109) is sharp.

In two dimensions truncation to simple banded form is in general not adequate, since the full operator will typically be block banded. However, both the crude truncation and the band projection generalizes easily to treat block banded form instead of just banded. Let M be the tensor product of two $N \times N$ matrices. Then we define truncation as

$$\mathrm{trunc}_2(M, \nu)_{ij} = \begin{cases} M_{ij}, & \text{if } 2\,|i - j - rN| \le \nu - 1 - |2r|, \\ 0, & \text{otherwise,} \end{cases} \qquad |2r| + 1 \le \nu. \tag{111}$$

This mimics the typical block structure of a discretized differential operator. For the band projection, the subspace in the one-dimensional definition, is simply replaced by the subspace of matrices with the block banded sparsity pattern defined in (111).

In two dimensions, untangling the various H components of (96) from $\bar{L}$ is more complicated than finding the H in (93) and (94) for one-dimensional problems. There is however no unsurmountable difficulty, and truncating H instead of $\bar{L}$ can be done also for two-dimensional problems.

Applicatons of the wavelet based homogenization technique to subgrid scale moddeling of wave guides and to multigrid methods can be found in [12].

5 Multiscale Finite Element Methods

Assume that a standard Galerkin finite element method, with lower order basis functions, is approxmating the model problem

$$L_\varepsilon u_\varepsilon = -\frac{d}{dx}\left(g(x/\varepsilon)\frac{d}{dx}\right)u_\varepsilon(x) = f(x), \qquad 0 < x < 1, \tag{112}$$

with Dirichlet boundary conditions,

$$u_\varepsilon(0) = u_\varepsilon(1) = 0. \tag{113}$$

Let the element size $h \gg \varepsilon$. The integrals in the stiffness matrix will then approximate integrals where $g(x/\varepsilon)$ is replaced by $\bar{g}$ in (51) (and not $\tilde{g}$ in (37)) resulting in the wrong solution as $h, \varepsilon \to 0$. It is essential that the coarse grid finite element method contains some important subgrid scale information.

In [16,17] Hughes introduces a general procedure for finite element methods applied to multiscale problems. Extra degrees of freedom, representing the smaller scales, are eliminated in order to get a coarse grid method. This elimination in the earlier work of Hughes is analogous to the Schur complement (15) that we also use.

We shall consider a similar multiscale finite element method developed by Hou and collaborators, [14,15]. In the one dimensional case this method can be seen as a projection method of the type discussed in this paper.

Consider the two-point boundary value problem (112, 113). Approximate the equations by a standard Galerkin method with piecewise linear basis functions on the grid

$$x_j = jh, \qquad j = 0, 1, \dots, J, \qquad Jh = 1. \tag{114}$$

This results in the linear system

$$LU = F, \tag{115}$$

where the stiffness matrix $L = \{\ell_{ij}\}$, with $1 \le i, j \le J-1$, is symmetric tridiagonal, $\ell_{ij} = 0$ for $|i-j| > 1$ and

$$\ell_{j,j+1} = \ell_{j+1,j} = -\int_{x_j}^{x_{j+1}} g(x/\varepsilon)dx, \qquad \ell_{jj} = -\int_{x_{j-1}}^{x_{j+1}} g(x/\varepsilon)dx. \tag{116}$$

Moreover, $U = (u_1, \dots, u_{J-1})^T$ and $F = (f_1, \dots, f_{J-1})^T$ where

$$u_j \approx u_\varepsilon(x_j), \qquad f_j = \int_{x_{j-1}}^{x_{j+1}} f(x)\varphi_j(x)dx. \tag{117}$$

Here, $\varphi_j(x)$ denotes the standard hat function centered at x_j. The step size h is chosen such that the ε-scale is resolved on the grid, $h < \varepsilon$.

There is also a coarser grid associated with the method,

$$X_k = kH, \qquad k = 1, 2, \dots, K, \qquad KH = 1, \qquad H = mh. \tag{118}$$

In the multiscale finite element mehtod the fine grid discretization is used to derive the coarse grid basis functions ψ_k. These functions are defined as Galerkin apprixmations of

$$L_\varepsilon \psi_k = 0, \qquad X_{k-1} < x < X_{k+1}, \quad x \neq X_k, \tag{119}$$

with boundary conditions, $\psi_k(X_j) = \delta_{kj}$. From [15] we get that the solution on the coarse and fine grids are the same in this special one dimensional case.

By appropriately defining the projection operator P we can identify this multiscale method with the different steps in our general framework. Let P be given by

$$PU = (u_m, u_{2m}, \dots, u_{(K-1)m})^T \tag{120}$$

and then Q is defined by

$$QU = (u_1, \dots, u_{m-1}, u_{m+1}, \dots, u_{2m-1}, u_{2m+1}, \dots, u_{J-1})^T. \tag{121}$$

Following (13) the matrix L will be transformed into

$$\mathcal{W}L\mathcal{W}^* = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{122}$$

with A block diagonal: $A = \mathrm{diag}(A_k)$ for $k = 1, \dots, K$. Each A_k is a tridiagonal submatrix of L,

$$A_k = \{\ell_{ij}\}, \qquad km \le i, j, < (k+1)m. \tag{123}$$

The inverse of A in the definition of the projection generated homogenized operator $\bar{L} = D - CA^{-1}B$, consist of the inverses of the separate A_k submatrices. This corresponds directly to solving for the different coarse grid basis functions ψ_k as in (119).

The final coarse space problem, $\bar{L}PU = \bar{F}$, is identical to the discrete linear system for the multiscale method with the basis functions ψ_k.

In the two dimensional case there is not as direct a correspondence. The matrix A in (122) is not block diagonal. The discrete linear system of the multiscale finite element method will only be an approximation of $\bar{L}$. The approximation is, however, very good due to a well chosen way of defining the basis functions, that is based on oversampling, see [14,15].

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