A Note on "Self-Organized Criticality: Analysis and Simulation of a 1D Sandpile"

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We prove a conjecture made in the paper *Self-Organized Criticality: Analysis and Simulation of a 1D Sandpile*, by Lorenz, Jackett and Qin, IMA preprint 1515, 1997, available at `http://www.ima.umn.edu/preprints/OCT97/1515.pdf`. Also published in Doedel, Eusebius (ed.) et al., *Numerical methods for bifurcation problems and large-scale dynamical systems.* Based on two workshops held as part of the 1997-1998 IMA academic year on emerging applications of dynamical systems. New York, NY: Springer. IMA Vol. Math. Appl. 119, 229-264 (2000).

The conjecture concerns the spectral radius of a block (P_{11}) of the Markov matrix. Terminology and notation are as in the original paper.

Introduce the functional

$$\beta : \mathcal{A}_L \to \mathbb{N}, \qquad \beta(u) = \sum_{s=1}^{L} \max(s - u_s, 0) \tag{1}$$

which can be seen as a measure how far a given set is from the set of recurrent states $\mathcal{R}_L$. We observe that

$$0 \le \beta(u) \le \sum_{s=1}^{L} s = \frac{L(L+1)}{2} \equiv N_L. \tag{2}$$

Two useful properties of β are the following.

Lemma 1 *For all* $u \in \mathcal{S}_L$

$$\beta(E_r u) \le \beta(u), \qquad 1 \le r \le L. \tag{3}$$

Proof: We split the toppling operator into $L+1$ suboperators $\{\tilde{T}_k\}$ such that

$$\tilde{T}_k : \mathcal{A}_L \to \mathcal{A}_L, \qquad (\tilde{T}_k u)_s = \begin{cases} u_s - 2, & s = k \le L \text{ and } u_k \ge u_{k-1} + 3, \\ u_s + 2, & s = k - 1 > 0 \text{ and } u_k \ge u_{k-1} + 3, \\ u_{s-1}, & s = L + 1 \text{ and } k = L + 1, \\ u_s, & \text{otherwise,} \end{cases} \tag{4}$$

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for $1 \le k \le L+1$. Then the toppling operator can be written

$$T = \tilde{T}_{L+1}\tilde{T}_L \cdots \tilde{T}_1 \tag{5}$$

If $u_k \le u_{k-1}+2$ or if $k = L+1$ we clearly have that $\beta(\tilde{T}_k u) = \beta(u)$. Otherwise, since $\max(x+2,0) - \max(x,0)$ is an increasing function of x,

$$\begin{aligned}
\beta(\tilde{T}_k u) - \beta(u) &= \max(k - u_k + 2, 0) - \max(k - u_k, 0) + \\
&\quad \max(k-1-u_{k-1}-2, 0) - \max(k-1-u_{k-1}, 0) \\
&\le \max(k - u_{k-1} - 3 + 2, 0) - \max(k - u_{k-1} - 3, 0) + \\
&\quad \max(k-1-u_{k-1}-2, 0) - \max(k-1-u_{k-1}, 0) \\
&= 0,
\end{aligned} \tag{6}$$

for $1 < k \le L$ and

$$\begin{aligned}
\beta(\tilde{T}_1 u) - \beta(u) &= \max(1 - u_1 + 2, 0) - \max(1 - u_1, 0) \\
&\le \max(1 - u_0 - 3 + 2, 0) - \max(1 - u_0 - 3, 0) \\
&= \max(u_0, 0) = 0.
\end{aligned} \tag{7}$$

So, for all $u \in \mathcal{A}_L$ and $1 \le k \le L+1$, we have that $\beta(\tilde{T}_k u) \le \beta(u)$ and by (5) this extends to $\beta(Tu) \le \beta(u)$. Moreover, for $1 \le r \le L$,

$$\beta(R_r u) - \beta(u) = \max(r - u_r - 1, 0) - \max(r - u_r, 0) \le 0, \qquad \forall u \in \mathcal{A}_L. \tag{8}$$

Since for $u \in \mathcal{S}_L \subset \mathcal{A}_L$ the evolution operator $E_r u = T^n R_r u$ for some n, the lemma follows. □

Lemma 2 *For each $u \in \mathcal{T}_L$ there exists an $r \ge 1$ such that $\beta(E_r u) = \beta(u) - 1$.*

Proof: This follows immediately from the beginning of Lemma 3.2, where it is asserted that for any $u \in \mathcal{T}_L$ there exists an $r \ge 1$ such that $u_r < r$ and $R_r u \in \mathcal{S}_L$. □

We can now state the theorem.

Theorem 1 *The diagonal block P_{11} of the Markov matrix P satisfies*

$$\rho(P_{11}) = (L-1)/L. \tag{9}$$

Proof: Let $\{V_k\}$ be the disjoint family of sets such that

$$V_k = \{u \in \mathcal{S}_L : \ \beta(u) = k\}. \tag{10}$$

Trivially, $V_0 = \mathcal{R}_L$ and by (2)

$$\mathcal{T}_L = \bigcup_{k=1}^{N_L} V_k. \tag{11}$$

Order the states in $\mathcal{T}_L$ according to which V_k they belong to, so that $u \in V_{N_L}$ come first. Then, in view of Lemma 1, P_{11} can be partitioned into blocks as

$$P_{11} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1,N_L} \\ 0 & A_{22} & \dots & A_{2,N_L} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & A_{N_L,N_L} \end{pmatrix}, \qquad A_{kk} \equiv \{a^k_{ij}\} \in \mathbb{R}^{m_k \times m_k}. \tag{12}$$

It is clear that $\rho(P_{11}) = \max_k \rho(A_{kk})$. What is more, the number of non-zero entries in each row of A_{kk} is at most $L-1$, because of Lemma 2. Therefore,

$$\rho(A_{kk}) \leq |A_{kk}|_\infty = \max_{1 \leq i \leq m_k} \sum_{j=1}^{m_k} |a^k_{ij}| \leq \frac{L-1}{L}, \qquad 1 \leq k \leq N_L. \tag{13}$$

So $\rho(P_{11}) \leq (L-1)/L$. That $\rho(P_{11}) \geq (L-1)/L$ is already stated in Lemma 4.3. □

Remark 1: The sets V_k used in the proof above can be seen as a ladder such that a sequence of states will start at a certain level and steadily go downwards, but never up. Note, however, that there is no guarantee that a sequence does not take two steps at a time. (For instance $\beta([0\ 0\ 2\ 4\ 6]) = 4$ and $\beta(E_3[0\ 0\ 2\ 4\ 6]) = 2$.) Hence, in the general case the index of the set does not signify the least number of evolution steps needed for its states to reach $\mathcal{R}_L$. (This is, however, true for $L < 5$.)

Remark 2: In the proof of Lemma 4.3, the matrix A is actually the same as P on level $L-1$, scaled by $(L-1)/L$. The result therefore follows directly from Theorem 4.1. Also $N_0 = \#\mathcal{S}_{L-1}$.