Chapter 14:
Stable sets, Bargaining set and the Sharpley Value

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Recapitulation

- The core:
  - Game outcomes that no coalition profitably can block
  - Game (N,v)
    - N the set of players
    - v: £N ! R, characteristic function
      - v(∅) = 0
    - Payoff vector α 2 R^N
    - S µ N, a coalition with payoff α(S)

Core

- The core of (N,v) is
  - α(S) ¸ v(N) for 8 S µ N

and

  - α(N) = v(N)
- If α do no exist there is no core, it is <u>not</u> empty

Chapter contents

- Deviations / objections to coalitions
- Stable sets: Coalitions that are stable
- Bargaining Set: Objections and counter objections. Conflict sets between players!
- Shapley value: Greedy negotiation

Stable sets

Imputation *x*

- Objection of a coalition S to the imputation y
  if $x_i > y_i$ for $\forall\, i \in S$ and $x(S) \leq v(S)$ named $x \succ_S y$
- I.e. S objects to y as it dominates it with the payoff x

The subset $Y \subset X$ of imputations is a **stable set** iff

– Internal stability: If $y \in Y$ then $\nexists\, z \in Y$ with a coalition for which $z \succ_S y$

– External stability: if $z \in X \backslash Y$ then $\exists\, y \in Y$ such that $y \succ_S z$ for some coalition S

Stable set properties

- The core is a subset of every stable set
- No stable set is a subset of another stable set
- If a core is a stable set then it is the only stable set

Bargaining set, Kernel, Nucleolus

- Objections and counter objections
  - Pair (y,S) is an objection of *i* against *j* to x if i2S and j∉ S and y_k>x_k for 8 k2S
  - Pair (z,T) is a counterobjection to the objection (y,S) of *i* against *j* if z_k>x_k, for 8k2T\S and z_k>y_k for 8k2TÅS
- Bargaining set of all objections to which there is a counter objection
  - The conflict set

The kernel

- For any coalition define
  - e(S,x) = v(S) – x(S) where x is the imputation
    - e>0 loss of S for x to be implemented
    - e<0 the gain of S compared to x
  - Dynamics
    - Coalition S in an objection of i against j to x if S includes i and not j and x_j>v({j})
    - Coalition T is a counterobjection to the objection if T includes j and not i and e(T,x) ¸ e(S,y)

Kernel II

- The set of dynamics outlined above
  - “i is against j but j can point to a coalition that is as good for all members without i but including j”
  - Define excess/gain/loss between ij as $s_{ij}(x)$ as the max excess of the any coalition
    - $S_{ij}(x) = \max\{e(S,x)$: i2 S, j 2 N\S}
  - Kernel set of imputation x for i,j such that

$$s_{ji}(x) \;\text{,}\; s_{ij}(x)$$

Kernel properties

- The kernel of a coalition game is a subset of the bargaining set
- Nucleolus
  – The set of objections to which there is a counter objection.
  – The nucleolus is a subset of the kernel

Shapley Value

- Objection:
  - Increase $\psi(i)$ as bail out would give
    - $\psi_j(N \setminus \{i\}, v^{N\setminus i})$ rather than x_j
  - Give me more to make sure you are not excluded causing me to obtain
    - $\psi_j(N \setminus \{j\}, v^{N\setminus j})$ rather than x_i
- Counterobjection the reverse
  - I can cause a similar objection

Shapley value (cont)

- The value that causes a balanced contribution is the shapley value!
- Characteristics
  - Symmetry: if i and j are interchangeable then
    - $\psi_i(v) = \psi_j(v)$
  - Dummy player
    - If i is a dummy then $\psi_i(v) = v(\{i\})$
  - Additivity
    - For two games v and w: $\psi_i(v+w) = \psi_i(v) + \psi_i(w)$ where v+w is the game defined by $(v+w)(S) = v(S)+w(S)$ for every S
- The shapley value is the only that satisfied all three.

Market economy as a game

- A market is:

  (T, G, A, U)

  - T the set of traders
  - G initial endowment G ½ $R^{|T|}$
  - Actions: A={a^i: i 2 T}½ G
  - Utility: U={u^i: i2 T}: u^i: G ! R

Market

- A market (T,G,A,U) can generate a game (N,v) as
  - N=T
  - $v(S) = \max_{X} \sum u^{i}(x^{i})$, 8SµN
- Named a market game