How to control a hot tub?

Our control problem is to use as little energy as possible to make the tub water meet a temperature suitable for a bath, at precise scheduled time points. We assume that the heating power is either on or off, and that the air temperature is constant during the heating period.

Dynamics.

The variables used are: (i) current time and water temperature: t and T. (ii) Required water temperature, T_b, at bathing time t_b. (iii) Environmental (air) temperature T_e.

System constants are: Time constant c and hypothetical balance temperature T_d, the asymptotic water temperature reached when power is on being $T_e + T_d$. Both quantities (c and T_d) can be estimated by measuring temperatures (air and water) during trial runs with power on and off. The 'half-life' is $\log 2/c$. During the half-life, the difference $T - T_e$ decreases to half its value if power is off, and $T_e + T_d - T$ decreases to half its value if heating power is on. In my 2m diameter wooden tub (and with a frigolite lid on) with 6kW heating the constant c is 0.0003 (time measured in minutes) and T_d is 130 degrees Celsius (centigrades). The balance point is thus above water boiling temperature and will not be reached, but the math and physics is still OK for water temperatures well below the boiling temperature and well above the freezing temperature. The half-life is around 1.65 days (or 40 hours).

We have assumed that the energy flow out of the water is proportional to the temperature difference $T - T_e$, and then the temperature development will be following a solution to a first order differential equation with constant coefficients with solutions:

$T(t) = C_{off} e^{-ct} + T_e$, when power is off

$T(t) = C_{on} e^{-ct} + T_e + T_d$, when power is on

The integration constants C_{off} and C_{on} are found by conditions T(t)=T and $T(t_b) = T_b$, respectively, i.e.,

$C_{off} = (T - T_e)e^{ct}$

$C_{on} = (T_b - T_e - T_d)e^{ct_b}$

Power should be turned on at time x when the two temperature curves meet:

$C_{off} e^{-cx} + T_e = C_{on} e^{-cx} + T_e + T_d$ **, i.e.,** $x = -\log(T_d/(C_{off} - C_{on}))/c$

However, if $x < t$ the water will not have time to warm up and if $x > t_b$ the water will not have sufficient time to cool down.

Scheduling.

If new readings are not pushed, we have a number of wake-up times, set by the scheduling times. Each time schedule is altered, we additionaly schedule a checking time which is the power-on time for the first scheduled bath, x above. If $x < t$, power is turned on, if $t < x$, power is turned off, and then we go to sleep. It can be suitable to schedule a few wake-up times just before t_b to avoid overheating if the calibration is inaccurate.

If new sensor readings are pushed, the method is the same, but the control decision is made whenever a new sensor reading wakes the scheduler. In Octave:

```
function x=power(T,t,Tb,tb,Te,Td,c)
Coff=(T-Te)/exp(-c*t); Con=(Tb-Te-Td)/exp(-c*tb);
x=-log(Td/(Coff-Con))/c;
schedulewakeup(x);
if x<t poweron else poweroff;
```

Calibration.

Two steps are involved in calibration, and T_e is the average air temperature during each step. Both steps should have durations of several hours, approaching the inverse guessed time constant (but with temperatures below T_b). It is probably a good idea to use separate calibrations for summer and winter use (since snow may constrict air flow round the tub, and T_e is very different in winter). The two steps can be reversed if it is convenient (*i.e.* if the water is cold when you start).

(i) Let the tub cool down without power to find the time constant; This phase runs from t_{c0} to t_{c1} while temperature drops from T_{c0} to T_{c1}. Do not take off the lid because then you get an incorrect time constant (thermal flow should be the same in both phases and also when the scheduling is used). Now c is found by:

$$c = \frac{\log \frac{T_{c0}-T_e}{T_{c1}-T_e}}{t_{c1}-t_{c0}}$$

(ii) Turn on power to heat water from T_{h1} to T_{h2} during the time interval from t_{h1} to t_{h2}, and compute T_d by:

$$T_d = \frac{T_{h1}-T_{h2}e^{c(t_{h2}-t_{h1})}}{1-e^{c(t_{h2}-t_{h1})}} - T_e$$

In Octave, this becomes:

```
function [c,Td]=calibrate(Tc0,tc0,Tc1,tc1,Th1,th1,Th2,th2,Te)
c=log((Tc0-Te)/(Tc1-Te))/(tc1-tc0);
ect=exp(c*(th2-th1));
Td=(Th1-ect*Th2)/(1-ect)-Te;
```

Above, all time points and temperatures must be measured in the same units (Centigrades, Fahrenheit, ..., respectively hours, minutes, seconds, ..., from a common reference point (*e.g.* the built in time counter of the computer system).

Enjoy!/Stefan Arnborg.