program whose optimal value coincides with the optimal value of *LP* (see Appendix A). Therefore, if we consider a minimization *integer* linear program *ILP* whose relaxation provides *LP*, any feasible solution of *DLP* has a measure no greater than the optimal measure of *ILP* (which, in turn, is no greater than the value of any feasible solution of *ILP*) and can, thus, be used as a lower bound when estimating the quality of an approximate solution of *ILP* (see Fig. 2.8).

A primal-dual algorithm exploits this property to find approximate solutions of an integer linear program *ILP*: in particular, it simultaneously maintains a (possibly unfeasible) integer solution $x$ of *ILP* and a (not necessarily optimal) feasible solution of *DLP*. At each step, $x$ and $y$ are examined and modified to derive a new pair of solutions $x'$ and $y'$ where $x'$ is "more feasible" than $x$ and $y'$ has a better measure than $y$. The algorithm ends when the integer solution becomes feasible: the quality of this solution is evaluated by comparing it with the final dual solution. This approach allows us to obtain faster algorithms because it is not necessary to optimally solve either *ILP* or *DLP*. Moreover, as we will see in the rest of this section, there are cases in which the method allows us to obtain solutions with a good performance ratio.

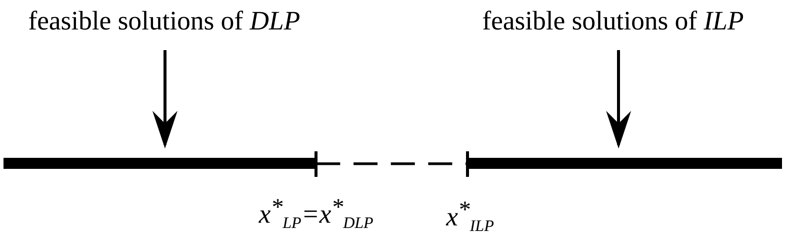


Figure 2.8
The space of values of feasible solutions of *ILP* and *DLP*

In particular, let us formulate a primal-dual algorithm for MINIMUM WEIGHTED VERTEX COVER. First observe that, given a weighted graph $G=(V,E)$, the dual of the previously defined relaxation $LP_{VC}$ is the following linear program $DLP_{VC}$:

$$
\begin{array}{lll}
\texttt{maximize} & \displaystyle\sum_{(v_i,v_j)\in E} y_{ij} & \\
\texttt{subject to} & \displaystyle\sum_{j:(v_i,v_j)\in E} y_{ij} \le c_i & \forall v_i \in V \\
 & y_{ij} \ge 0 & \forall (v_i,v_j)\in E.
\end{array}
$$

Note that the empty set is an unfeasible integer solution of MINIMUM WEIGHTED VERTEX COVER (that is, of the initial integer linear program) while the solution in which all $y_{ij}$ are zero is a feasible solution with value 0 of $DLP_{VC}$. The primal-dual algorithm starts from this pair of solutions