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Abstract

This thesis deals with combinatorics in connection with Coxeter groups, finitely generated but not necessarily finite. The representation theory of groups as nonsingular matrices over a field is of immense theoretical importance, but also basic for computational group theory, where the group elements are data structures in a computer. Matrices are unnecessarily large structures, and part of this thesis is concerned with small and efficient representations of a large class of Coxeter groups (including most Coxeter groups that anyone ever payed any attention to.)

The main contents of the thesis can be summarized as follows.

- We prove that for all Coxeter graphs constructed from an n-path of unlabelled edges by adding a new labelled edge and a new vertex (sometimes two new edges and vertices), there is a *permutational representation* of the corresponding group. Group elements correspond to integer n-sequences and the nodes in the path generate all $n!$ permutations. The extra node has a more complicated action, adding a certain quantity to some of the numbers.
- We derive formulas and interpretations of many important group concepts in the permutational representations, such as *length* and *Bruhat order*.
- Our computer implementation of the permutational representations have been used successfully for various investigations. We obtain interesting variants of Peter Ungar's theorem about the minimal number of partial flip moves required to accomplish a total reversal of a permutation. The theorems give sharp bounds for the complexity of *stack-based sorting* - a well-known problem in computer science.
- Is the language of reduced Coxeter group expressions regular? After much research, giving only partial answers, the question was recently settled in the affirmative. We are now able to construct simple finite automata for all Coxeter groups.
- The original *pebbling game* starts with one pebble on the lower left square of a potentially infinite chessboard. Any pebble with empty squares above and to the right may be split into two and moved to these two squares. We are able to enumerate and characterize legal positions, not only in the plane, but in higher dimension and on general posets. In particular, if played on a Coxeter group, the legal positions give unexpected connections between the two poset structures, weak order and Bruhat order.
- The *chip-firing game* has been a rich source of combinatorical results in recent years. We find a surprising connection between minimal recurrent games and conjugacy classes of Coxeter elements.

Keywords: permutational representation, game, pebbling, polygon property, allowable sequence, essential chain, reachability, Coxeter groups, weak order, Bruhat order, polygon poset, chip-firing, Coxeter element, conjugacy class

Sammanfattning

Ämnet för denna avhandling är kombinatorik i samband med coxetergrupper, ändligt genererade men inte nödvändigtvis ändliga. Teorin för grupprepresentation med ickesingulära matriser över en kropp är av högsta teoretiska betydelse, men också grundläggande för datorräkningar med grupper, där gruppelementen är datastrukturer i datorns minne. Matriser är här onödigt utrymmeskrävande och en del av avhandlingen behandlar mindre och effektivare representationer av en stor klass av coxetergrupper (de flesta grupper som någon intresserat sig för torde ingå).

Avhandlingen har i stora drag följande innehåll.

- Varje coxetergraf som man kan bilda av en n-stig av omärkta kanter genom att lägga till en ny märkt kant (ibland två nya kanter) och en ny nod har en *permutationsrepresentation*. Gruppelementen motsvaras av n-vektorer av heltal och stigens noder genererar alla $n!$ permutationer av komponenterna. Den extra noden har mer komplicerad verkan och adderar en viss storhet till vissa av talen.
- Vi härleder ett otal formler och tolkningar av viktiga gruppbegrepp, såsom *längd* och *bruhatordningen*.
- Vår datorimplementation av permutationsrepresentationerna har framgångsrikt använts för olika undersökningar. Vi har erhållit intressanta varianter av Peter Ungars sats om minimala antalet partiella omvändningar som behövs för att vända hela permutationen bak och fram. Man får därmed skarpa gränser för komplexiteten hos *sortering med stack*, ett känt problem inom datalogin.
- Utgör de reducerade uttrycken i en coxetergrupp ett *reguljärt språk*? Många har sysslat med frågan under de senaste åren och många partiella resultat har ernåtts. Vi kan här för godtycklig coxetergrupp beskriva en ändlig automat som känner igen reducerade uttryck.
- Det ursprungliga *pebblingspelet* börjar med en sten på nedre vänstra rutan i ett potentiellt oändligt schackbräde. Om rutorna norr och öster om en sten är tomma får stenen klyvas och flyttas till dessas rutor. Vi lyckas räkna och karakterisera legala ställningar, inte bara i planet utan i högre dimension och på allmänna pomängder. Om spelet utförs på en coxetergrupp ger legala ställningar oväntade samband mellan svaga ordningen och bruhatordningen.
- *Chipsspelet* har på senare år varit en rik källa till kombinatoriska resultat. Vi finner ett överraskande samband mellan minimala rekurrenta spel och konjugatklasser för coxeterelement.

Nyckelord: permutationsrepresentation, spel, polygonegenskapen, coxetergrupper, nåbarhet, svaga ordningen, bruhatordningen, polygonpomängd, chipsspel, konjugatklass

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Preface

I am a victim of the enigmatic *ADE-phenomenon*, so named by the famous Russian mathematician V.I.Arnol'd [2]. Even a casual reader of this thesis must notice the ubiquitous pictures of small circles connected with straight lines. The most common types are named A, D and E, and the challenge is to explain why these little pictures pop up all over the place, not only in combinatorics. For some years, I tried to ignore them, but as you can see, they prevailed.

The ADE-diagrams symbolise *Coxeter groups*, named after Harold Scott Macdonald Coxeter, one of my all-time favourite mathematicians. His name is always associated with beautiful maths and maths that is fun. I thoroughly enjoyed his work in recreational and aesthetic mathematics (in particular [33] and [31]) long before I could imagine that I would one day be writing a thesis with his name in the title. Being an embarrassingly over-age Ph.D. candidate, I find it especially encouraging that professor Coxeter at ninety is still active and top-ranking in his field.

Mathematical research is, in principle, a very lonely occupation. However, to be successful you need fellow–creatures, for advice, criticism and encouragement. I would like to express my gratitude to all those who have helped me in any of these ways.

I count myself fortunate to have had *Anders Björner* as my supervisor. He taught me everything I know in combinatorics and he introduced me to the world of international mathematics. I also want to thank my younger son *Kimmo Eriksson* for countless discussions of mathematics (mainly at night) ever since his childhood, and my elder son *Viggo Kann* for solving all my LaTeX problems.

Henrik Eriksson
Stockholm
July 18, 2015.

Part 1. Combinatorial representations of Coxeter groups

Chapter 1

Computational Coxeter group theory

1. Computational group theory

A modern encyclopedia (Nationalencyklopedin, 1992) defines *computer* as an automatic machine for *numeric calculation* and *symbolic manipulation*. It is a strange fact that in computational group theory, computers are used for *symbolic calculation* and the methods that we are going to introduce could well be called *numeric manipulation*!

The first mathematician to think of using computers for group-theoretical calculations was Alan Turing. During World War II, Turing and other British mathematicians were engaged in cryptographic activities, mainly in breaking the German secret codes. At the end of the war, Turing started to work on the design of an automatic computing engine. In a document circulated at the end of 1945, Turing suggested several tasks that his engine would be able to take on. One of these was the enumeration of all groups of order 720.

However, the fundamental algorithms of computational group theory predate the advent of the computer by almost a century. In fact, calculation with groups was an established activity from the middle of the nineteenth century, although the definition of an abstract group did not appear until 1893 (Weber [81]). Cayley never gave the definition, but he certainly knew something about groups. The famous theorems of Sylow date from 1872 and were originally results about certain sets of permutations. Paradoxically stated, computational group theory is the mother of group theory. The father? There is always some doubt, but a candidate is the surge of axiomatising, abstraction and striving for purity and clarity, so evident in Hilbert's talk in Paris at the turn of the century. This emphasis on structure with some disregard for substance had its climax in the nineteen-sixties, when many areas of mathematics were rewritten in the categorical language of morphisms.

Since then, a remarkable change of attitude is evident. Much more attention is now paid to concrete representations of mathematical objects. Chronologically, this change of attitude coincides with the entry of computers into the academic world. It could not be termed far-fetched to suggest a causal connection.

Conjecture 1. The current upswing in interest for concrete representations of mathematical abstractions is a mental side-effect, when the computers go marching in.

(It should be noted that the author in an earlier paper, [38], 1985, has denied any impact whatsoever of computers on university mathematics.)

1.1. Presentation and representation of groups.

A typical problem in computational group theory begins something like this: *"Given a group G, determine ..."*. This problem may be easy or difficult depending on how G is specified. The recent survey by Sims [72] states that there are really only three methods in common use. We add one here, the game method, that we believe to be of increasing importance.

(1) The simplest way is to start with a list of *concrete objects*, that we already know how to multiply, and define G as the group generated by these elements. The most common such objects are permutations and square matrices. Note that the matrix elements can belong to any ring, for examples be polynomials in x and y with integer coefficients modulo 15.

One may be led to think that all problems about representation are solved with this concrete specification, but that is not always true. The concrete objects may be unnecessarily large, making the representation unefficient. In some cases, Coxeter groups are naturally defined by $n \times n$-matrices and these are often large objects. A common situation is the specification of n such reflection matrices and, most often, these reflections will generate super-exponentially many other matrices, say $\sim n^n$. Even though n is seldom greater than 10, memory space will soon be exhausted. That is one of the reasons why we are going to derive permutational representations, where each matrix can be replaced by an n-vector, resulting in space savings by a factor n.

(2) The next simplest way of specifying the group is to give a list of *legal moves* in a game. The group consists of all legal move sequences, where two sequences are considered equivalent if they lead to the same final position (starting from the same position). Here, the game positions are the concrete objects, but the move sequences may be represented in many ways, some of which may be very space consuming (for instance, a dynamic stack of simple moves). Some may be very space efficient (for instance, the final position only) but hopelessly time consuming (how do you multiply two final positions?).

This is precisely the state of things, when one uses the numbers game (soon to be defined) to represent a Coxeter group. Our solution to this problem is to trade a small amount of memory space (our permutational vectors have $n+1$ components when the numbers game vectors have n components) for a large gain in time efficiency.

(3) The third specification method uses one specific combinatorial or geometric object and defines G as its group of *automorphisms* or symmetries. The object may be a graph, very easily represented in a number of ways, or a regular polytope, very difficult to represent at all (the coordinates of the vertices will not do in general, for they are irrational and not easily representable in the computer).

Almost all finite Coxeter groups are symmetry groups of regular polytopes, but few attempts have been made at using this in computer programs. In principle, it could be done, and for automorphism groups, there is at least a concrete object to start with. [1]

(4) Not so in the last and most well-known specification of an abstract group, the finite presentation by *generators and relations*. A set of symbols, for example $\{x, y\}$, called generators is given together with relations, for example $x^2 = y^2 = e$, where e is the identity, and $xyx = yxy$.

The concept of a Coxeter group is defined in this way, with a certain kind of relations. The two-generators group above is the Coxeter group denoted by A_2. It has only six elements, but from the presentation, it may be impossible to see if it is infinite or perhaps trivial with only one element.

The study of groups given by presentations is called *Combinatorial group theory*. A good book with many examples is Coxeter & Moser [31]. The overlap between the two CGT-areas is smaller than one might think. First of all, many fundamental problems about finitely presented groups have been shown to be algorithmically unsolvable, and we shall soon return to that theme. Second, many of the most interesting problems are sufficiently general, that only marginal progress can be made by calculations with specific groups.

In a seminal problem survey ([48], 1987) Roger Lyndon stated twenty important problems in combinatorial group theory. Only two of these can be tackled by computational methods. The others are of the following kind: "Is every finitely presented torsion group finite?". Nevertheless, computational group theory has had its greatest successes in connection with finitely presented groups.

[1] A more promising object is the *Coxeter complex*, an abstract simplicial complex (no need for co-ordinates) whose type-preserving automorphisms constitute the Coxeter group (see Tits [77]).

1.2. **Coset enumeration and other algoritms.** The most basic of the computational group algorithms is called *coset enumeration*. It appeared in 1936, long before computer implementations were even thought of, in a paper by Coxeter and Todd [28], but is now part of every existing soft-ware package for group theory. For a very clear and readable description of the algorithm, we refer to Atkinson [3], for a critical comparison of soft-ware implementations, see Neubüser's paper in [19]. Joacim Neubüser wrote the first published paper on the use of computers in group theory in 1960 and is still a leading scientist in that area.

We have to restrict ourselves to a description of what the algorithm does. As input, it first takes a finite presentation of G by generators and relations, for example $x^2 = y^2 = e$, where e is the identity, and $xyx = yxy$. Further, it takes a subgroup $H \subset G$, where H is given by a set of expressions that generate H, for example $H = [xy]$. After a finite period of time, the algorithm has found the subgroup H and all its cosets aH. (In our small example, $|G : H| = 2$, i.e. there are two cosets, one of which is H itself.) A special case is when $H = \{e\}$, in which case coset enumeration means element enumeration.

A disadvantage with this algorithm is that *no time limit* can be given for its completion. Computing cosets is simple, the hard part is recognizing when two seemingly different cosets are in fact one and the same. But for Coxeter groups, there is no such problem. The numbers game provides an auxiliary algorithm that works in time proportional to the order of the group and gives immediate information about whether two elements are equal.

This thesis contains many results that have been proved by letting the computer execute a program. Some of these programs are written for computer algebra systems (mostly Maple V) but most of them (about twenty) were written in a general-purpose, high-level language called Modula-3. The advantage of Modula-3 is its high degree of modularisation and data abstraction, making it very easy to recycle program modules. The most common algorithms used are standard computer science methods, like back-tracking, breadth-first search and hashing. An excellent textbook is Robert Sedgewick's [68] but behind Sedgewick one perceives a giant, Donald Knuth, [59], [60],[61].

Since some theorems in this thesis have computer proofs, we consider these programs as an appendix of the report, not included in print but available upon request.

1.3. **Computability and complexity.** The main objective of computational group theory is devising algorithms for studying the structure of groups given by finite presentations. Unfortunately, this is impossible! For many important questions, there are no algorithms for finding the answer and it can be proved that no such algorithms can exist.

One of the most basic of these questions is the *word problem*. Let x and y be two words in the generators of the finitely presented groups — how can we decide whether or not x and y denote the same group element? Sometimes it is very simple; we can apply one or two of the relations and get from x to y. In fact, *if* the two words define the same element, we can always prove it by presenting such a chain of relation applications. But there are groups in which we can never prove that two words define different group elements. We express this fact by saying that the question is *undecidable*. (For full definition, see Sims [72], Ch.3.)

Many other important questions are even more undecidable. One such question is this one: Are x and y conjugate? If we could decide all such questions, then in particular we could decide whether or not xy^{-1} and e are conjugate, that is the word problem for x and y. And that, we cannot do. We cannot even decide whether a finitely presented group has more than one element!

All of these questions are decidable for Coxeter groups, but there is still the question about *complexity*. As already mentioned, the numbers game solves the word problem in time proportional to the length of the words. It is even easier to answer the question about the group's being trivial — the answer is no! But the conjugacy complexity is still open, or at least, we do not know the answer. The exponential bound recently proved by A. Cohen [23], certainly is not sharp. In Chapter 5, we show how to do it in time proportional to the number of generators, but only for a special kind of element, the Coxeter elements.

There is some controversy about using the complexity measure of computer science (asymptotic growth) in computer algebra, since the value of the parameter (number of generators, word length etc) is seldom more than 10 in practise. It seems unnatural then to consider asymptotics. On the other hand, that is exactly what makes theoretical computer science theoretic.

2. Coxeter groups

The prime object of our study is an amazingly versatile contraption. There are so many radically different viewpoints, that communication of ideas is often rendered difficult. At the same time, a change of viewpoint sometimes makes a hard problem much easier.

Our main reference is Humphreys [56], which focuses on reflection groups and root systems. Root systems are important for Bourbaki [15] too, but the indefatigable count also finds time and space for material about Coxeter complexes and hyperplane arrangements. For symmetries of polytopes, we should listen to Coxeter himself [32], [31], for the Lie algebras and their generalisations, Kac [57] is a good source and for the combinatorical aspects — well, that book is still waiting to be written.

The definition of an abstract Coxeter group by generators and relations is straightforward. With our combinatorial disposition, we feel free to assume all groups *finitely generated.* Following Bourbaki, we shall use the term Coxeter system for a Coxeter group with a distinguished set of generators. The generators must all be *involutions*, that is satisfy $s^2 = e$, and also satisfy mutual *Coxeter relations*, that is $s_1 s_2 = s_2 s_1$ or $s_1 s_2 s_1 = s_2 s_1 s_2$ or $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$ etc. By convention, the occurring mutual relations are coded in a *Coxeter graph*, where vertices correspond to generators and edges are drawn between vertices corresponding to noncommuting generators. An edge is left unlabelled if a relation of type $s_1 s_2 s_1 = s_2 s_1 s_2$ holds, it is labelled by 4 etc if the relation is $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$ etc. Finally, the edge is labelled by ∞ if no relation exists.

Example. Here is the graph of the group B_3: $\overset{a}{\circ}\!\!-\!\!\!-\!\!\overset{b}{\circ}\!\!\overset{4}{-\!\!\!-}\!\!\overset{c}{\circ}$. It has got three generators, a, b, c, six relations: $a^2 = b^2 = c^2 = e$, $aba = bab$, $ac = ca$, $bcbc = cbcb$ and 48 elements.

Definition. The group W, finitely presented by the generators $s_1, \ldots, s_n$ and relations $s_i^2 = e$, $(s_i s_j)^{m_{ij}} = e$ specified in a Coxeter graph is called the Coxeter group of the graph. The graph together with its group is called a Coxeter system (W, S). This abstract definition is not the way Coxeter groups entered mathematics. They have a solid geometric background, that we shall now sketch briefly.

2.1. **Reflections and symmetries.** The theory of reflection groups has its origin in research into *regular polyhedra* in n dimensions and *regular tesselations* of n-space. The symmetry groups of these objects are reflection groups. In 1934, Coxeter [26] was able to enumerate all finite irreducible reflection groups and found that they were Coxeter groups. The connection is the following.

All reflections are generated by some *simple reflections* s_i in hyperplanes H_i, which bound the *fundamental polyhedron* P. The Coxeter relations $(s_i s_j)^{m_{ij}} = e$ are determined by the angles between the faces of P supported by H_i and H_j, in fact, that angle is π/m_{ij}. If the faces do not intersect, $m_{ij} = \infty$.

In 1935, Coxeter [27] completed the classification of finite Coxeter groups by proving that there are only the finite reflection groups. The established names of these groups and their graphs are shown in table 1. (Note that B_n and C_n name the same group.)

Hyperplanes through the origin correspond to linear reflections, affine hyperplanes correspond to affine reflections and some such affine arrangements divide space into congruent polytopes, *alcoves*. The corresponding affine reflection groups are also Coxeter groups and were also enumerated by Coxeter. They are all obtained by addition of an extra node to the graph of one of the finite groups, and named by tildefication.

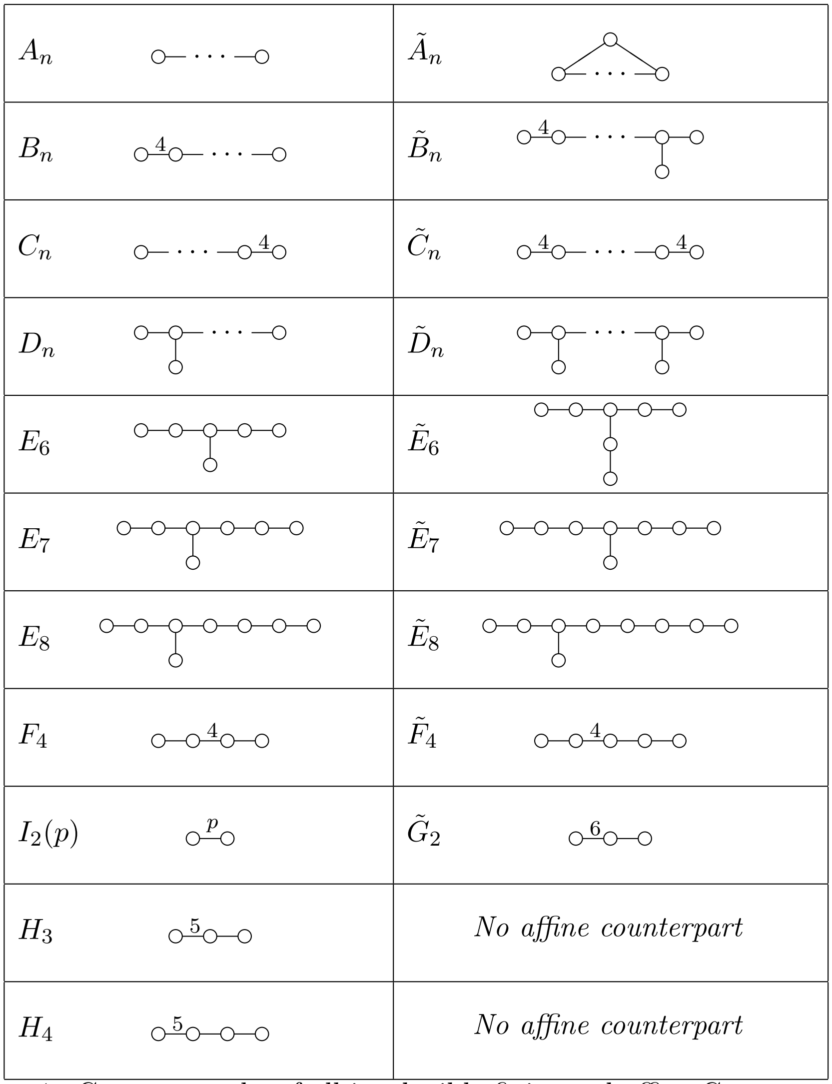


TABLE 1. Coxeter graphs of all irreducible finite and affine Coxeter groups

2.2. **The standard geometric representation.** After Coxeter's pioneering work and Witt's [82] complementing investigations , the situation was the following. Finite Coxeter groups had a very nice geometric representation as finite reflection groups, affine Coxeter groups also had a very nice, but different, geometric interpretation as tesselations of n-space, a few of the others (the hyperbolic groups) could be viewed as tesselations of hyperbolic n-space, but most infinite Coxeter groups had no geometric model.

Coxeter was able to find a linear representation in $\mathbf{R}^n$ of an arbitrary Coxeter group, where all generators map to (not necessarily orthogonal) linear reflections, but he gave no proof of the faithfulness of this representation. The Bourbaki volume on Lie theory [15], that appeared in 1968, contained this faithfulness proof, so we can state the following theorem, see [56], p 110.

Theorem 2. [Standard geometric representation] *Let (W,S) be a Coxeter system and $\{\alpha_i\}$ any basis for $\mathbf{R}^n$. Then W has a faithful linear representation in which all s_i act as reflections*

$$s_i : \; x \mapsto x - 2\langle x, \alpha_i \rangle \alpha_i$$

The symmetric bilinear form $\langle , \rangle$ is defined on the basis by $\langle \alpha_i, \alpha_j \rangle = -\cos \pi / m_{ij}$.

The cosine-matrix appearing in the theorem is the key to the classification of Coxeter groups. It is positive definite for the finite groups, positive semidefinite for the affine groups and indefinite for all other groups.

2.3. **Root systems and hyperplane arrangements.** Almost all finite Coxeter groups had occurred in the work of Cartan and Weyl, 1925–1927, on semisimple Lie groups. The term *Weyl group* is still in use for these groups. The concept of a root and of root systems emanates from that background. Every finite reflection group defines an arrangement of hyperplanes and each hyperplane is defined by any perpendicular vector, a *root*. The length of this root is arbitrary if we just want it to define a hyperplane, but in a *root system*, root lengths are chosen such that roots are mapped to roots under all reflections.

The roots corresponding to simple reflections are called *simple roots*. It is a theorem that all roots are unique linear combinations of simple roots and that the nonzero coefficients are either all positive or all negative. A reflection maps positive roots to positive roots, with one exception — the positive root α perpendicular to the hyperplane is of course mapped to $-\alpha$.

Why should people be more interested in these root artefacts than in the natural reflections themselves? The answer is simple from the viewpoint of computational group theory. We have replaced a bulky $n \times n$-matrix by a slim n-vector and with no loss of information. It is even better than that. All finite groups have some particularly nice root systems where most roots have only one or two nonzero components. This fact is exploited by John Stembridge in his excellent Maple packages *coxeter* and *weyl* [75]. He does not store the zero components but uses the Maple datastructure for linear expressions in the basis vectors instead. These expressions then have only one or two terms.

The definition of a root can be generalized to infinite groups via the standard geometric realization. In Chapter 3 we are going to use these roots to construct recognizing automata for the language of reduced words.

Hyperplane arrangements are less attractive from the computational point of view, but have the advantage of a geometrically intuitive length concept. The *length* of an element w is the length of the shortest word in the generators s_i, that is equal to w. A reflection group hyperplane arrangement divides the unit sphere into simplicial *chambers*, each face of which can be labelled by one of the s_i, such that reflections map chambers onto chambers while respecting the labelling. A word in the s_i can be represented by a walk through a *gallery* of chambers, namely by letting each s_i in the word determine which wall to pass next. In this correspondence, shortest possible galleries between two given chambers correspond to shortest possible words representing the reflection that maps the first chamber onto the second. We also have the following theorem.

Theorem 3. *The length of an element w is equal to the number of hyperplanes that separates a chamber from its w-image.*

The well-known intuitive geometric proof, that seems first to have appeared in Coxeter's book on polytopes [32], generalizes nicely to *any* hyperplane arrangement, see [11], Sec. 4.2.

3. The numbers game

The numbers game was originally presented to the contestants of the International Olympiad of Mathematics, 1986: *Five integers are arranged on a circle. The player may pick any negative number, add it to its neighbours and reverse its sign.* It was generalized by S. Mozes to general graphs in 1987 and it was further generalized and studied by K. Eriksson in a series of papers and in his thesis [42], 1993.

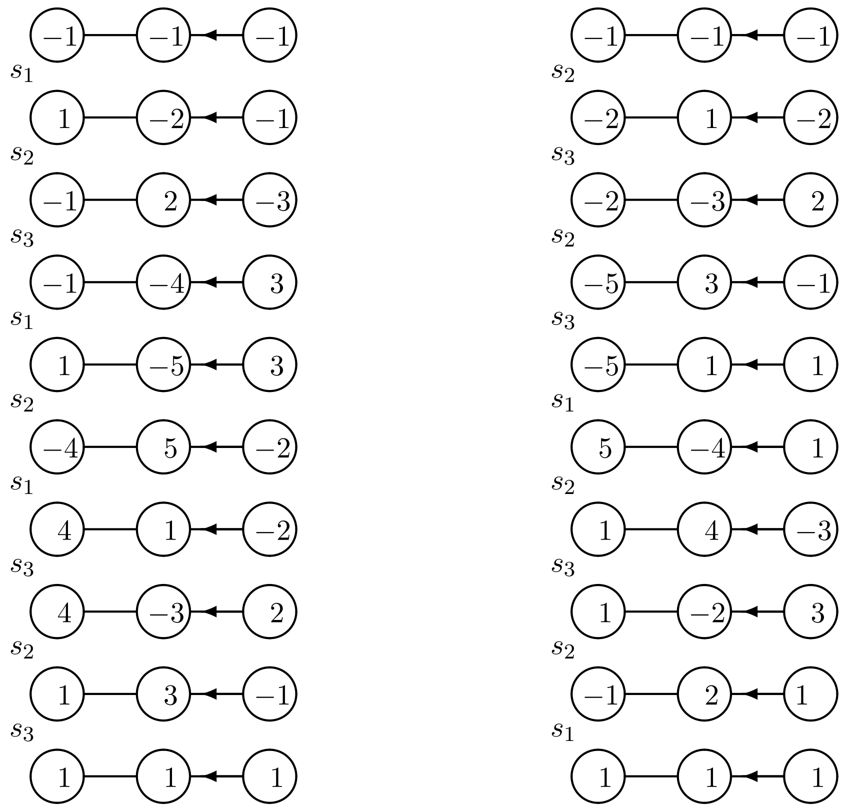
The key property of the numbers game is *strong convergence* (a concept due to A. Björner), the purport being that if some legal game terminates, all legal games will terminate in the same final position after the same number of moves. This is illustrated in the example below.

We are primarily interested in the following representation result ([42], p 80).

Theorem 4. *Start a numbers game on a Coxeter graph with the value* -1 *on each node. Every word in the generators* s_i *can be seen as describing a play sequence. This correspondence establishes a faithful map from group elements to reachable game positions. A reduced word (shortest possible) corresponds to a legal play sequence (only negative numbers played).*

The theorem is incomplete as far as edge labels are concerned. The game rules must then be modified accordingly. A number moving along a labelled edge (from right to left, say) is multiplied by 2 if the edge label is 4, it is multiplied by 3 if the edge label is 6 and by 4 if the label is ∞. In general, the multiplier is $4\cos^2 \pi/m_{ij}$. Note that this multiplication does not take place in the other, left to right, direction.

Example. Two games on $B_3\colon \overset{a}{\circ}\!-\!\!-\!\overset{b}{\circ}\!\overset{4}{-\!\!-}\!\overset{c}{\circ}$ are shown. The directed edge has multiplier 2.



Remark 5. K.Eriksson characterized game graphs where all games terminate and also graphs with looping games. They turn out to be exactly the finite resp. the affine groups in Figure 1.

Regarding a Coxeter group as a numbers game is very different from interpreting it as a hyperplane arrangement, a root system, a tesselation, reflections in n-space or symmetries of a polytope. But there are even more exotic interpretations. In the 1977 paper by Hazewinkel et al. [51] (an introduction to the ADE-problem), the theory of *quivers* is one such item. A quiver is a diagram like $\bullet \!\longrightarrow\! \bullet \!\longleftarrow\! \bullet$, symbolizing linear mappings of vectorspaces. When the (undirected) graph is of type A, D or E, there are only finitely many types of indecomposable quivers, same type meaning same matrices in well chosen bases, indecomposable meaning not direct sum of other quivers (P. Gabriel [44], 1973). The ADE-graphs also emerged when V.I.Arnol'd classified *singularities of algebraic hypersurfaces*. Who knows where they are going to materialize next time!

4. Length, order and permutations

This section deals with Coxeter group concepts of special interest to combinatorialists. A good part of combinatorics takes place inside a Coxeter group, namely the group of all permutations of the symbols $1, 2, \ldots, n$. It is called the symmetric group and usually denoted by S_n, but in the classification of Coxeter groups, it is A_{n-1}. The generators $s_1, s_2, \ldots, s_{n-1}$ are the *adjacent transpositions*, s_1 transposes the first and second elements, s_2 the second and third etc. One can check the Coxeter relations: $s_i^2 = e$, $s_i s_j s_i = s_j s_i s_j$ if i, j are neighbours and $s_i s_j = s_j s_i$ otherwise. (A general reference for all cited facts is Humphreys [56].)

We have already defined the *length function* as the length of the shortest possible word $s_1 s_2 \cdots s_k = w$. The length of the identity element is defined as zero. For the symmetric group, it is an elementary fact that $l(w) = \#(\text{inversions})$. Recall that an inversion is an occurrence of a larger number preceding a smaller number in the permutation. All finite groups have an *element of maximal length*, w_o. In the symmetric group, w_o is the permutation $n, \ldots, 2, 1$ and $l(w_o) = n(n-1)/2$. (Depending upon point of view, $l(w_o)$ can also be regarded as the number of hyperplanes or the number of positive roots or the length of the longest numbers game.)

If $l(ws) = l(w) + 1$, where $s \in S$, we can write $w < ws$. This relation generates a partial order on W, the *weak order*. If $l(ws) = l(w) - 1$ (the only other possibility), we say that s belongs to the *descent set* of w and write $s \in D(w)$. We have $D(e) = \emptyset$ and $D(w_o) = S$. In the symmetric group, the descent set is indicated by the descents of the permutation, that is the adjacent inversion pairs.

Choose any reduced word for w and take $s \in D(w)$, so that ws can be written with one letter less. The *exchange condition* states that this can be achieved simply by deleting an appropriate letter s' in w. (If we then write s at the end of the word, we again have a reduced word for w obtained by exchanging s' for s.) The interpretation for permutations is simple: if, after a sequence of adjacent transpositions, there is an inversion of two symbols, then one of the transpositions must have transposed that very pair of symbols, and deleting that transposition has the same effect as transposing the pair in the final position.

Conjugates of generators are called *reflections*. A reflection can always be written as a palindrome $t = s'_k \cdots s'_2 s'_1 s'_2 \cdots s'_k$, with $s'_i \in S$. In the symmetric group, a reflection is a transposition, not necessarily adjacent. The weak order, generated by $w < ws$, where $s \in S$ and $l(ws) = l(w) + 1$, can be expanded to the *Bruhat order*, generated by $w < wt$, where t is any reflection such that $l(wt) = l(w) + 1$. There are two weak orders, one defined by $w < ws$, the other by $w < sw$, but only one Bruhat order, for $wt = t'w$, where $t' = wtw^{-1}$ is a reflection if t is. For permutations, the following criterion can decide whether a permutation π precedes another permutation σ in Bruhat order. It is mentioned by Lehmann [63] and can be seen as a special case of Deodhar's criterion [36] for general Coxeter groups.

Tableau criterion: Let π_{ij} be the element obtained by sorting the first j symbols of π in increasing order and then picking the ith symbol. Then $\pi \leq \sigma$ in Bruhat order if and only if $\pi_{ij} \leq \sigma_{ij}$ whenever $1 \leq i \leq j \leq n$.

In terms of reduced words, $u \leq w$ in Bruhat order simply means that a reduced word for u can be obtained by deleting some letters from a reduced word for w; this is the *subword property*. The weak order and the Bruhat order interact in the *lifting property*.

Lifting property: Let $u < w$ in Bruhat order and assume that for some $s \in S$, $u < us$ but $ws < w$. Then $u \leq ws$ and $us \leq w$.

A new result connecting weak order and Bruhat order is Lemma 116 in chapter 4.

The symmetric group A_{n-1} has two siblings, B_n and D_n, both representable by *signed permutations* of $1, \ldots, n$, that is by vectors $(x_1, \ldots, x_n)$, such that $(|x_1|, \ldots, |x_n|)$ is an ordinary permutation. As in A_{n-1}, $s_1, \ldots, s_{n-1}$ mean adjacent transpositions, but s_n means something different. In B_n, s_n acts by sign-inversion of x_1. In table 1, s_n is the leftmost node, followed by s_1 etc (sorry about that, but it suits our purpose), and so the relation $s_n s_1 s_n s_1 = s_1 s_n s_1 s_n$ must hold. In D_n, s_n acts by sign-inversions and transposition of x_1, x_2. The relation $s_n s_2 s_n = s_n s_2 s_n$ must and does hold.

There are $2^n n!$ signed permutations, so that is the number of elements in B_n. In D_n, there will only appear signed permutations with an even number of minus-signs, so D_n is only half as big as B_n. For reference, we include a table of important data for the finite groups.

Remark 6. As always with group actions, one should state whether the groups acts from the left or from the right. For example, with permutations, does $s_1 s_2$ mean that transposal of the first pair followed by transposal of the second pair, or is it the other way around? The conventional evaluation order for operators is right-to-left, but this seems to be in conflict with the reading and writing habits of the western world, and even if an author proclaims that she is using right-to-left action, there is often some accidental occidental anomaly.

To avoid this effect, we are going to be rather vague about our intended interpretation. Whenever possible, statements are written so as to fit both views. Otherwise, we stick to the natural left-to-right order except when matrix notation is used. The action of a matrix $\boldsymbol{A}$ will be $\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x}$, as usual.

Group	A_{n-1}	B_n	D_n	E_6	E_7	E_8	F_4	$I_2(p)$	H_3	H_4
$\lvert W \rvert$	$n!$	$2^n n!$	$2^{n-1} n!$	51840	2903040	696729600	1152	$2p$	120	14400
$l(w_o)$	$\binom{n}{2}$	n^2	$n(n-1)$	36	63	120	24	p	15	60

TABLE 2. Orders of finite Coxeter groups and number of reflection elements.

Chapter 2

Permutational representations of Coxeter groups

5. Introduction and outline of results

The permutations of $\{1, \ldots, n\}$ represent the symmetric group S_n in such a way that the generators s_i in the Coxeter graph A_{n-1}: $\overset{1}{\circ}\!-\!\overset{2}{\circ}\!-\!\overset{3}{\circ}\cdots\overset{n-1}{\circ}$ correspond to transpositions of adjacent elements. In this chapter, we consider similar representations for Coxeter graphs obtained by adding one or two extra edges and nodes to A_{n-1}. In all these cases, group elements are represented by certain ordered n-subsets of some set of numbers. The well-known representations of B_n and D_n by signed permutations (see page 10) are the generic examples.

It turns out that all finite and affine Coxeter groups have graphs of this kind and simple permutational representations. If all edge labels are either 3, 4, 6 or ∞, that is if the group is *crystallographic*, there is a permutational representation involving integers only.

Example. The infinite group $\tilde{C}_4$ with Coxeter graph $\begin{smallmatrix} \circ & - & \circ & - & \circ \\ 4| & & & & |4 \\ \circ & & & & \circ \end{smallmatrix}$ can be represented as a set of integer lattice points in $\mathbf{Z}^4$. The identity element is $(1, 2, 3, 4)$, the three nodes in the horizontal path are transpositions of adjacent components, the lower left node reverses the sign of the first component x_1 and the lower right node replaces x_4 by $(10 - x_4)$. This representation will be derived in section 8.

Combinatorial, geometric and algebraic representations and interpretations of Coxeter groups are already abundant, so what is the raison d'être for this study of representations? First, it is a unified approach to all finite and affine Coxeter groups. Second, for each group element it gives explicit formulas for the length of a reduced decomposition and related concepts. Third, it provides an embedding of all finite Coxeter groups into the symmetric group S_n and all affine groups into S_∞ (bijections from $\mathbf{Z}$ to $\mathbf{Z}$).

Example. $\tilde{C}_4$ can be considered as a group of permutations of $\mathbf{Z}$ in the following way. Let there be mirrors at co-ordinates $x = 0$ and $x = 5$. Now, the transposition $1 \leftrightarrow 2$ has reflected images $-1 \leftrightarrow -2$ and $8 \leftrightarrow 9$ and doubly reflected images $-9 \leftrightarrow -8$ and $11 \leftrightarrow 12$ etc. Let s_1 be the combination of these transpositions, let s_2 be the mirror images of $2 \leftrightarrow 3$ and s_3 the mirror images of $3 \leftrightarrow 4$. The lower left node s_4 is $-1 \leftrightarrow 1$, $9 \leftrightarrow 11$ etc and the lower right node s_5 is $4 \leftrightarrow 6$, $14 \leftrightarrow 16$ etc.

In this way, every generator s_i will correspond to a combination of transpositions. It will be proved in section 10 that the generated group of permutations of $\mathbf{Z}$ is indeed isomorphic to $\tilde{C}_4$.

The material in this chapter is organized in sections, one for each type of Coxeter graph. All useful results appear as propositions in these sections, but before losing the reader in a jumble of details, we take the opportunity to state a summarizing theorem.

Theorem 7. *For every Coxeter graph that can be transformed into a path* $\overset{1}{\circ}\!-\!\overset{2}{\circ}\!-\!\overset{3}{\circ}\cdots\overset{n-1}{\circ}$ *by contraction of an edge (labelled or unlabelled), there is a faithful group representation on* $\mathbf{R}^n$ *in which the path nodes correspond to transpositions of adjacent co-ordinates. By choosing an appropriate orbit, one can faithfully represent group elements as certain vectors in* $\mathbf{R}^n$*, and the length of an element is easily calculable from its representing vector.*

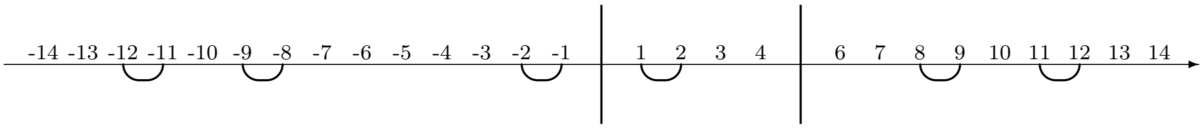
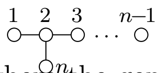


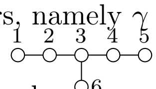
FIGURE 1. The action of $s_1 \in \tilde{C}_4$ as transpositions on $\mathbf{Z}$.

Although the affine Coxeter graphs are not of this type, all mentioned results are valid for the affine groups too.

6. Representations of D_n, E_6, E_7 and E_8 and friends

The groups D_n:  already have well-known representations as *signed permutations* of $\{\pm 1, \ldots, \pm n\}$, where the generator s_n acts by *transposition and sign change* of the first two elements, $s_n : (x_1, x_2, \ldots) \mapsto (-x_2, -x_1, \ldots)$. To see how this can be generalized to other groups, in particular to E_6, E_7 and E_8, we start with a description of D_n, suitable for that generalization.

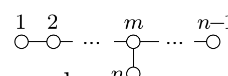
Each element of D_n corresponds to an ordered n-subset $x_1, \ldots, x_n$ of $\{-n, \ldots, n\}$, the identity e to $(1, 2, \ldots, n)$. The generators $s_1, \ldots, s_{n-1}$ act as transpositions of adjacent numbers, while s_n adds a quantity γ to the first two numbers, namely $\gamma = -x_1 - x_2$.

In the analogous representation of E_6: , the elements correspond to certain ordered six-subsets of $\{-8, \ldots, 8\}$, in particular, e corresponds to $(3, 4, 5, 6, 7, 8)$ and w_o to $(-3, -4, -5, -6, -7, -8)$. The generators $s_1, \ldots, s_5$ act as transpositions of adjacent numbers, while s_6 adds a quantity γ to the first three numbers, namely $\gamma = \frac{1}{3} \sum x_i - x_1 - x_2 - x_3$.

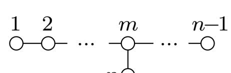
Example. $345678 \overset{s_3}{\mapsto} 346578 \overset{s_6}{\mapsto} 124578 \overset{s_3}{\mapsto} 125478 \overset{s_6}{\mapsto} 236478 \overset{s_3}{\mapsto} 234678 \overset{s_6}{\mapsto} 345678$, as expected, since $(s_6 s_3)^3 = e$.

The representations of E_7 and E_8 are very similar. The identity gets another representation, but the action of the s_i is exactly as for E_6, with the same expression $\gamma = \frac{1}{3} \sum x_i - x_1 - x_2 - x_3$. All these graphs are of the special kind considered in the following section.

6.1. **Analysis of permutational representations.**

We can try this game for any group , letting it act on $\mathbf{R}^n$ in the following way. As before, the generators $s_1, \ldots, s_{n-1}$ transpose adjacent vector components, so that $s_i : (\ldots, x_i, x_{i+1}, \ldots) \mapsto (\ldots, x_{i+1}, x_i, \ldots)$, while s_n acts by adding a quantity γ to $x_1, \ldots, x_m$. Here γ is a linear expression in the x_i, chosen such that the commutation relations involving s_n are maintained. We shall see that the only such expression is $\gamma = \sum_{m+1}^n x_i - \frac{2}{m} \sum_1^n x_i + \text{const}$.

First, s_n commutes with $s_1, \ldots, s_{m-1}$ and with $s_{m+1}, \ldots, s_{n-1}$, therefore γ must be invariant under permutations of $x_1, \ldots, x_m$ and under permutations of $x_{m+1}, \ldots, x_n$, so we can write $\gamma = \alpha(x_1 + \cdots + x_m) + \beta(x_{m+1} + \cdots + x_n) + \text{const}$. Second, $(s_n)^2 = e$, and one easily checks that this is equivalent to $\alpha = -\frac{2}{m}$. Third, $(s_n s_m)^3 = e$, and some elementary algebra proves this to be equivalent to $\beta = \frac{m-2}{m}$. Thus, the final expression can be written $\gamma = \frac{m-2}{m} \sum_1^n x_i - \sum_1^m x_i + \text{const}$.

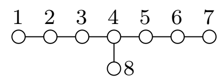
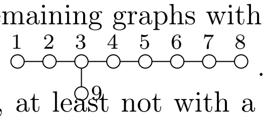
Proposition 8. *A Coxeter group with graph*  *has a faithful representation on $\mathbf{R}^n$ in which $s_1, \ldots, s_{n-1}$ act as transpositions of adjacent components, while s_n acts by adding a certain real number γ to the first m components, $x_1, \ldots, x_m$, namely $\gamma = \frac{m-2}{m} \sum_1^n x_i - \sum_1^m x_i + c$, where c is an arbitrary constant.*

Proof. As we have seen, this expression for γ ensures the desired commutation relations between the s_i-actions. What remains to be shown is faithfulness, and for that we can use the standard geometric representation for general Coxeter groups (page 6). In our case, s_n corresponds to an affine transformation, so to begin with we must get rid of the constant. In all cases, except when $m^2 = (m-2)n$, the constant can be disposed of by a translation of the origin. In fact, let $(x'_1, x'_2, \ldots) = (x_1 - d, x_2 - d, \ldots)$, where d is chosen such that $(m^2 - (m-2)n)\, d = mc$. Then the group action on the x'-coordinates is the same as before, except that the constant has disappeared from γ.

We now introduce basis vectors α_i, namely $\alpha_1 = (1, -1, 0, \ldots)$, $\alpha_2 = (0, 1, -1, 0, \ldots)$ etc, $\alpha_{n-1} = (\ldots, 0, 1, -1)$ and $\alpha_n = (\underbrace{\ldots, -1, -1}_{m}, 0, \ldots)$, and also a symmetric bilinear form defined by

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } s_i \text{ and } s_j \text{ commute} \\ -\frac{1}{2} & \text{if } (s_i s_j)^3 = e \end{cases}$$

It is easily checked that all s_i act as reflections $x \mapsto x - 2\langle x, \alpha_i \rangle \alpha_i$ and the theorem is applicable. For instance, the transposition s_2 should map $(1, -1, 0, \ldots)$ to $(1, 0, -1 \ldots)$ and this is exactly what $\alpha_1 \mapsto \alpha_1 - 2\langle \alpha_1, \alpha_2 \rangle \alpha_2 = \alpha_1 + \alpha_2$ means.

The remaining graphs with $m^2 = (m-2)n$ are the *affine groups*, $\tilde{E}_7$:  and $\tilde{E}_8$: . Here, no translation $(x'_1, x'_2, \ldots) = (x_1 - d, x_2 - d, \ldots)$ will make s_n linear, at least not with a constant d. But with $d = \frac{c}{m} + \frac{1}{n} \sum_1^n x_i$, the group action on the x'-coordinates becomes linear. The transpositions $s_1, \ldots, s_{n-1}$ act as before, but the s_n-action is more complicated. Since $\sum_1^n x_i$ increases by $m\gamma$, the d-expression will increase by $m\gamma/n$. But $\gamma - m\gamma/n = 2\gamma/m$, so the action of s_n is

$$(x'_1, \ldots, x'_n) \mapsto (x'_1 + \frac{2\gamma}{m}, \ldots, x'_m + \frac{2\gamma}{m}, x'_{m+1} - \frac{m\gamma}{n}, \ldots, x'_n - \frac{m\gamma}{n}) \; .$$

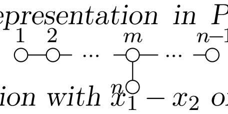
Straightforward calculation shows that $\gamma = -x'_1 - \ldots - x'_m$, so s_n is indeed linear. It is easily checked that all s_i act as reflections $x \mapsto x - 2\langle x, \alpha_i \rangle \alpha_i$, so again the theorem is applicable and we find that the action on the x'-coordinates is a faithful linear representation of the group.

There may still be some doubt concerning the mapping from x-coordinates to x'-coordinates. How do we know that it is invertible? We don't and it isn't! It is easily seen that the mapping is the orthogonal projection onto the affine hyperplane $x_1 + \cdots + x_n = -nc/m$. Any vector in $x \in \mathbf{R}^n$ splits uniquely as $x = (x'_1, \ldots, x'_n) + d(1, 1, \ldots)$, with $x'_1 + \cdots + x'_n = -nc/m$, and the transpositions $s_1, \ldots, s_{n-1}$ as well as the affine reflection s_n are compatible with this split.

So, the representation being faithful after the projection, it must have been even more so before! For the special case $c = 0$, the projection would make the representation collapse, but then s_n is a linear action, so that case is already covered. □ □

In section 3, an alternative model of the general Coxeter group, the *numbers game*, was presented. In this game, real numbers are placed on the nodes of the Coxeter graph, so a game position is a vector in $\mathbf{R}^n$. For a graph with unlabelled edges (such as those considered here) the generators s_i transform the positions according to the following rule: the value on node i is added to the values on the neighbouring nodes j and then the sign is reversed on node i itself. The resulting linear representation is faithful (Theorem 4, p 8).

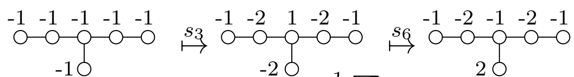
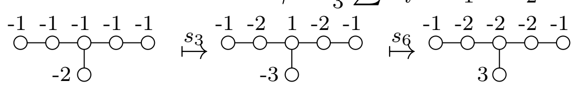
There is a close connection between the numbers game and permutation representations of Coxeter groups. Loosely speaking, the node numbers in the game are the *differences* in the permutation.

Proposition 9. *The permutational representation in Proposition 8 can be interpreted as a numbers game on the Coxeter graph*  *in the following way. Every vector $x \in \mathbf{R}^{n+1}$ defines a numbers game position with $x_1 - x_2$ on the first node, $x_2 - x_3$ on the second node etc., and finally $\gamma = \frac{m-2}{m} \sum_1^n x_i - \sum_1^m x_i + c$ on the nth (lower) node. The action of s_i on the x-vector corresponds to playing the ith node.*

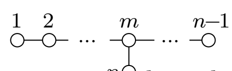
Proof. The action $s_2 : (x_1, x_2, x_3, x_4, \ldots) \mapsto (x_1, x_3, x_2, x_4, \ldots)$ adds the number on node two to its neighbours, for $x_1 - x_3 = (x_1 - x_2) + (x_2 - x_3)$ and $x_2 - x_4 = (x_3 - x_4) + (x_2 - x_3)$. There is also a sign change on node two itself, for $x_3 - x_2 = -(x_2 - x_3)$. The γ-value is affected only by s_m, which is the way it should be in the numbers game. Finally, s_n adds γ to $x_1, \dots, x_m$, which does not affect differences, except for $(x_m - x_{m+1})$, which increases by γ. The only complicated (but still trivial) verification concerns the action of s_n on the γ-expression. We leave it to the reader to check that this action is just a sign-change. □ □

Note that the correpondence is one–way; many permutations have the same difference vector, and there is no natural way of associating one permutation to a numbers game position. This observation is what makes our investigation worthwhile! The permutation encodes some extra information, which is not readily obtainable from the difference vector. After all, the permutation has one component more than the difference vector.

We give two examples of this correspondence for E_6, one with $c = 0$ in the expression for γ, the other with $c = -1$.

Example.
$345678 \overset{s_3}{\mapsto} 346578 \overset{s_6}{\mapsto} 124578$ corresponds to  .
Here we used $\gamma = \frac{1}{3}\sum x_i - x_1 - x_2 - x_3$, but we can redo it with $\gamma = \frac{1}{3}\sum x_i - x_1 - x_2 - x_3 - 1$.
$345678 \overset{s_3}{\mapsto} 346578 \overset{s_6}{\mapsto} 013578$ corresponds to  .

Our real interest lies in finding an integer representation by choosing an orbit consisting only of integer lattice points. From the numbers game perspective, it is clear that if the start vector has integer components and integer γ-value, then everything stays integer. It is also important that the orbit is faithful, i.e. that it does not double up on itself, and there is a useful numbers game result about that too.

Proposition 10. *A Coxeter group of type*  *has a faithful representation as a set of vectors in* $\mathbf{Z}^n$*, compatible with the group action described in Proposition 8.*

A sufficient condition for faithfulness is that some vector in the set has increasing integer components and a negative integer γ*-value.*

Proof. Starting the numbers game with negative values on all nodes ensures a bijective correspondence between game positions and group elements (Theorem 4, p 8). The theorem also states that all positions can be reached by playing nodes with negative values only, and we shall make use of that later. To see that some increasing integer vector has a negative integer γ-value, we use $(1, 2, \dots, n)$ and compute

$$\gamma = \frac{m-2}{m} \cdot \frac{n(n+1)}{2} - \frac{m(m+1)}{2} + c\ .$$

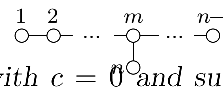
By choosing c appropriately, we can give γ any desired value, for instance $\gamma = -1$. We may have to use a noninteger c, but that is all right; no fractional numbers will ever creep into the vector. □ □

In all cases except the two affine groups $\tilde{E}_7$ and $\tilde{E}_8$, we can get rid of the constant c by translating the start vector, as in the proof of Proposition 8. Again, this may produce noninteger values as vector components, and in such cases, there are two ways to restore integrity. Either multiply all numbers by the least common denominator or replace $\gamma = -1$ by another negative integer value. The procedure is best clarified in a few examples.

Example. The group E_{10} characterized by $m = 3, n = 10$ has an integral representation containing the vector $(1, 2, \dots, 10)$ and with the corresponding $\gamma = -1$, namely for $c = -40/3$. One can eliminate c by translation $(1-d, 2-d, \dots, 10-d)$, where $d = mc/(m^2 - (m-2)n) = 40$, so the perfect start vector is $(-39, -38, \dots, -30)$.

The group E_{11} also has an integral representation containing $(1, 2, \ldots, 11)$ and with $\gamma = -1$, namely for $c = -17$. But here, to eliminate c, one must translate the vector by $d = 51/2$, i.e. to $(-24.5, -23.5, \ldots, -14.5)$. We can get rid of the fractions by doubling, thus starting with $(-49, -47, -45, \ldots, -29)$ instead. It is immediate by induction that all vector components stay odd, that their sum stays divisible by three and that γ stays even.

Alternatively, one can use $\gamma = -2$ and get $c = -18$, $d = 27$ and a consecutive startvector $(-26, -25, \ldots, -16)$. This would usually seem preferable, and our next proposition states that this technique is always available.

Proposition 11. *All Coxeter groups of type* 

, except those specified below, admit integer permutational representations with $c = 0$ *and such that the identity element is represented by an increasing sequence of consecutive integers. Exceptions are*
- *the affine groups* $\tilde{E}_7$ *and* $\tilde{E}_8$*,*
- *cases where* m *and* n *are even and* m *contains a higher power of two than* n *does.*

Proof. As in the above examples, we start with the vector $(1, 2, \ldots, n)$ and an *unspecified* negative integer γ-value. By putting together the formulas involving γ, c and d, we find the following expression for the translation d eliminating c:

$$d = \frac{(m-2)n(n+1)/2 - m^2(m+1)/2 - \gamma m}{(m-2)n - m^2} .$$

Usually, this value can be made integer for some suitable γ, in particular if the denominator and m are relatively prime, so we only have to consider the case where m and $(m-2)n$ have common factors.

Factors other than two are of no significance, as they must divide all three terms in the numerator. Also when n is odd and m even, the common factor two appears in all these three terms. We are left with the case when both m and n are even. Assume that 2^r is the highest power of two that divides n but that m is divisible by at least 2^{r+1}. Divide all terms by 2^r and note that now the first term of the numerator is odd while all other terms are even. The conclusion follows. □ □

For the ordinary signed permutations representation of D_n, it is evident that the expression $\sum x_i^2$ has the same value for all elements of the group, i.e. is *invariant*. The generalization of this quadratic invariant constitutes our next proposition.

Proposition 12. *The quadratic expression* $Q(x) = m^2 \sum x_i^2 - (m-2)(\sum x_i)^2 - 2mc \sum x_i$ *is invariant in the permutational representations of proposition 8. The quadric surface* $Q(x) =$ const. *is bounded (an ellipsoid) if* $m^2 > (m-2)n$*, otherwise unbounded. The bounded case occurs for the finite groups, i.e.* D_n, E_6, E_7, E_8*.*

Proof. It is an easy exercise to check that $Q(x)$ is invariant when γ is added to the first m components.

The eigenvalues of the quadratic form can be found and are $\lambda_1 = \ldots = \lambda_{n-1} = m^2$, $\lambda_n = m^2 - (m-2)n$, so the form is positive definite only when $m^2 > (m-2)n$, that is for combinations of m and n that produce the listed finite groups. □ □

6.2. **Representations of** D_n, E_6, E_7, E_8**.** For the finite groups, the condition for faithfulness given in Proposition 10 is in fact necessary, not only sufficient. But it is possible to state more explicit criteria that recognize vectors x, for which the group action on the orbit of x is faithful.

As a trivial example, consider the ordinary signed permutations representation for D_4 and look at the orbit of $x = (2, -1, 7, -2)$. Is it faithful? Certainly not, since after application of s_3 and s_2 we reach $(2, -2, -1, 7)$ which is left unchanged by s_4. How about $x = (0, 1, 2, 3)$? The zero component may seem jeopardous, but as seen from the following proposition, it is perfectly all right.

Proposition 13. *For D_n, the orbit of a vector x under the group action defined in Proposition 8 is faithful if and only if two criteria are fulfilled.*

(1) *No two x_i are equal.*
(2) *No two x_i sum to c.*

For E_6 and E_7, there are three criteria, and a fourth one for E_8.

(1) *No two x_i are equal.*
(2) *No three x_i sum to $\frac{1}{3}\sum x_j + c$.*
(3) *No six x_i sum to $\frac{2}{3}\sum x_j + 2c$.*
(4) *No x_i equals $3c$. (E_8 only)*

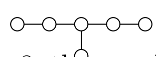
Proof. A necessary provision for faithfulness is that the x_i are different from each other, otherwise some product of the s_i corresponding to a transposition of the two equal numbers will leave x unchanged.

Also, we must have $\gamma \neq 0$, otherwise s_n will leave x unchanged. Thus, the necessity of conditions 1 and 2 is clear.

The third condition for the E_n-case emerges if one applies s_n to a vector with $x_1 + \cdots + x_6 = \frac{2}{3}\sum x_j + 2c$, for afterwards $x_4 + x_5 + x_6$ will violate the second condition (with the updated value of $\sum x_j$). The necessity of the fourth condition for E_8 is also clear, for after application of s_n to a vector with $x_1 = 3c$, the new value of $x_1 + x_4 + x_5 + x_6 + x_7 + x_8$ will violate the third condition.

This set of necessary conditions is invariant under further application of $s_1, \ldots, s_{n-1}$ (trivial) and s_n (takes some checking). That is at least an indication of the fact that they are also sufficient. The length function, defined in section 6.4 below, will give a simple proof of this, so we shall postpone the sufficiency proof until then. □ □

In each orbit that represents the group, as described in the proposition, there is exactly one vector with increasing x_i and negative γ. This fact is also a consequence of the existence of a length function, and will be proved in section 6.4. From now on, we assume that this vector represents the identity e. According to Proposition 10, each such e-vector also satisfies the conditions of Proposition 13.

In most cases, such e-vectors also have positive components. The exception is E_6, and to deal with that, a digression on the *graphic automorphism* is called for. This is the group automorphism induced by the reflection map of the symmetric graph  , that is $s_1 \leftrightarrow s_5$, $s_2 \leftrightarrow s_4$. In the matrix representation on $\mathbf{R}^6$ defined in Proposition 8, the graphic automorphism corresponds to a certain inner automorphism $W \mapsto UWU^{-1}$.

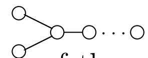
Proposition 14. *Let U be the 6×6-matrix defined by*

$$U_{ij} = \begin{cases} -2/3 & \text{if } i+j=7 \\ 1/3 & \text{otherwise} \end{cases} .$$

The inner automorhism $W \mapsto UWU^{-1}$ induces the graphic automorphism on the E_6 matrix group, defined by setting $c = 0$ in Proposition 8. If $e = (e_1, \ldots, e_6)$ satisfies the criteria in Proposition 13, then so does Ue.

Proof. Let S_i be the matrix representing s_i, e.g. $S_1 = \begin{pmatrix} 0 & 1 & \cdots \\ 1 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$. Switching the first two rows of U means the same as switching the last two columns, therefore $S_1U = US_5$. For the same reason, $S_2U = US_4$ and $S_3U = US_3$. Finally, computation shows that $S_6U = US_6$.

A faithful orbit is mapped onto a faithful orbit by the matrix U. In fact, the vector We representing a group element w maps to $UWe = (UWU^{-1})Ue$, so Ue complies with the criteria of proposition 13. □ □

The D_n-graphs are also symmetric if drawn like this  and there is a corresponding graphic automorphism, defined by $s_1 \leftrightarrow s_n$. The analog of the previous proposition is almost trivial, but stated for reference.

Proposition 15. *Let U be the $n \times n$ diagonal matrix defined by*

$$U_{ii} = \begin{cases} -1 & \text{if } i = 1 \\ 1 & \text{otherwise} \end{cases} .$$

The inner automorhism $W \mapsto UWU^{-1}$ induces the graphic automorphism on the D_n matrix group, defined by setting $c = 0$ in Proposition 8. If $e = (e_1, \dots, e_n)$ satisfies the criteria in Proposition 13, then so does Ue.

Proof. As the proof of the previous proposition. □ □

Our next goal is a characterization of the vector set corresponding to a fixed such e-vector. For simplicity, we assume that the constant c has been eliminated.

Proposition 16. *Let the identity element of one of the finite Coxeter groups under consideration correspond to an increasing vector $(e_1, \dots, e_n)$ with negative γ. For E_6 and D_n, we also assume that the first component of the graphically automorphic vector Ue is less than or equal to e_1 (otherwise Ue is used instead of e). Then,*

- *all components of e are positive,*
- *all vectors in the e-orbit have all components in the interval $[-e_n, e_n]$.*

Proof. To begin with, all e_i are positive in E_7, for in $\gamma = \frac{1}{3}\sum e_j - e_1 - e_2 - e_3 < 0$, we can use $\frac{e_2+e_3+e_4}{3} > e_2$ and $\frac{e_5+e_6+e_7}{3} > e_3$ and conclude that $e_1 > 0$. The same computation and the same conclusion holds for E_8. In E_6, the first component e_1 may be negative, but then $(Ue)_1 = \frac{1}{3}\sum e_j - e_6 = e_1 + e_2 + e_3 + e_4 + e_5 - \frac{2}{3}\sum e_j > e_4 + e_5 - e_1 - e_2 - e_3 > 0$, so Ue is positive. In D_n, $\gamma < 0$ means $e_1 + e_2 > 0$, so if e_1 is negative, Ue will be a positive, increasing vector.

For the signed permutations representation of D_n, the second statement is trivial. For E_6, E_7, E_8, it can be proved by induction, but only if we strengthen it by adding two more statements for E_6, E_7 and a third one for E_8. The strengthened hypothesis states that for all $i \le n$,

(1) $|x_i| \le e_n$,
(2) $|\frac{1}{3}\sum x_j - x_{i_1} - x_{i_2}| \le e_n$,
(3) $|\frac{2}{3}\sum x_j - x_{i_1} - x_{i_2} - x_{i_3} - x_{i_4} - x_{i_5}| \le e_n$,
(4) $|x_i - x_j| \le e_n$ (E_8 only).

If we can verify that these conditions are true for $x = e$, and that they stay true (i.e. are invariant under all s_i), the proposition will follow.

Monotonicity and $\frac{1}{3}\sum e_j - e_1 - e_2 < e_3$ imply $\frac{1}{3}\sum e_j - e_{i_1} - e_{i_2} < e_n$, which is the second inequality except for the absolute value sign. The other half, $\frac{1}{3}\sum e_j - e_{i_1} - e_{i_2} > -e_n$, is easier doubled: $-\frac{2}{3}\sum e_j + 2e_{i_1} + 2e_{i_2} - 2e_n < -\frac{2}{3}\sum e_j + e_{i_1} + e_{i_2} < -\frac{2}{3}\sum e_j + e_4 + \cdots + e_n = \frac{1}{3}\sum e_j - e_1 - e_2 - e_3 < 0$

The first half of the third inequality is $\frac{2}{3}\sum e_j - e_{i_1} - \cdots - e_{i_5} \le \frac{2}{3}\sum e_j - e_1 - \cdots - e_5 < \frac{1}{3}\sum e_j - e_4 - e_5 < e_n$. For E_7, E_8, the other half follows from $\frac{2}{3}\sum e_j - e_{i_1} - \cdots - e_{i_5} \ge e_1 + e_2 - \frac{1}{3}\sum e_j > -e_3 > -e_n$ and for E_6 it is just another way of writing $(Ue)_1 = \frac{1}{3}\sum e_j - e_6 \le e_1$.

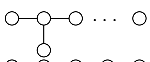
Finally, the fourth condition is evident, all e_i being positive.

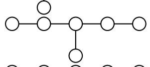
Assuming that these conditions hold for some vector x, we must prove that they stay true after application of s_i. It is obvious that the conditions are invariant under permutations, so we can concentrate on the s_n-action. Let $x \mapsto x'$ under s_n, i.e. $x'_i = \begin{cases} x_i + \gamma & \text{if } i \le 3 \\ x_i & \text{otherwise} \end{cases}$. We must show that, for example, $|x_1 + \gamma| \le e_n$, but this can be written as $|\frac{1}{3}\sum x_j - x_2 - x_3| < e_n$, which is the second condition for x exactly. All four conditions for x' are equivalent to some or other of the same four conditions for x, in fact, this is how these conditions have been constructed. Depending on whether both, one or none of i_1, i_2 are in $\{1,2,3\}$, condition two for x' turns into condition one, two or three for x. And depending on whether three or two of $i_1, \ldots, i_5$ are in $\{1,2,3\}$, condition three for x' turns into condition two or three. For E_8, there are also the possibilities of one or none of the $i_1, \ldots, i_5$ belonging to $\{1,2,3\}$ and these correspond to condition four or three for x. For E_7 with exactly one of $i_1, \ldots, i_5$ in $\{1,2,3\}$, we get condition one for x. Finally, condition four turns into condition three, four or three, depending on whether both, one or none of i, j belong to $\{1,2,3\}$. □ □

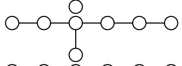
Remark 17. The resemblance between these four numbered inequalities and the four criteria in proposition 13 is no coincidence, but a consequence of their invariance under the s_i. Specifically, the invariance under s_n of the first criterion means $x_i' \neq x_j'$, so that, for example, $x_1 + \gamma \neq x_4$. In this way, the second criterion appears; the invariance of the second criterion produces the third criterion etc. The inequalities of the proof come up in the same way.

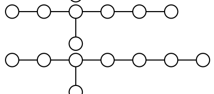
The most natural choice of permutational representation of the identity element e seems to be an increasing sequence of consecutive positive integers, chosen as small as possible. The corresponding vector sets are completely characterized in the next proposition. As before, we state the result only for $c = 0$.

Proposition 18. *The following choices of representing vector for the identity element are minimal among increasing sequences of consecutive integers.*

D_n:  $e = (0, 1, 2, \ldots, n-1)$

E_6:  $e = (3, 4, 5, 6, 7, 8)$

E_7:  $e = (9, 10, 11, 12, 13, 14, 15)$

E_8:  $e = (22, 23, 24, 25, 26, 27, 28, 29)$

The representation involves exactly those integer vectors $(x_1, \ldots, x_n)$ that satisfy the inequalities of Proposition 13 and give the quadratic form in Proposition 12 the same value as the e-vector does.

Proof. The statement about D_n is almost evident. For the other groups, the proof of the pudding is a computer check. A program ran through and counted all combinations of integers satisfying the criteria. For any legal integer vector, all permutations of it are legal too, so we only had to count vectors with increasing components. The totals obtained were 72, 576 and 17280, in agreement with the table on page 10, namely $|E_6| = 72 \cdot 6!$, $|E_7| = 576 \cdot 7!$, $|E_8| = 17280 \cdot 8!$.

It would be nice to be able to give a mathematical explanation of the fact that all integer lattice points on a certain ellipsoid and outside certain hyperplanes belong to one and the same orbit. As things stand, however, we must admit the possibility that chance, rather than mathematical necessity, is responsible. □ □

It is interesting to see how the parabolic subgroup relations $E_8 \supset E_7 \supset E_6 \supset D_5$ are reflected in these representations. Fix the last component of the E_8-vector e and consider only the E_7-action. Now the γ-expression for E_8 can be interpreted as

$$\gamma = \frac{1}{3}\sum_1^8 x_i - x_1 - x_2 - x_3 = \frac{1}{3}\sum_1^7 x_i - x_1 - x_2 - x_3 + c, \text{ where } c = e_8/3 \text{ is constant.}$$

So the subgroup E_7 gets a representation of our type. Note also that a coset xE_7 consists of the vectors reachable from x by E_7-action only, i.e. fixing the x_8-component. Since $|E_8 : E_7| = 240$ (see Humphreys, [56], p 44), we can anticipate that there are 240 different values occuring as x_8 of a vector in the E_8-representation. Similar conclusions follow from $|E_7 : E_6| = 56$ and $|E_6 : D_5| = 27$. All these numbers reappear in Proposition 19.

The other embedding of D_5 as the subgroup $[s_2, s_3, s_4, s_5, s_6] \subset E_6$, corresponds to the reinterpretation

$$\gamma = \frac{1}{3}\sum_1^6 x_i - x_1 - x_2 - x_3 = -x_2 - x_3 + c, \text{ where } c = \frac{1}{3}\sum_1^6 x_i - x_1$$

This makes sense only if the expression for c is constant in D_5, and, in fact, it is. Transpositions $s_2, \ldots, s_5$ do not change it, and s_6 adds $\frac{1}{3}(\gamma+\gamma+\gamma) - \gamma$ to c.

6.3. **Interpretation of D_n, E_6, E_7, E_8 as subgroups of S_k.** The signed permutations model of D_n using $e = (1, 2, \ldots, n)$ admits an interpretation as a subgroup of S_{2n}. Let e' be the antisymmetric vector $(-n, \ldots, -1, 1, \ldots, n)$, then every signed permutation x induces an ordinary permutation $(-x_n, \ldots, -x_1, x_1, \ldots, x_n)$ of e'. The action of s_1 now transposes not only x_1 and x_2 but also $-x_2$ and $-x_1$ and similarly for s_2 to s_{n-1}. The action of s_n is a transposition of $-x_2$ and x_1 and also of $-x_1$ and x_2. In this way, D_n is interpreted as the subgroup of all antisymmetric permutations of e', but of course, such a permutation is completely specified by its right half.

If instead one had used $e = (0, 1, 2, \ldots, n-1)$, the set would have contained two different zeroes. Still, looking through the right half window one would be able to identify the whole permutation, and in particular decide which of the zeroes is visible in the window.

It is useful to have similar interpretations of the permutational representations of E_6, E_7, E_8, and here they are. For the sake of simplicity, they are stated only for the case $c = 0$.

A word of caution: when the proposition states “a total of 27 values”, it means that 27 different linear expressions in the e_i occur. For special choices of the e_i, some of these values may coincide, just as in the case of two different zeroes above. As an example, $e = (3, 4, 5, 6, 7, 8)$ gives rise to seventeen different values only, namely $-8, \ldots, 8$, but out of these, $-4, \ldots, 4$ come in two flavours and 0 even in a third flavour. The resulting homomorphism $E_6 \to S_{17}$ is not one-to-one, so different flavours should be distinguished. For the purpose of finding embeddings in S_k, one can as well assume that the e_i are rationally independent or even that they belong to a transcendental field extension $\mathbf{R}(e_1, e_2, \ldots)$.

Proposition 19. *All component values in the permutational representations considered are given by the following expressions in the e-components.*

- E_6 : *All six e_i, all fifteen $(\frac{1}{3}\sum e_j - e_{i_1} - e_{i_2})$ for any two e_i, and all six $(\frac{2}{3}\sum e_j - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5})$ for any five e_i. A total of 27 values.*
- E_7 : *All fourteen $\pm e_i$, all forty-two $\pm(\frac{1}{3}\sum e_j - e_{i_1} - e_{i_2})$ for any two e_i. In all, 56 values.*
- E_8 : *All sixteen $\pm e_i$, all fifty-six $\pm(\frac{1}{3}\sum e_j - e_{i_1} - e_{i_2})$ for any two e_i, all one hundred and twelve $\pm(\frac{2}{3}\sum e_j - e_{i_1} - e_{i_2} - e_{i_3} - e_{i_4} - e_{i_5})$ for any five e_i, and all fifty-six $(e_{i_1} - e_{i_2})$ for any two e_i. Altogether 240 values.*

Proof. Let the linear expressions in the proposition be represented by row vectors, with $(1, -1, 0, \ldots)$ meaning $e_1 - e_2$ etc. Originally, the only existing values are the e_i, so we start with the vectors $u_1^\top = (1, 0, \ldots)$, $u_2^\top = (0, 1, 0, \ldots)$ etc. Since $s_1, \ldots, s_{n-1}$ only permute $e_1, \ldots, e_n$, we can concentrate on the action of s_n, that can produce three new values. After the first application of s_n, these are $u_1^\top S_n e$, $u_2^\top S_n e$ and $u_3^\top S_n e$. The first value is $e_1 + \gamma = \frac{1}{3}\sum e_j - e_2 - e_3$, but we can get any value of the type $\frac{1}{3}\sum e_j - e_{i_1} - e_{i_2}$ as $u_1^\top S_n S_{k_1} S_{k_2} \cdots S_{k_r} e$, with appropriate transpositions S_k, and that explains those expressions in the proposition.

This algorithm is iterative. Whenever $v^\top$ is one of our row vectors, so is $v^\top S_n$ and all its permutations. In practice, no matrix calculations are needed, for S_n acts very simply on expressions:

$$\boxed{\begin{array}{l} e_i \mapsto e_i + \gamma \text{ if } i = 1,2,3 \\ e_i \mapsto e_i \text{ otherwise} \\ \gamma \mapsto -\gamma \\ \frac{1}{3}\sum e_j \mapsto \frac{1}{3}\sum e_j + \gamma \end{array}}$$

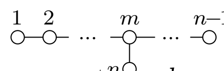
As always, γ denotes $\frac{1}{3}\sum e_j - e_1 - e_2 - e_3$.

Let us apply these rules to $\frac{1}{3}\sum e_j - e_{i_1} - e_{i_2}$. First assume that $i_1 \le 3, i_2 > 3$. The γ from the sum and the γ from e_{i_1} cancel, so the expression is left intact. Then assume that both $i_1, i_2 \le 3$. The expressions is changed into $\frac{1}{3}\sum e_j - e_{i_1} - e_{i_2} - \gamma = e_{i_3}$, an old value. Finally, assume both $i_1, i_2 > 3$. The expressions turns into $\frac{1}{3}\sum e_j + \gamma - e_{i_1} - e_{i_2} = \frac{2}{3}\sum e_j - e_1 - e_2 - e_3 - e_{i_1} - e_{i_2}$, and all possible permutations hereof.

The analysis ends here for E_6, but must be iterated a few more times for E_7 and E_8. □ □

The proposition provides an embedding $E_6 \subset S_{27}$, that can be interpreted as the incidence-preserving permutations of the famous 27 lines on the general cubic surface in projective space (see the Atlas of finite groups [25] under $U_4(2) \cong S_4(3)$). For E_8, the expressions come in plus-minus pairs and the actions of all s_i commute with sign inversion. Therefore, the usual signed permutations model of B_n defines an embedding $E_8 \subset B_{120} \subset S_{240}$. The same thing is true for E_7, but here we even have $E_7 \subset D_{28} \subset S_{56}$. That is because all s_i transform an even number of positive expressions into negative expressions.

6.4. **The length function for D_n, E_6, E_7, E_8.** The length $l(w)$ of an element of any Coxeter group (W, S) is defined as the length of a reduced expression for w as a product of the generators s_i. For S_n, the length of an element is the number of inversions in the corresponding permutation. The *descent set* of a permutation is defined as $\{i \mid x_i > x_{i+1}\}$ and this notion is generalized to any Coxeter group in the following way. The product of an element w by a generator s has length $l(w) \pm 1$ and the descent set $D(w)$ consists of the generators that shorten w.

Proposition 20. *For a Coxeter group of type* 

, represented as in Proposition 10 with an integer e-vector with increasing components and negative γ, the descent set of an element corresponding to a vector x includes all s_i such that $x_i > x_{i+1}$ and s_n if the value of γ is positive.

Proof. Restated in the terminology of the numbers game, only nodes with negative numbers are legal in play sequences corresponding to reduced expressions ([42], p 81). □ □

Finite Coxeter groups have a unique element w_o of maximal length. Which vectors represent this w_o for D_n, E_6, E_7, E_8?

Proposition 21. *If the identity is represented by the vector $(e_1, e_2, \ldots)$, then the element of maximal length will be represented as*

D_n: $w_o = (w_1, c - e_2, c - e_3, \ldots)$, *where* $w_1 = \begin{cases} c - e_1 & \text{if } n \text{ is even} \\ e_1 & \text{if } n \text{ is odd} \end{cases}$
E_6: $w_o = (e_6 + b, e_5 + b, \ldots)$, *where* $b = 2c - \frac{1}{3}\sum e_i$
E_7: $w_o = (3c - e_1, 3c - e_2, \ldots)$
E_8: $w_o = (6c - e_1, 6c - e_2, \ldots)$

Proof. The statement about D_n is well-known in the case $c = 0$ and the constant can always be disposed of by a translation of the origin. In fact, let $(x'_1, x'_2, \ldots) = (x_1 - d, x_2 - d, \ldots)$, where d is chosen such that $(m^2 - (m-2)n)\, d = mc$. Then the group action on the x'-coordinates is the same as before, except that the constant has disappeared from γ. With $m = 2$ we get $d = c/2$, so $x'_i = x_i - c/2$ changes sign to $-x_i + c/2$ in w'_o, which corresponds to $c - x_i$ in w_o.

The verification of the w_o-formulas for E_6, E_7, E_8 are best left to the computer. It is sufficient to check the case $c = 0$, for translation of the origin works as above. □ □

We can now derive formulas for $l(w)$ in the finite Coxeter groups considered. We use the shorthand $\#(x_{i_1} + x_{i_2} + x_{i_3} < \frac{1}{3}\sum x_j + c)$ for the number of three-subsets of the x_i which sum to less than $\frac{1}{3}\sum x_j + c$ and $\#(\text{inversions})$ for the number of pairs $i < j$ with $x_i > x_j$.

Proposition 22. *Let $(x_1, \ldots, x_n)$ represent a group element w and assume that the e-vector has been chosen with increasing components and negative γ.*

In D_n, the length of w is given by $l(w) = \#(\text{inversions}) + \#(x_{i_1} + x_{i_2} < c)$.
In E_6 and E_7, the formula is
$l(w) = \#(\text{inversions}) \ + \ \#(x_{i_1} + x_{i_2} + x_{i_3} < \frac{1}{3}\sum x_j + c) \ + \ \#(x_{i_1} + \cdots + x_{i_6} < \frac{2}{3}\sum x_j + 2c)$.
For E_8, the expression contains one more term, $l(w) = \ldots + \#(x_i < 3c)$.

Proof. One way to prove the formula is to show that it changes by ± 1 when w is multiplied by s_i. To complete the proof, it is then sufficient to check that $l(e) = 0$ and that $l(w_o)$ is correct. In that case, our l-expression must also be correct for any initial segment of any reduced expression $s_{i_1}, \ldots, s_{i_k} = w_o$, and a reduced expression for any w is an initial segment of some such w_o-expression.

The assumptions about the e-vector forces $l(e)$ to be zero. In D_n, we have $\gamma = c - e_1 - e_2 < 0$ and since e is increasing, all $e_{i_1} + e_{i_2} > c$. The same argument gives $e_{i_1} + e_{i_2} + e_{i_3} > \frac{1}{3}\sum e_j + c$ in E_6, E_7, E_8. Adding the inequality $e_{i_4} + e_{i_5} + e_{i_6} > \frac{1}{3}\sum e_j + c$, we see that the contribution of the third term in $l(e)$ is zero too. Finally, adding $e_{i_7} + e_{i_8} + e_i > \frac{1}{3}\sum e_j + c$, we see that the extra term in E_8 does not contribute either.

Applying the formulas to our w_o-expressions, we get maximal contribution of all terms:

$$\boxed{\begin{array}{l} D_n\!: \ l(w_o) = n(n-1)/2 + n(n-1)/2 = n(n-1) \\ E_6\!: \ l(w_o) = 15 + 20 + 1 = 36 \\ E_7\!: \ l(w_o) = 21 + 35 + 7 = 63 \\ E_8\!: \ l(w_o) = 28 + 56 + 28 + 8 = 120 \end{array}}$$

From the theory of root systems, it is clear that $l(w_o)$ is the number of positive roots, so that $2l(w_o) = \#(\text{roots})$. Our results are consistent with the table of the number of roots in all finite Weyl groups, that can be found in Humphreys, [56] p 44.

The remaining step in the proof is showing that the expression changes by ± 1 when w is multiplied by s_i. For all transpositions, $s_1, \ldots, s_{n-1}$, the only change is an inversion more or less, but for s_n things are more complicated. The γ-expression is negative when s_n is applied and positive afterwards, and that contributes $+1$ to the second term. Many other changes may occur, however, and we have to check that these always cancel out. Let us look at an example in E_6.

The element w represented by $(5, 4, 3, -2, -3, -4)$ has $l(w) = 15 + 10 + 0 = 25$. The γ-value is -11, so ws_6 corresponds to $(-6, -7, -8, -2, -3, -4)$ with $l(ws_6) = 6 + 19 + 1 = 26$. Nine of the fifteen inversions have changed into second term contributions and one of the second term contributions has turned into a third term contribution. A disappearing inversion means that, for instance, $x_1 > x_4$ but $x_1 + \gamma < x_4$. Here x_1 may as well be x_2 or x_3 and x_4 may as well be x_5 or x_6. The second inequality is equivalent to $x_2 + x_3 + x_4 > \frac{1}{3}\sum x_j + c$, meaning that $x_2 + x_3 + x_4$ did not contribute to the second term before, and the first inequality is equivalent to $x_2 + \gamma + x_3 + \gamma + x_4 < \frac{1}{3}\sum x_j + \gamma + c$, meaning a second term contribution after application of s_6.

One more case for E_6, E_7 and another one for E_8 has to be checked in the same way. We leave out these details. For D_n, there are no complications, so that concludes the proof. $\square$ $\square$

We have stated some rather complicated formulas with no motivation. The proof, using verification by manipulation, was not very illuminative either. Also, a comparison of the formulas for $l(w)$ with the inequalities in Proposition 13 makes it reasonable to ask for a proof that provides some insight into what is going on here. Our next section aims to present such an approach.

6.5. **Geometric interpretation of the formulas.** The connection between propositions 13 and 22 is due to the fact (Theorem 2) that every Coxeter group can be realized as a group generated by (skew) reflections in $\mathbf{R}^n$. The reflections correspond in turn to some hyperplane arrangement, dividing space into cones, where the cones correspond to the elements of the group. Our representation picks one interior point from each cone and the inequalities in Proposition 13 state that the point must not lie on any hyperplane in the arrangement. For the finite groups, the invariant positive definite quadratic form means that all points in the orbit lie on an ellipsoid.

Seen in this light, Proposition 18 states that the cell decomposition of the ellipsoid surface induced by the hyperplane arrangement contains exactly one lattice point in each $(n-1)$-dimensional cell.

Let us find the geometric content of the subgroup relations $E_8 \supset E_7 \supset E_6 \supset D_5$ considered at the end of section 6.2. Fixing the last component of an E_8-vector x and considering only the E_7-action, means intersecting the ellipsoid with a hyperplane $x_8 = \text{const}$. The intersection is of course an ellipsoid surface in one dimension less. This is straightforward. Considering D_5 as the parabolic subgroup $[s_2, s_3, s_4, s_5, s_6]$ in E_6, does not mean fixing the first component x_1, for s_6 certainly affects x_1. But, as we found, the expression $\frac{1}{3}\sum_1^6 x_i - x_1$ is invariant under $s_2, \ldots, s_6$, and so, the corresponding hyperplane $\frac{1}{3}\sum_1^6 x_i - x_1 = \text{const.}$ intersects the E_6-ellipsoid in a D_5-ellipsoid.

Proposition 22 equates the length $l(w)$ with the number of hyperplanes separating the points representing e and w. This equality is valid not only for finite Coxeter groups, but for all linear and affine representations such that all reflection elements ws_iw^{-1} have different hyperplanes. The treatment of the affine case in Humphreys [56], p 91, is general enough to show this.

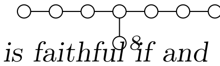
For the affine groups $\tilde{E}_7$ and $\tilde{E}_8$, the quadratic form in Proposition 12 is positive semidefinite, so the quadric surface will be a circular cylinder if $c = 0$ and a rotation paraboloid if $c \neq 0$. The axis direction is the eigenvector $(1, 1, \ldots, 1)$ corresponding to the eigenvalue zero. As we will see below, for $c \neq 0$, the hyperplanes are all different and the length formula valid. For $c = 0$, however, infinitely many reflection elements have the same hyperplane, and thus we get no simple length formula. A pity, for the cylindric representation is attractive in other respects.

In the proof of Proposition 8, we noted that it is possible to transform the affine reflection s_n into a linear reflection by projecting the paraboloid in its axis direction onto the hyperplane $x_1 + \cdots + x_n = -\frac{nc}{m}$. If we wish, we can then get a representation in dimension $n-1$, for instance by simply suppressing the last co-ordinate. The disadvantage is that s_{n-1} cannot be interpreted as a transposition any more. Instead, it becomes an affine reflection. For completeness, this slimmed–down representation will be stated in the next section, but it is not a permutational representation in our sense.

Projection along the cylinder axis obviously would make the structure collapse, so in the case $c = 0$, the dimension cannot be lowered.

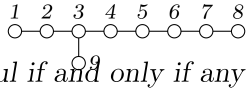
For other infinite groups, the set of hyperplanes is more complicated, but in principle the same method can produce length formulas. Starting with the hyperplanes of type $x_i = x_{i+1}$ and $x_1 + \cdots + x_m = \frac{m-2}{m}\sum x_i + c$, and letting co-ordinate permutations and s_n act on them over and over again will produce the complete hyperplane arrangement. The length of the element represented by x is the number of these hyperplanes separating it from the e-vector.

6.6. **Representations of $\tilde{E}_7$ and $\tilde{E}_8$.** For the affine groups, elimination of the constant c by a change of origin in the co-ordinate system is neither possible nor desirable. However, the next proposition remains true for $c = 0$ if the congruences modulo $2c$ or $3c$ are replaced by ordinary equality.

Proposition 23. *For the group $\tilde{E}_7$:* 

, the permutational representation using $\gamma = \frac{1}{2}\sum x_i - x_1 - x_2 - x_3 - x_4 + c$ is faithful if and only if any vector $x = (x_1, \ldots, x_8)$ in the orbit satisfies two conditions :

1. *No two x_i are congruent modulo $|2c|$.*
2. *No sum of four x_i is congruent to $(\frac{1}{2}\sum x_j + c)$ modulo $|2c|$.*

For $\tilde{E}_8$: 

, the permutational representation using $\gamma = \frac{1}{3}\sum x_i - x_1 - x_2 - x_3 + c$ is faithful if and only if any orbit vector $x = (x_1, \ldots, x_9)$ satisfies two conditions :

1. *No two x_i are congruent modulo $|3c|$.*

(2) *No sum of three x_i is congruent to $(\frac{1}{3}\sum x_j + c)$ modulo $|3c|$.*

Proof. Exactly as in the proof of Proposition 13, one finds that this is the set of affine hyperplanes obtained by iterated reflections of the hyperplanes corresponding to the simple reflections s_i. In $\tilde{E}_7$, for example, application of s_8 to the hyperplane $x_4 = x_5 + 2kc$ gives $x_4 + \frac{1}{2}\sum x_j - x_1 - x_2 - x_3 - x_4 + c = x_5 + 2kc$. But this is a case of the congruence $x_1 + x_2 + x_3 + x_5 \equiv \frac{1}{2}\sum x_j + c \pmod{|2c|}$.

The discussion in the previous section and in Proposition 10 applies. The vector representing the identity e cannot reappear as the vector representing some other element w, for these two vectors are separated by exactly $l(w)$ of the hyperplanes in the proposition. That argument is of course valid only when $c \neq 0$. For $c = 0$, we get a finite number of hyperplanes only, all intersecting along the $(1, 1, 1, \ldots)$-ray, the axis of a circular cylinder containing the orbit. The cylinder is divided into finitely many infinitely long slices by the hyperplane wedges, and each slice contains an infinite number of orbit points.

A proof of the proposition for this case can be based upon the theory of looping numbers games in Eriksson [42], p 59. Looping games occur only on the affine Coxeter graphs, these graphs have adjacency matrices with largest eigenvalue 2 and a corresponding eigenvector with only positive components. For $\tilde{E}_7$ it is $(1, 2, 3, 4, 3, 2, 1, 2)$ and for $\tilde{E}_8$ it is $(2, 4, 6, 5, 4, 3, 2, 1, 3)$. A looping game has a *shot vector* that is a multiple of this eigenvector, the kth component of the shot vector being the sum of the numbers fired from node k. In terms of the permutational representation, s_1 fires $x_1 - x_2$, s_2 fires $x_2 - x_3$ etc and s_n fires γ. The shot vector's nth component is the sum of the γ-values fired in the course of the game.

Now, consider the change in $\sum x_j$ caused by an s_i. It is only s_n that makes the sum increase by $m\gamma$. The total increase is m times the nth component of the shotvector, i.e. a nonzero increase. But that demonstrates that we are not back where we started. Although the differences $x_i - x_{i+1}$ are the same as before, the sum has increased, so the vector has been translated some distance along the cylinder axis $(1, 1, 1, \ldots)$. □ □

The subgroup relations $E_7 \subset \tilde{E}_7$ and $E_8 \subset \tilde{E}_8$ appear in the following way. Fix x_1 in the $\tilde{E}_7$–representation (with $c = 0$) and let $[s_2, \ldots, s_8]$ be the parabolic subgroup isomorphic to E_7. The γ-expression can be reinterpreted as

$$\gamma = \frac{1}{2}\sum_{1}^{8} x_i - x_1 - x_2 - x_3 - x_4 = \frac{1}{3}\sum_{2}^{8} -x_2 - x_3 - x_4 + c, \text{ where } c = \frac{1}{6}\sum_{2}^{8} x_i - \frac{1}{2}x_1 \text{ is constant.}$$

Also, the inequalities for E_8 in Proposition 13 follow directly from the inequalities for $\tilde{E}_8$ (with $c = 0$) by fixing x_8 and putting $3c = x_8$.

The natural representation of the identity as an increasing sequence of consecutive integers is not possible if $c = 0$, by the second condition.

Proposition 24. *Simple permutational representations of $\tilde{E}_7$ and $\tilde{E}_8$ are given by*

$\tilde{E}_7$: $e = (1, 2, 3, 4, 5, 6, 7, 8)$ *and* $c = -9$
or $e = (1, 2, 3, 4, 5, 6, 7, 17)$ *and* $c = 0$
$\tilde{E}_8$: $e = (1, 2, 3, 4, 5, 6, 7, 8, 9)$ *and* $c = -10$
or $e = (1, 2, 3, 4, 5, 6, 7, 8, 30)$ *and* $c = 0$

For the representations with $c \neq 0$, the descent set of an element is determined by the inversions and by the sign of γ, as stated in Proposition 20.

Proof. All node values of the corresponding numbers game are negative. □ □

The following conjecture can be computer tested for $c = 0$, for the orbit points on the cylinder are periodic, but we have disdained to do so – there should be some mathematical proof hiding in the undergrowth.

Conjecture 25. The representation involves exactly those integer vectors $(x_1, \ldots, x_n)$ that satisfy the inequalities of Proposition 23 and give the quadratic form in Proposition 12 the same value as the e-vector does.

We also have formulas for $l(w)$ in the natural representations with $c \neq 0$. Note our convention that every pair x_{i_1}, x_{i_2} generates one term in the sum $\sum \left\lfloor \frac{|x_{i_1}-x_{i_2}|}{|2c|} \right\rfloor$.

Proposition 26. *Let $(x_1, \ldots, x_n)$ represent a group element w, in the representations of Proposition 24 with $c \neq 0$.*
In $\tilde{E}_7$, the length of w is given by

$$\#(\text{inversions}) + \#(x_{i_1}+x_{i_2}+x_{i_3}+x_{i_4} < \tfrac{1}{2}\sum x_j + c) + \sum \left\lfloor \frac{|x_{i_1}-x_{i_2}|}{|2c|} \right\rfloor + \sum \left\lfloor \frac{\left|\frac{1}{2}\sum x_j + c - x_{i_1} - x_{i_2} - x_{i_3} - x_{i_4}\right|}{|2c|} \right\rfloor$$

In $\tilde{E}_8$, the formula is

$$\#(\text{inversions}) + \#(x_{i_1}+x_{i_2}+x_{i_3} < \tfrac{1}{3}\sum x_j + c) + \sum \left\lfloor \frac{|x_{i_1}-x_{i_2}|}{|3c|} \right\rfloor + \sum \left\lfloor \frac{\left|\frac{1}{3}\sum x_j + c - x_{i_1} - x_{i_2} - x_{i_3}\right|}{|3c|} \right\rfloor$$

Proof. The formulas evidently count the number of hyperplanes separating e and w, so the discussion in section 6.5 gives the proof. □ □

Remark 27. In analogy with the finite case (see section 6.3), these affine groups can be embedded in S_∞, the group of bijections on $\mathbf{Z}$. In this way, the linear algebra flavour of the representations is completely replaced by a combinatorial setting. We postpone this line of investigation until section 10, where all affine Coxeter groups will be reconsidered.

For completeness, we shall also state the analogous results for the slimmed–down representations, mentioned at the end of the previous section.

Proposition 28. *The group $\tilde{E}_7$ has a representation on $\mathbf{R}^7$ with $s_1, \ldots, s_6$ acting as transpositions and*

$$\begin{aligned} s_7 : (x_1, \ldots, x_7) &\mapsto (x_1, \ldots, x_6, -(2c + x_1 + \ldots + x_7)), \\ s_8 : (x_1, \ldots, x_7) &\mapsto (x_1 + \frac{\gamma}{2}, \ldots, x_4 + \frac{\gamma}{2}, x_5 - \frac{\gamma}{2}, x_6 - \frac{\gamma}{2}, x_7 - \frac{\gamma}{2}), \end{aligned}$$

where $\gamma = -x_1 - x_2 - x_3 - x_4$. It is faithful if and only if any vector $x = (x_1, \ldots, x_7)$ in the orbit satisfies three conditions :

1. *No two x_i are congruent modulo $|2c|$.*
2. *No sum of four x_i is congruent to zero modulo $|2c|$.*
3. *No x_i is congruent to $-\sum x_j$ modulo $|2c|$.*

For $\tilde{E}_8$, the representation on $\mathbf{R}^8$ with $s_1, \ldots, s_7$ acting as transpositions and

$$\begin{aligned} s_8 : (x_1, \ldots, x_8) &\mapsto (x_1, \ldots, x_7, -(3c + x_1 + \ldots + x_8)) \\ s_9 : (x_1, \ldots, x_8) &\mapsto (x_1 + \frac{2\gamma}{3}, x_2 + \frac{2\gamma}{3}, x_3 + \frac{2\gamma}{3}, x_4 - \frac{\gamma}{3}, \ldots, x_8 - \frac{\gamma}{3}), \end{aligned}$$

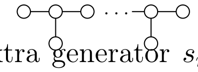
where $\gamma = -x_1 - x_2 - x_3$, is faithful if and only if any orbit vector $x = (x_1, \ldots, x_8)$ satisfies four conditions :

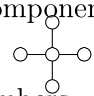
1. *No two x_i are congruent modulo $|3c|$.*
2. *No sum of three x_i is congruent to zero modulo $|3c|$.*
3. *No sum of six x_i is congruent to zero modulo $|3c|$.*
4. *No x_i is congruent to $-\sum x_j$ modulo $|3c|$.*

Proof. As for Proposition 23. □ □

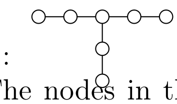
The reader should be able to find length formulas, similar to the previous ones.

7. Representations of $\tilde{D}_n$, $\tilde{E}_6$ and friends

In this section, we shall find that the affine group $\tilde{D}_n$:  has a permutation representation of vectors $(x_1, \ldots, x_n)$, just like D_n. The extra generator s_{n+1} acts in analogy with s_n but on the *last* two components.

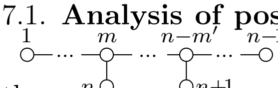
The simplest case is $\tilde{D}_4$: , with e-vector $(0, 1, 2, 3)$. The generators s_1, s_2, s_3 are transpositions of adjacent numbers, s_4: $(x_1, x_2, x_3, x_4) \mapsto (-x_2, -x_1, x_3, x_4)$ as before and s_5: $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, 6-x_4, 6-x_3)$.

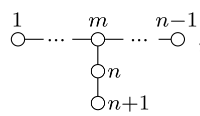
The hyperplanes in the arrangement are $x_i \equiv \pm x_j \pmod 6$, and the length function follows directly from this.

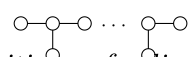
Another Coxeter graph with two extra edges joined to a path is $\tilde{E}_6$: . We shall find a permutational representation with e-vector $(3, 4, 5, 6, 7, 8)$. The nodes in the horizontal path are transpositions of adjacent numbers, the middle center node s_6 adds $\gamma = \frac{1}{3}\sum x_i - x_1 - x_2 - x_3$ to x_1, x_2 and x_3, as in the E_6-representation. The lower center node s_7 subtracts the number $\mu = \frac{1}{3}\sum x_i + 12$ from all six components of the vector.

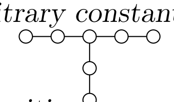
The hyperplanes in the arrangement are $x_i \equiv x_j \pmod{12}$, $\sum x_j \equiv 0 \pmod{12}$ and $x_{i_1} + x_{i_2} + x_{i_3} \equiv \frac{1}{3}\sum x_j \pmod{12}$ and the length function follows directly from this.

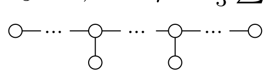
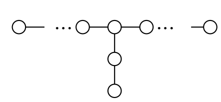
7.1. Analysis of possible permutational representations.

Let us consider groups of type  (we can assume $m, m' \geq 2$), acting on $\mathbf{R}^n$ as in section 6.1, but with the new generator s_{n+1} adding a quantity γ' to the last m' components. It is clear that, in analogy with γ, we must have $\gamma' = -\frac{2}{m'}\sum_1^n x_i - \sum_1^{n-m'} x_{n-i} + c'$. Now, it is easy to check that the commutation relation between s_n and s_{n+1} forces $m = 2$ and $m' = 2$, so permutational representations exist only for $\tilde{D}_n$.

In the same way, we can check all groups of type , and find permutational representations only for $n = 6$, $m = 3$ (the $\tilde{E}_6$–case) and for $n = 8$, $m = 2$ (an unusual rendering of $\tilde{E}_8$).

Proposition 29. *The Coxeter group $\tilde{D}_n$ with graph*  *has a faithful representation on $\mathbf{R}^n$ in which $s_1, \ldots, s_{n-1}$ act as transpositions of adjacent components, while $s_n : (x_1, x_2, \ldots) \mapsto (c - x_2, c - x_1, \ldots)$ and $s_{n+1} : (\ldots, x_{n-1}, x_n) \mapsto (\ldots, c' - x_n, c' - x_{n-1})$. Here $c \neq c'$ are otherwise arbitrary constants.*

The group $\tilde{E}_6$ with graph  *has a faithful representation on $\mathbf{R}^6$ in which $s_1, \ldots, s_5$ are adjacent transpositions, s_6 adds γ to x_1, x_2, x_3 and s_7 subtracts μ from all six x_i. Here $\gamma = \frac{1}{3}\sum x_i - x_1 - x_2 - x_3 + c$, and $\mu = \frac{1}{3}\sum x_i + c'$ for arbitrary constants $c' \neq -2c$.*

No other groups of type  *or*  *have permutational representations of this kind.*

Proof. It is straightforward to check that these expressions for γ and μ ensure the desired commutation relations. What remains to be shown is faithfulness. By a translation of the origin, $x'_i = x_i - c/2$, we can accomplish that $c = 0$ in the $\tilde{D}_n$-case. Then $s_1, \ldots, s_n$ are orthogonal linear reflections while s_{n+1} is an orthogonal affine reflection. A known result about affine Coxeter groups ([56], p 95) states that these can be faithfully represented in $\mathbf{R}^n$, such that $s_1, \ldots, s_n$ correspond to orthogonal reflections in hyperplanes through the origin, while s_{n+1} corresponds to an orthogonal affine reflection. This is exactly our situation, so we conclude that $\tilde{D}_n$ is faithfully represented.

By translating the origin, $x'_i = x_i - c$, we can get rid of c in the $\tilde{E}_6$-case too. Now s_6 is a linear reflection and s_7 is still an affine orthogonal reflection. We cannot immediately use the same theorem though, for s_6 is not orthogonal. However, in the next section we shall derive a length function $l(w)$ that counts the number of hyperplanes between e and w. It will be apparent that $l(w) > 0$ for $w \neq 0$, so faithfulness is evident. □ □

7.2. **Representations of $\tilde{D}_n$ and $\tilde{E}_6$.** As can be seen from the proof, we are dealing with affine reflections in dimension one lower than the number of generators. Thus, there is no quadratic invariant in these cases.

The arrangement of affine hyperplanes can be found as in proposition 23, and immediately tells us which e-vectors are useful. For simplicity, the statements are formulated for the case $c = 0$ only.

Proposition 30. *For $\tilde{D}_n$, the orbit of the vector x under the group action defined in Proposition 29 is faithful if and only if for any two components, $x_i \not\equiv \pm x_j \pmod{|c'|}$.*
For $\tilde{E}_6$, there are three criteria:

(1) $x_i \not\equiv x_j \pmod{|c'|}$
(2) $x_{i_1} + x_{i_2} + x_{i_3} \not\equiv \frac{1}{3}\sum x_j \pmod{|c'|}$.
(3) $\frac{1}{3}\sum x_j \not\equiv 0 \pmod{|c'|}$.

Proposition 31. *Simple vectors satisfying the faithfulness criteria are*

$\tilde{D}_n$: $e = (0, 1, \ldots, n-1)$, $c = 0$, $c' = 2n - 2$.
$\tilde{E}_6$: $e = (3, 4, 5, 6, 7, 8)$, $c = 0$, $c' = -12$.

The descent set corresponding to a vector x includes all s_i such that $x_i > x_{i+1}$. Furthermore, for $\tilde{D}_n$, s_n is included if $x_1 + x_2 < 0$ and s_{n+1} if $x_{n-1} + x_n > c'$. For $\tilde{E}_6$, s_n is included if the value of γ is positive and s_{n+1} if μ is positive.

Proposition 32. *Let $(x_1, \ldots, x_n)$ represent w, in the representations of Proposition 31.*

In $\tilde{D}_n$, the length of w is given by

$$\#(\textit{inversions}) + \#(x_{i_1} + x_{i_2} < 0) + \sum \left\lfloor \frac{|x_{i_1} - x_{i_2}|}{|c'|} \right\rfloor + \sum \left\lfloor \frac{|x_{i_1} + x_{i_2}|}{|c'|} \right\rfloor$$

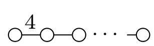
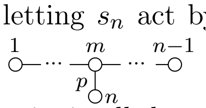
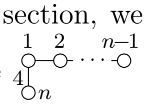
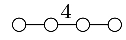
In $\tilde{E}_6$, the length of w is given by

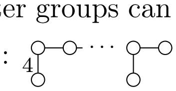
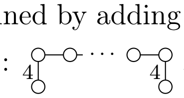
$$\#(\textit{inversions}) + \#(x_{i_1} + x_{i_2} + x_{i_3} < \tfrac{1}{3}\textstyle\sum x_j) + \sum \left\lfloor \frac{|x_{i_1} - x_{i_2}|}{|c'|} \right\rfloor + \sum \left\lfloor \frac{|\frac{1}{3}\sum x_j - x_{i_1} - x_{i_2} - x_{i_3}|}{|c'|} \right\rfloor +$$
$$+ \#(\textstyle\sum x_j < 0) + \left\lfloor \frac{|\sum x_j|}{|3c'|} \right\rfloor \quad \textit{(where } \#(\textstyle\sum x_j < 0) \textit{ is either zero or one).}$$

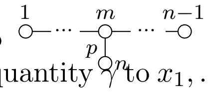
Proof. The formulas evidently count the number of hyperplanes separating e and w, so the discussion in section 6.5 gives the proof. □ □

Remark 33. The algebra involved in the action of s_n and s_{n+1} can be avoided completely by embedding the groups in S_∞, the group of bijections on $\mathbf{Z}$. This program is carried out in section 10, where all affine Coxeter groups will be reconsidered.

8. Representations of B_n, $\tilde{B}_n$, $\tilde{C}_n$ and friends

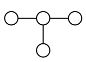
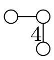
The finite group B_n:  can be represented by signed permutations of $(1, \ldots, n)$, letting s_n act by reversing the sign of x_1. In this section, we generalize to all groups of type , where B_n is the particular case . Permutational representations exist in all these cases. Note that all finite Coxeter groups except F_4:  are of this general type.

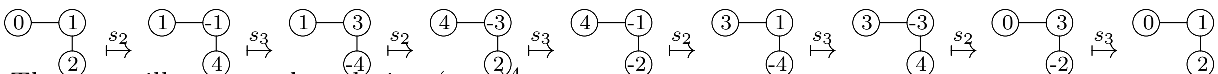
In section 6.6. we saw that the affine groups $\tilde{D}_n$ and $\tilde{E}_6$ can be represented by permutations. Two more classes of affine Coxeter groups can be obtained by adding two extra edges and nodes to a horizontal path, namely $\tilde{B}_n$:  and $\tilde{C}_n$: , and these groups can also be represented by permutations. The only affine groups not covered are $\tilde{A}_n$ and $\tilde{F}_4$, but they are the subject of the next section.

8.1. **Analysis of possible representations.** As before, for any group , we let $s_1, \ldots, s_{n-1}$ act on $\mathbf{R}^n$ by adjacent transpositions and s_n by adding a quantity γ to $x_1, \ldots, x_m$. Here γ is a linear expression in the x_i, chosen such that the commutation relations involving s_n are maintained. As we saw in section 6.1, the relations $(s_n)^2 = e$ and $s_n s_i = s_i s_n$ for $i \neq m$ forces the following expression $\gamma = \beta(x_{m+1} + \cdots + x_n) - \frac{2}{m}(x_1 + \cdots + x_m) + c$.

Here, β and c must be chosen such that $(s_n s_m)^p = e$. This may seem like a simple exercise, but is in fact a formidable task, not feasible without computer algebra even for $p = 4$.

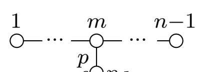
So, it is fortunate that one can avoid all calculation by recasting the permutations into an edge weighted numbers game (see p 8) on the Coxeter graph. The number on node i is $x_i - x_{i+1}$ and the number on node n is γ. The rules are the usual, that is the played node adds its number to the neighbours and then reverses the sign of its own number, except for one case: When node m is played, its number is multiplied by a constant before it is added to node n. The multiplier is the downwards edge weight. Eriksson ([42], p 31) showed that the value of this constant must be $4\cos^2\frac{\pi}{p}$ to produce the Coxeter relation $(s_n s_m)^p = e$.

Example.  has $p = 3$, so the edge weight should be $4\cos^2\frac{\pi}{3} = 1$, i.e. unweighted.  has edge weight $4\cos^2\frac{\pi}{4} = 2$ in the downwards direction. A sample game follows.



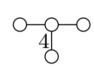
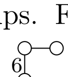
The game illustrates the relation $(s_2 s_3)^4 = e$.

Returning to the permutations, we must choose β such that playing node m increases the value γ on node n by $4\cos^2\frac{\pi}{p}$ times the value $(x_m - x_{m+1})$ on node m. But s_m is a transposition of x_m and x_{m+1}, so the value of $\gamma = \beta(x_{m+1} + \cdots + x_n) - \frac{2}{m}(x_1 + \cdots + x_m) + c$ increases by $(\beta + \frac{2}{m})(x_m - x_{m+1})$. The result follows.

Proposition 34. *A Coxeter group with graph*  *has a faithful representation on* $\mathbf{R}^n$ *in which* $s_1, \ldots, s_{n-1}$ *act as transpositions of adjacent components, while* s_n *acts by adding a certain real number* γ *to the first* m *components,* $x_1, \ldots, x_m$*, namely*

$$\gamma = -\frac{2}{m}\sum_{1}^{n} x_i + 4\cos^2\frac{\pi}{p}\sum_{m+1}^{n} x_j + c\ ,$$

where c *is an arbitrary constant.*

Proof. As in section 6.1, we can usually get rid of the constant c by a translation of the origin to $(d, d, \ldots)$, where d is chosen such that $(2n - 4m(n-m)\cos^2\frac{\pi}{p})d = mc$. For a few combinations of p, m and n, this cannot be done, namely when $n = 2m(n-m)\cos^2\frac{\pi}{p}$. All these cases are affine Coxeter groups. For $p = 3$, we found $\tilde{E}_7$ and $\tilde{E}_8$. For $p = 4$, we get $\tilde{B}_3$:  and for $p = 6$, we have $\tilde{G}_2$:  . These two affine groups will get special treatment in sections 8.3 and 8.4, so for the moment, we can ignore them.

In all other cases, we can apply the linear representation theorem for general Coxeter groups. As in the proof of Proposition 8, we introduce basis vectors $\alpha_1 = (1, -1, 0, \ldots)$, $\alpha_2 = (0, 1, -1, 0, \ldots)$ etc, $\alpha_n = (\underbrace{\ldots, -2\cos\frac{\pi}{p}, -2\cos\frac{\pi}{p}}_{m}, 0, \ldots)$, and also a symmetric bilinear form defined by

$$\langle \alpha_i, \alpha_j \rangle = -\cos\frac{\pi}{p} \quad \text{if } (s_i s_j)^p = e$$

It is easily checked that all s_i act as reflections $x \mapsto x - 2\langle x, \alpha_i\rangle\alpha_i$ and the theorem is applicable. □

□

In Proposition 12, we derived a quadratic invariant of the representing vector x, namely $Q(x) = m^2 \sum x_i^2 - (m-2)(\sum x_i)^2 - 2mc \sum x_i$. There is a quadratic invariant also in the more general situation, we just have to look for a second degree symmetric polynomial in the x_i, which stays the same when γ is added to $x_1, \dots, x_m$.

Proposition 35. *For each representation in Proposition 34, the following quadratic expression is invariant.*

$$Q(x) = 2m^2 \cos^2 \frac{\pi}{p} \sum x_i^2 + (1 - 2m \cos^2 \frac{\pi}{p})(\sum x_i)^2 - mc \sum x_i$$

The quadric surface $Q(x) = const.$ is bounded (an ellipsoid) if $n > 2m(n-m)\cos^2 \frac{\pi}{p}$, otherwise unbounded. The bounded case occurs for the finite groups $A_n, B_n, D_n, E_6, E_7, E_8$, H_3, H_4 and $I_2(p)$.

Proof. The matrix of the quadratic form is of the following general type

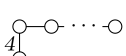
$$\begin{pmatrix} a+b & b & b & b \\ b & a+b & b & b \\ b & b & a+b & b \\ b & b & b & a+b \end{pmatrix}$$

where $a = 2m^2 \cos^2 \frac{\pi}{p}$ and $b = 1 - 2m \cos^2 \frac{\pi}{p}$. The eigenvalues are $\lambda_1 = \ldots = \lambda_{n-1} = a$, $\lambda_n = a + nb$. Since a is positive, the sign of $a + nb$ determines positive definiteness. □ □

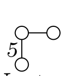
The combinations of m, n and p that satisfy the inequality of Proposition 35 correspond to all finite Coxeter groups except F_4 (which does not have a graph of this kind). Some of these have already been analyzed in section 6.3 so we can concentrate on the groups with a labelled edge. The only really new feature is the noncrystallographic groups, H_3 and H_4, which can have no integer permutational representation.

Recall that the geometric significance of a positive definite quadratic invariant is that the orbits are bounded and so contain only finitely many integer points. For H_3 and H_4, there are faithful orbits that live in the intersection of the invariant ellipsoid and the dense lattice of points with co-ordinates in $\{i + j\sqrt{5} \mid i, j \in \mathbf{Z}\}$. Apart from that, everything works the same way as before.

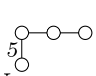
8.2. **Representations of B_n, H_3, H_4 and $I_2(p)$.** In these cases, we have $m = 1$, i.e. the labelled edge is attached at the end of the horizontal path. We can write $\gamma = -2x_1 + (4\cos^2 \frac{\pi}{p} - 2)(x_2 + x_3 + \cdots) + c$, so the action of s_n simplifies to $x_1 \leftarrow -x_1 + 2\cos\frac{2\pi}{p}(x_2 + x_3 + \cdots) + c$. The quadratic invariant is simplified to $Q(x) = x_1^2 + x_2^2 + \cdots - 2\cos\frac{2\pi}{p}(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n) - c(x_1 + \cdots + x_n)$

Proposition 36. *For B_n,*  *, the orbit of the vector x under the group action defined in Proposition 34 is faithful if and only if three criteria are fulfilled.*

(1) *No two x_i are equal.*
(2) *No two x_i sum to c.*
(3) *No x_i equals $c/2$.*

For H_3,  *, there are four criteria. The golden ratio $\frac{1+\sqrt{5}}{2}$ is denoted by α.*

(1) *No two x_i are equal.*
(2) *No x_i satisfies $(\alpha+1)x_i = (\alpha-1)\sum x_j + c$.*
(3) *No two x_{i_1}, x_{i_2} satisfy $x_{i_1} + \alpha x_{i_2} = (\alpha-1)\sum x_j + c$.*
(4) *No three $x_{i_1}, x_{i_2}, x_{i_3}$ satisfy $x_{i_1} + x_{i_2} + (\alpha-1)x_{i_3} = (\alpha-1)\sum x_j + c$.*

For H_4,  *, the same four criteria and four more apply.*

(5) *No x_i satisfies $x_i = (2-\alpha)\sum x_j - (\alpha+2)c$.*

(6) *No x_i satisfies $x_i = (3\alpha + 2)c$.*
(7) *No two x_{i_1}, x_{i_2} satisfy $x_{i_1} + x_{i_2} = (\alpha - 1)\sum x_j - 2\alpha c$.*
(8) *No two x_{i_1}, x_{i_2} satisfy $x_{i_1} = (\alpha - 1)x_{i_2} + (\alpha + 1)c$.*

For $I_2(p)$, there are p illegal hyperplanes, namely for $k = 1, \ldots, p$

$$\mathcal{H}_k\colon\ x_1 \cos\frac{(k-1)\pi}{p} = x_2 \cos\frac{(k+1)\pi}{p} + \frac{\sin(k\pi/p)}{\sin\pi p}\frac{c}{2}$$

Proof. As we have seen twice already, the important point is finding the set of reflecting hyperplanes. In all cases, $x_i = x_{i+1}$ and $\gamma = 0$ are the hyperplanes belonging to the simple reflections s_i. The other hyperplanes are found by iterated application of the s_i to the planes already found. After a few such iterations, the set of planes obtained is invariant under all reflections and is therefore complete.

Note that $\mathcal{H}_k = \mathcal{H}_{k+p}$ and that $\mathcal{H}_0 : x_1 = x_2$, $\mathcal{H}_1 : \gamma = 0$. The hyperplanes transform according to $s_1(\mathcal{H}_k) = \mathcal{H}_{p-k}$ and $s_2(\mathcal{H}_{p-k}) = \mathcal{H}_{k+2}$. $\square$ $\square$

For simplicity, the remaining results in this section are stated for $c = 0$ only. As before, we let α denote the golden ratio.

Proposition 37. *The set of values occurring as co-ordinates in the representations can be expressed in the components e_i of the start vector as follows.*

B_n : *All $\pm e_i$. A total of $2n$ values.*
H_3 : *As for B_n and also $\pm[\alpha e_i - (\alpha - 1)\sum e_j]$. In all, 12 values.*
H_4 : *As for H_3 and further*
$\pm[(\alpha+1)e_i - (\alpha-1)\sum e_j]$, $\pm[(\alpha+1)e_{i_1} + e_{i_2} - \sum e_j]$, $\pm[\alpha e_{i_1} - \sum e_j]$
$\pm[(\alpha+1)e_{i_1} + \alpha e_{i_2} - \sum e_j]$, $\pm[e_{i_1} - e_{i_2}]$, $\pm[e_{i_1} - \alpha e_{i_2}]$. *Altogether, 120 values.*
$I_2(p)$: *Only p values, namely $v_k = \frac{\sin k\varphi}{\sin\varphi}e_1 - \frac{\sin(k-1)\varphi}{\sin\varphi}e_2$, where $\varphi = \frac{2\pi}{p}$.*

Proof. For B_n, H_3, H_4, the proof is exactly as in Proposition 19. For $I_2(p)$, we note that $v_{k+p} = v_k$ and that $v_1 = e_1$, $v_2 = e_2$. The rest is trickonometry. $\square$ $\square$

In order to avoid cumbersome formulas, we have stated these results only for the case $c = 0$. However, if one wants the general expression, it is not necessary to redo the calculation from the start. There is a simple way to obtain the missing c-term in these expressions. Set all $e_i = 1$ and determine c such that $\gamma = 0$. Then no other value than 1 can ever occur, so every expression must be supplemented with a c-term that makes the the total value equal to 1. For example, in B_n the complete expressions are e_i and $c - e_i$, for from $\gamma = -2e_1 + c = 0$ we see that $e_i = 1, c = 2$ should make all expressions equal to 1.

Example. For $I_2(6)$, also denoted by G_2, we learn from Proposition 36 that (with $c = 0$)

$$\mathcal{H}_0 : x_1 = x_2,\ \mathcal{H}_1 : 2x_1 = x_2,\ \mathcal{H}_2 : x_1 = 0,\ \mathcal{H}_3 : x_1 = -x_2,\ \mathcal{H}_4 : 0 = x_2,\ \mathcal{H}_5 : x_1 = 2x_2.$$

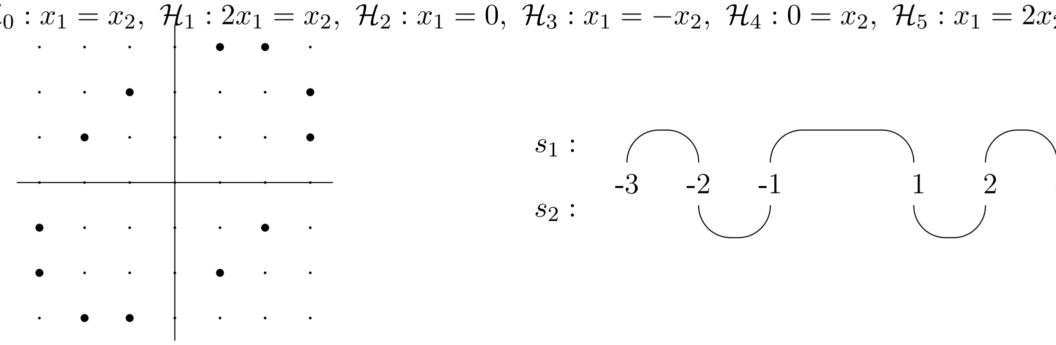


FIGURE 2. The group G_2 represented as points in $\mathbf{Z}^2$ and as permutations of six numbers.

A legal start is $e = (2, 3)$ and then Proposition 37 tells us that six values will occur, namely

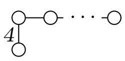
$$v_k = \frac{2\sin(k\pi/3) - 3\sin((k+1)\pi/3)}{\sin(\pi/3)},$$

i.e. $v_0 = 3$, $v_1 = 2$, $v_2 = -1$, $v_3 = -3$, $v_4 = -2$, $v_5 = 1$. It is indeed easy to construct all twelve vectors using $s_2 : x_1 \leftarrow -x_1 + x_2$ (and the transposition s_1).

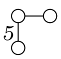
$$(2,3) \overset{s_2}{\mapsto} (1,3) \overset{s_1}{\mapsto} (3,1) \overset{s_2}{\mapsto} (-2,1) \overset{s_1}{\mapsto} \text{etc.}$$

A plot of these points reveals the invariant quadric, $Q(x) = x_1^2 + x_2^2 - x_1x_2 = 7$. The embedding $G_2 \subset S_6$ is defined by the combined transpositions effected by s_1 and s_2 (see figure).

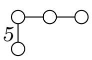
Proposition 38. *Simple vectors satisfying the faithfulness criteria are*

B_n: 

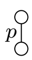
, $e = (1, 2, \dots, n)$

H_3: 

, $e = (2, 1{+}\alpha, 2\alpha)$, *where* $\alpha = \frac{1+\sqrt{5}}{2}$

H_4: 

, $e = (3\alpha, 5, 2{+}2\alpha, 7{-}\alpha)$

$I_2(p)$: 

, $e = (2\cos\frac{\pi}{p} - 1, 1)$

Proof. According to Proposition 36, we only have to check a few criteria. Apart from that, the vectors have been chosen so as to make sure that the length function in the next proposition is correct. In other words, e is chosen as the element of minimal length. $\square$

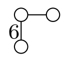
Conjecture 39. The representations of H_3 and H_4 involve exactly those $(x_1, \dots, x_n)$ with components in $\{i + j\alpha \,|\, i, j \in \mathbf{Z}\}$ that satisfy the inequalities of Proposition 36 and give the quadratic form in Proposition 35 the same value as the e-vector does.

Proposition 40. *Let $(x_1, \dots, x_n)$ represent a group element w, with e-vector as in Prop. 38.*
In B_n, the length of w is given by $l(w) = \#(\text{inversions}) + \#(x_i < 0) + \#(x_{i_1} + x_{i_2} < 0)$.
In H_3, the formula is $l(w) = \#(\text{inversions}) \;+\; \#((\alpha+1)x_i < (\alpha-1)\sum x_j) \;+$
$+\; \#(x_{i_1} + \alpha x_{i_2} < (\alpha-1)\sum x_j) \;+\; \#(x_{i_1} + x_{i_2} + (\alpha-1)x_{i_3} < (\alpha-1)\sum x_j)$.
For H_4, the expression contains four more terms, $l(w) = \ldots + \;\#(\alpha x_i < (\alpha-1)\sum x_j) \;+$
$+\; \#(x_i < 0) \;+\; \#(x_{i_1} + x_{i_2} < (\alpha-1)\sum x_j) \;+\; \#(x_{i_1} < (\alpha-1)x_{i_2})$.
For $I_2(p)$, we have $l(w) = \#(x_1 \cos\frac{(k-1)\pi}{p} < x_2 \cos\frac{(k+1)\pi}{p})$.

Proof. In all cases, $l(w)$ is the number of hyperplanes separating w from e. $\square$ $\square$

Remark 41. The length formula for B_n has an elegant alternative form, due to Brenti [16], namely

$$l(w) = \#(\text{inversions}) \;-\; \sum_{x_i<0} x_i \;.$$

8.3. **A special study of $\tilde{G}_2$.** One of the infinite groups with an extra labelled edge merits a separate treatment, and that is the affine group $\tilde{G}_2$ 

. We start by specializing the results of section 8.1 to the case $m = 1, p = 6$.

Proposition 42. *The faithful representation on $\mathbf{R}^3$ of the affine Coxeter group $\tilde{G}_2$ (in which $s_1 : x_1 \leftrightarrow x_2$ and $s_2 : x_2 \leftrightarrow x_3$ transpose co-ordinates and $s_3 : x_1 \leftarrow -x_1 + x_2 + x_3 + c$ changes the first co-ordinate) has the following properties.*

- *The quadratic expression $Q(x) = x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_1x_3 + x_2x_3) - c(x_1 + x_2 + x_3)$ is invariant under the action of $\tilde{G}_2$.*
- *The quadric surface $Q(x) = \text{const.}$ is paraboloidal if $c \neq 0$ and cylindrical if $c = 0$.*
- *The orbit of the vector x is faithful if and only if for $i \neq j$*

(1) $x_i \not\equiv x_j \pmod{|c|}$.
(2) $x_i \not\equiv \frac{1}{3}(\sum x_j + c) \pmod{|c|}$.
If $c = 0$, congruence is to be interpreted as equality.

- *The set of component values occurring in the orbit is* $\{(1-j-k)e_1 + je_2 + ke_3 + (j^2+jk+k^2-j-k)c \mid j,k \in \mathbf{Z}\}$, *where e_i are the components of the start vector.*
- *A simple legal e-vector is $(2,3,4)$ with $c = -6$ or alternatively $(0,1,3)$ with $c = 0$.*
- *For an increasing e-vector and $c \neq 0$, the length function is*

$$\#(\textit{inversions}) + \#\left(x_i < \tfrac{1}{3}(\textstyle\sum x_j + c)\right) + \sum \left\lfloor \frac{|x_{i_1} - x_{i_2}|}{|c|} \right\rfloor + \sum \left\lfloor \frac{\left|\frac{1}{3}(\sum x_j + c) - x_i\right|}{|c|} \right\rfloor$$

- *For $e = (0,1,3)$ and $c = 0$, the orbit consists of the intersection of the integer lattice and the elliptic cylinder $Q(x) = x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_1x_3 + x_2x_3) = 7$*

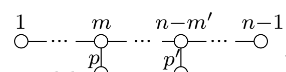
Proof. Apply the same methods as in the last section! The equation of the cylinder may be written $(x_1 - x_3)^2 + (x_2 - x_3)^2 + (x_1 - x_3)(x_2 - x_3) = 7$, so all horizontal cross–sections look like the G_2–invariant ellipse in figure 2, but with center (x_3, x_3). Therefore, the last statement is evident. □ □

The expression for the component values occurring in an orbit is valid regardless of whether the e_i are integers. Using this expression, one can describe the orbit even more explicitly and in the noninteger case too.

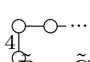
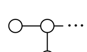
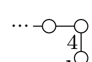
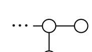
Proposition 43. *The orbit of (e_1, e_2, e_3) under the $\tilde{G}_2$–action of the previous proposition consists of all vectors (x_1, x_2, x_3) such that $\{x_1, x_2, x_3\} = \{V_{j,k},\ V_{j+1,k},\ V_{j,k+1}\}$ or $\{x_1, x_2, x_3\} = \{V_{j,k},\ V_{j-1,k},\ V_{j,k-1}\}$ for some $j, k \in \mathbf{Z}$, where $V_{j,k} = (1-j-k)e_1 + je_2 + ke_3 + (j^2+jk+k^2-j-k)c$.*

Proof. One can check that s_1, s_2 and s_3 transform a vector of either type into a vector of the other type. The vector e itself is of the first type, with $j = k = 0$. □

Remark 44. If one could choose integers e_1, e_2, e_3 and c such that the values $V_{j,k}$ were all different, then the last two propositions would define an embedding of $\tilde{G}_2$ into S_∞, the autobijection group of $\mathbf{Z}$. Unfortunately, this is not possible, as $V_{j,-k} = V_{-j,k}$ if $j = c + e_1 - e_3$ and $k = c + e_1 - e_2$.

8.4. **Two extra edges, $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{D}_n$.** Let us consider groups of type , acting on $\mathbf{R}^n$ as in the previous section, but with the new generator s_{n+1} adding a quantity γ' to the last m' components. It is clear that, in analogy with γ, we must have

$$\gamma' = -\frac{2}{m'} \sum_1^n x_i + 4\cos^2\left(\frac{\pi}{p'}\right) \sum_1^{n-m'} x_j + c' \ .$$

Now, it is easy to check that s_n and s_{n+1} commute only if $\frac{2}{m} = 4\cos^2(\frac{\pi}{p})$ and $\frac{2}{m'} = 4\cos^2(\frac{\pi}{p'})$. After testing some combinations of m and p, we find that there are only two possibilities, namely $m = 1,\ p = 4$ and $m = 2,\ p = 3$, so permutational representations exist only for Coxeter graphs starting like  or  and ending like  or . The conclusion is that the affine groups $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{D}_n$ are all we get. The analysis of $\tilde{D}_n$ in section 7 applies to $\tilde{B}_n$ and $\tilde{C}_n$ with small changes, so we can state the following results.

Proposition 45. *The Coxeter groups $\tilde{B}_n$ and $\tilde{C}_n$ have faithful representations as groups of affine transformations on $\mathbf{R}^n$ in which $s_1, \ldots, s_{n-1}$ act as transpositions of adjacent components while s_n and s_{n+1} act in the following way.*

$\tilde{B}_n$: $(x_1, x_2, \ldots) \overset{s_n}{\mapsto} (c - x_1, x_2, \ldots)$ *and* $(\ldots, x_{n-1}, x_n) \overset{s_{n+1}}{\mapsto} (\ldots, c' - x_n, c' - x_{n-1})$
$\tilde{C}_n$: $(x_1, x_2, \ldots) \overset{s_n}{\mapsto} (c - x_1, x_2, \ldots)$ *and* $(\ldots, x_{n-1}, x_n) \overset{s_{n+1}}{\mapsto} (\ldots, x_{n-1}, c' - x_n)$.

Here $c \neq c'$ are otherwise arbitrary constants.

Proof. The group generated by the s_i has two affine reflections, s_n and s_{n+1}, but we can always make s_n linear by a co-ordinate translation $(x'_1, x'_2, \ldots) = (x_1 - \frac{c}{2}, x_2 - \frac{c}{2}, \ldots)$. Then $s_1, \ldots, s_n$ are orthogonal linear reflections while s_{n+1} is an orthogonal affine reflection. But the analysis of affine reflecion groups in Humphreys, [56] p 95, makes it clear that we have a faithful representation of an affine Coxeter group. □ □

For simplicity, the following results are stated only for the case $c = 0,\ c' \neq 0$. The general case can be obtained by adding $\frac{c}{2}$ to all x_i in the formulas.

Proposition 46. *For $\tilde{B}_n$, the orbit of the vector x under the group action defined in Proposition 45 is faithful if and only if for any two components, $x_i \not\equiv \pm x_j \pmod{|c'|}$ and further $x_i \not\equiv 0 \pmod{|c'|}$. For $\tilde{C}_n$, the last condition is modified to $2x_i \not\equiv 0 \pmod{|c'|}$.*

Proposition 47. *The simplest permutational representations are given by*

$\tilde{B}_n$: $e = (1, 2, \ldots, n),\ \ c = 0,\ \ c' = 2n.$
$\tilde{C}_n$: $e = (1, 2, \ldots, n),\ \ c = 0,\ \ c' = 2n + 1.$

The descent set corresponding to a vector x includes all s_i such that $x_i > x_{i+1}$ and it includes s_n if $x_1 < 0$. For $\tilde{B}_n$, s_{n+1} is in the descent set if $x_{n-1} + x_n > c'$, but for $\tilde{C}_n$, s_{n+1} is included if $x_n > c'$.

Proposition 48. *Let $(x_1, \ldots, x_n)$ represent w, in the representations of Proposition 47.*

In $\tilde{B}_n$, the length of w is given by

$$l(w) = \#(\text{inversions}) + \#(x_{i_1} + x_{i_2} < 0) + \#(x_i < 0) + \sum \left\lfloor \frac{|x_{i_1} - x_{i_2}|}{|c'|} \right\rfloor + \sum \left\lfloor \frac{|x_{i_1} + x_{i_2}|}{|c'|} \right\rfloor + \sum \left\lfloor \frac{|x_i|}{|c'|} \right\rfloor$$

For $\tilde{C}_n$, the last term is modified to $\ldots + \sum \left\lfloor \frac{|2x_i|}{|c'|} \right\rfloor$.

Proof. The formulas evidently count the number of hyperplanes separating e and w, so the discussion in section 6.5 gives the proof. □ □

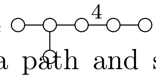
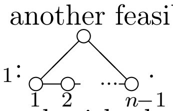
Remark 49. The e-vectors in Proposition 47 may be simplest possible, but the most pleasing representation for $\tilde{B}_n$ and $\tilde{C}_n$ is the mirror model, mentioned in the introduction and developed in section 10.

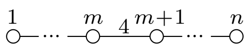
9. Representations of F_4, $\tilde{F}_4$, $\tilde{A}_n$ and friends

The finite group F_4: $\circ\!-\!\circ\overset{4}{-}\circ\!-\!\circ$ and the infinite group $\tilde{F}_4$: $\circ\!-\!\circ\overset{4}{-}\circ\!-\!\circ\!-\!\circ$ are different from the Coxeter groups considered earlier in this paper. They cannot be constructed from an S_n-path by adding an edge or two. Rather, they consist of a pair of paths connected by a four–edge. Still, it is possible to represent F_4 on $\mathbf{R}^5$ by letting s_1, s_3 and s_4 be adjacent transpositions, while $s_2 : (x_1, x_2, y_1, y_2, y_3) \mapsto (x_1 + x_2 - y_1,\ x_2,\ 2x_2 - y_1, y_2, y_3)$. With $e = (7, 8, 9, 10, 11)$ one gets a faithful representation as vectors in $\mathbf{Z}^5$, where the element of greatest length is $w_o = (-7, -8, -9, -10, -11)$.

The construction can be generalized to all Coxeter graphs obtained by connecting two paths by a marked edge. Integer representations exist in the crystallographic cases, i.e. when the connecting edge is marked four or six.

The permutations of S_n and the representing vectors of F_4, $\tilde{F}_4$ etc have a common feature, namely a *linear invariant*. Just as for all permutations in S_n the sum $\sum x_i$ is invariant, for all $(x_1, x_2, y_1, y_2, y_3)$ in the F_4–representation the linear expression $2x_1 + 2x_2 - y_1 - y_2 - y_3$ is invariant. As these vectors are restricted to some hyperplane, they are really representations in degree one less. It turns out that this slimmed–down representation is affine for $\tilde{F}_4$ and linear for the other groups considered.

Alternatively, one can use the superfluous dimension to represent an extra node and edge, just as we have done for S_n–paths in previous sections. In this way, permutational representations are obtained for groups like  . In a few cases, namely for the affine groups, we were able to add two edges to a path and still get a permutational representation. As might be expected, this scheme never works for F–paths. But there is another feasible way of adding two edges to an ordinary path, namely the last affine group $\tilde{A}_{n-1}$:  . It can be represented in $\mathbf{Z}^n$, with $s_1, \ldots, s_{n-1}$ transposing adjacent components and with the extra generator s_n : $(x_1, x_2, \ldots, x_n) \mapsto (x_n - c, x_2, \ldots, x_{n-1}, x_1 + c)$. The simplest choice is $e = (1, 2, \ldots, n)$ and $c = n$.

9.1. **Possible representations of F–type groups.** Let us call a group with Coxeter graph  , an F–type group. The generators act on vectors $(x_1, \ldots, x_m, y_1, \ldots, y_{n-m+1})$ in the following way. We let $s_1, \ldots, s_{m-1}$ be adjacent transpositions of $(x_1, \ldots, x_m)$ and $s_{m+1}, \ldots$ be adjacent transpositions of $(y_1, \ldots)$. What is the appropriate linear or affine transformation corresponding to s_m? In full generality, we can put

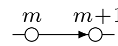
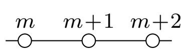
$s_m : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto S_m \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix}$, for some $n \times n$-matrix S_m and some constant vector $\begin{pmatrix} u \\ v \end{pmatrix}$.

The relation $s_m^2 = e$ means an identity $S_m^2 \begin{pmatrix} x \\ y \end{pmatrix} + S_m \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \equiv \begin{pmatrix} x \\ y \end{pmatrix}$, that is $S_m^2 = I$ and $S_m \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} u \\ v \end{pmatrix}$. Other relations are $(S_m S_{m-1})^3 = I$, $(S_m S_{m+1})^4 = I$ and $S_m S_i = S_i S_m$ for all other i. Using computer algebra (Maple V) and hours and hours of computer time, one finds the following unique class of solutions:

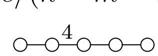
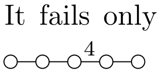
$(x_1, \ldots, x_{m-1}, x_m, y_1, y_2, y_3, \ldots) \overset{s_m}{\mapsto} (x_1 + a\gamma, \ldots, x_{m-1} + a\gamma, x_m - b\gamma, y_1 + 2(a^2 - b^2)\gamma, y_2, y_3, \ldots)$,

where $\gamma = \frac{1}{a^2 - b^2}(ax_m + b(x_1 + \cdots + x_{m-1}) + c - y_1)$ and a, b, c are constants, with $a^2 - b^2 \neq 0$ but otherwise arbitrary. If we choose the simplest possible parameter values $a = 1$, $b = c = 0$, the action can be described in words as follows.

$$\boxed{s_m \text{ adds } (x_m - y_1) \text{ to } x_1, \ldots, x_{m-1} \text{ and } 2(x_m - y_1) \text{ to } y_1.}$$

Before the computer era, we would probably have obtained the same result in a few seconds! The numbers game modelling the group has an edge weight 2 on  , like on page 8. The numbers on the nodes are to be the differences, $x_1 - x_2$ etc; in particular, the three nodes  , carry the values $(x_{m-1} - x_m)$, $(x_m - y_1)$ and $(y_1 - y_2)$ respectively. After firing the middle node, we get $(x_{m-1} - y_1)$, $(-x_m + y_1)$ and $(2x_m - y_1 - y_2)$ and these differences belong to a permutation vector $(\ldots, x_{m-1} + (x_m - y_1), x_m, y_1 + 2(x_m - y_1), y_2, y_3, \ldots)$. Q.E.D.

The possibility of a constant c in the γ-expression is unessential, for a translation of the origin $(x_1', x_2', \ldots) = (x_1 - c, \ldots, x_m - c, y_1, y_2, \ldots)$ makes away with c. Note, however, that we have a representation of F_4 in $\mathbf{R}^5$, that is, in one dimension too many. The explanation is the existence of a *linear invariant* $\sum y_j - \frac{2}{m-1} \sum x_i$. Transpositions within the x_i or the y_j do not alter the value, nor does the s_m-action, for $\sum y_j$ increases by 2γ and $\sum x_i$ increases by $(m-1)\gamma$. A representation in one degree less is obtained by leaving out the last co-ordinate, but not until after making a mental note of the invariant value, let us call it c. Then, we can always reconstruct the lost co-ordinate if need be — and there be need whenever s_n-action shall be taken! The old s_n-action replaced y_{n-m} by y_{n-m+1}, so the new action replaces y_{n-m} by $c + \frac{2}{m-1} \sum x_i - (y_1 + \cdots + y_{n-m})$.

As so many times before, we can try to get rid of this new constant c too, by translating the origin as $(x_1', \ldots, y_1', \ldots) = (x_1 - d, \ldots, y_1 - d, \ldots)$, and that works if $d = c/(n - m - 1 - \frac{2}{m-1})$. It fails only when the denominator happens to be zero and that is for  and for  , two renderings of the affine group $\tilde{F}_4$.

For all other cases, we now have a linear reflection representation in $\mathbf{R}^n$ and with the following basis vectors, we can again use the standard geometric representation on page 6. (We use $\|$ to separate x from y.)

$$\begin{array}{rcl}
\alpha_1 & = & (1, -1, 0, \ldots) \\
\alpha_2 & = & (0, 1, -1, 0, \ldots) \\
\vdots & & \\
\alpha_{m-1} & = & (\ldots, 0, 1, -1 \parallel 0, \ldots) \\
\alpha_m & = & (-1, \ldots, -1, 0 \parallel -2, 0, \ldots) \\
\alpha_{m+1} & = & (\ldots, 0 \parallel \sqrt{2}, -\sqrt{2}, \ldots) \\
\alpha_{m+2} & = & (\ldots, 0 \parallel 0, \sqrt{2}, -\sqrt{2}, \ldots) \\
\vdots & & \\
\alpha_{n-1} & = & (\ldots, \sqrt{2}, -\sqrt{2}) \\
\alpha_n & = & (\ldots, 0, \sqrt{2})
\end{array}$$

With the symmetric bilinear form defined by

$$\langle \alpha_i, \alpha_j \rangle = -\cos\frac{\pi}{p} \quad \text{if } (s_i s_j)^p = e \quad ,$$

it is easily checked that all s_i act as reflections $x \mapsto x - 2\langle x, \alpha_i \rangle \alpha_i$ and the theorem is applicable.

Proposition 50. *The Coxeter groups of F-type* $\overset{1}{\circ}\!-\cdots-\!\overset{m}{\circ}\!\overset{4}{-}\!\overset{m+1}{\circ}\!-\cdots-\!\overset{n}{\circ}$ *(assuming $1 < m < n$) have faithful representations as groups of affine transformations on $\mathbf{R}^m \times \mathbf{R}^{n-m} = \{(x, y)\}$, in which $s_1, \ldots, s_{m-1}$ and $s_{m+1}, \ldots, s_{n-1}$ act as transpositions of adjacent components, s_m acts by adding $(x_m - y_1)$ to the first $m-1$ components in $\mathbf{R}^m$ and $2(x_m - y_1)$ to the first component in $\mathbf{R}^{n-m}$ and s_n replaces the last component in $\mathbf{R}^{n-m}$ by $c + \frac{2}{m-1}\sum x_i - (y_1 + \cdots + y_{n-m})$. Here c is an arbitrary constant.*

Proof. The only case remaining to be proved is $\tilde{F}_4$ when the constant c is nonzero. Here, no translation $(x'_1, x'_2, y'_1, y'_2, y'_3) = (x_1–d, x_2–d, y_1–d, y_2–d, y_3–d)$ will make s_5 linear, at least not with a constant d. But with $d = c + 2x_1 + 2x_2 - y_1 - y_2 - y_3$, the group action on (x', y') becomes linear. As can be seen, the new components satisfy $c + 2x'_1 + 2x'_2 - y'_1 - y'_2 - y'_3 = 0$, and we are in fact dealing with a projection onto that hyperplane. The d-expression is invariant under s_1, s_2, s_3, s_4, so these actions are compatible (i.e. commute) with the projection, but the s_5-action is more complicated. Straightforward calculation shows that s_5 is linear, namely $(x'_1, x'_2, y'_1, y'_2, y'_3) \overset{s_5}{\mapsto} (x'_1 - y'_3, x'_2 - y'_3, y'_1 - y'_3, y'_2 - y'_3, -y'_3)$. Now, the linear representation theorem can be invoked again, with the basis vectors

$$\begin{array}{rcl}
\alpha_1 & = & (1, -1, 0, 0, 0) \\
\alpha_2 & = & (-1, 0, -2, 0, 0) \\
\alpha_3 & = & (0, 0, \sqrt{2}, -\sqrt{2}, 0) \\
\alpha_4 & = & (0, 0, 0, \sqrt{2}, -\sqrt{2}) \\
\alpha_5 & = & (\sqrt{2},\ \sqrt{2},\ \sqrt{2},\ \sqrt{2},\ 2\sqrt{2})
\end{array}$$

for one can check that all s_i act as reflections $x \mapsto x - 2\langle x, \alpha_i \rangle \alpha_i$. And a representation that becomes faithful after projection must have been even more so before! $\square$ $\square$

This slimmed-down representation is not a favourite of ours. We prefer to have s_n act as a transposition, even if that requires an extra component. The main reason for our analysis of the slim version is its usefulness in the proof of the next proposition.

Proposition 51. *A Coxeter group of F-type* $\overset{1}{\circ}\!-\cdots-\!\overset{m}{\circ}\!\overset{4}{-}\!\overset{m+1}{\circ}\!-\cdots-\!\overset{n}{\circ}$ *has a faithful representation on $\mathbf{R}^{n+1}$ in which $s_1, \ldots, s_{m-1}$ and $s_{m+1}, \ldots, s_n$ act as transpositions of adjacent components while s_m acts by adding $(x_m - x_{m+1})$ to the first $m-1$ components and $2(x_m - x_{m+1})$ to component $m+1$.*

Proof. There is a forgetful mapping onto the slim representation, and since the slim one is faithful, so is the fat one. $\square$ $\square$

The statement about the quadratic invariant takes a slightly different form, now that there is an extra dimension.

Proposition 52. *For the* $(n+1)$*-dimensional representation in Proposition 51, the following quadratic expression*

$$Q = 2\sum x_i^2 - \frac{2}{m-1}\left(\sum x_i\right)^2 + \sum y_j^2$$

is invariant, as is the linear expression

$$c = y_1 + \cdots + y_{n-m+1} - \frac{2}{m-1}\sum x_i \ .$$

By eliminating y_{n-m+1} *between these equations, one obtains a quadratic invariant for the* n*-dimensional representations of proposition 50.*

In both cases, the relevant quadric surface is the intersection of the plane with $Q = const$ *. It is bounded (an ellipsoid) if* $n < m+1+\frac{2}{m-1}$*, otherwise unbounded. The bounded case occurs for the finite groups* F_4, B_3 *and* B_4*. For the affine group* $\tilde{F}_4$*, the surface is a cylinder.*

Proof. One checks that Q is left invariant by s_m, and then the invariance claim is established. After elimination of y_{n-m+1}, the quadratic form gets more complicated, but it is still possible to find its eigenvalues. The eigenvalue $\lambda = 2$ has multiplicity $m-1$, $\lambda = 1$ has multiplicity $n-m-1$, and then there are two more eigenvalues, namely the roots of the equation

$$\lambda^2 - 2\lambda\left(\frac{m+1}{(m-1)^2} + \frac{n-m+1}{2}\right) + \frac{2}{m-1}\left(m+1-n+\frac{2}{m-1}\right) = 0 \ .$$

All eigenvalues are positive if the constant term is positive, that is if $n < m+1+\frac{2}{m-1}$. For $m = 2$, we get $n = 3$ or $n = 4$, for $m = 3$, we get $n = 4$ and no other solutions exist.

The positive semidefinite case with a zero eigenvalue occurs when $n = m+1+\frac{2}{m-1}$, which means $n = 5$ and $m = 2$ or $m = 3$. □ □

The only groups of F–type, worthy of closer study, are F_4 and $\tilde{F}_4$, so these will be our next theme.

9.2. **Representations of F_4 and $\tilde{F}_4$.** We are using the fat permutational representation, where F_4-vectors are $(x_1, x_2, y_1, y_2, y_3)$ and $\tilde{F}_4$-vectors are $(x_1, x_2, y_1, y_2, y_3, y_4)$. The action of s_2 adds $(x_2 - y_1)$ to the x_1-component and twice as much to the y_1-component. For F_4, we can assume that the invariant value $c = -2x_1 - 2x_2 + \sum y_j$ is zero, for a translation of the origin will make c vanish, except in the affine case.

Proposition 53. *For* F_4, $\circ\!-\!\circ\overset{4}{-}\circ\!-\!\circ$ *, the orbit of the vector* $(x_1, x_2, y_1, y_2, y_3)$ *under the group action defined in Proposition 51 is faithful if and only if four criteria are fulfilled.*

1. *No two components are equal.*
2. *No component is zero.*
3. *No* y_j *equals* $x_1 + x_2$*.*
4. *No* y_j *equals* $2x_i$ *for any* x_i*.*

For $\tilde{F}_4$: $\circ\!-\!\circ\overset{4}{-}\circ\!-\!\circ\!-\!\circ$ *, the criteria for faithfulness must be stated as congruences modulo the invariant* $c = -2x_1 - 2x_2 + y_1 + y_2 + y_3 + y_4$*. If* $c = 0$*, congruence is to be interpreted as ordinary equality.*

1. *No two components are congruent modulo* $|c|$*.*
2. *No two* y_{j_1}, y_{j_2} *satisfy* $y_{j_1} + y_{j_2} \equiv x_1 + x_2 \pmod{|c|}$*.*
3. *No two* y_{j_1}, y_{j_2} *satisfy* $y_{j_1} + y_{j_2} \equiv 2x_i \pmod{|c|}$*, for any* x_i*.*

Proof. Finding the set of reflecting hyperplanes presents no complications. In both cases, $x_1 = x_2$, $x_2 = y_1$ and $y_j = y_{j+1}$ are the hyperplanes belonging to the simple reflections s_i. The other hyperplanes are found by iterated application of the s_i to the planes already found. After a few such iterations with F_4, we get this set of twentyfour planes through the origin, invariant under all reflections and therefore complete.

In the affine case with $c \neq 0$, we first get thirty-three hyperplanes through the origin and then their translations by an arbitrary multiple of c. As an example, the plane $y_1 = y_4$ is transformed by s_2 into $y_1 + y_4 = 2x_2$, that plane is transformed by $s_1 s_3 s_5$ into $y_2 + y_3 = 2x_1$ and that plane is transformed by s_2 into $y_1 = y_4 - c$. □ □

The dual procedure of enumerating the component values, considered as linear expressions in the components of the e-vector, is equally simple. The resulting total of 48 values is less than thrilling, for that is also the number of roots in F_4 (see table in in Humphreys, [56] p 44). Any finite Coxeter group is a permutation group on its set of roots, so our representation did not produce a more effective embedding in the symmetric group.

Proposition 54. *The set of values occurring as co-ordinates in the F_4-representation can be expressed in the components e_i of the start vector as follows.*

All twenty $\pm(e_i - e_j)$.
All ten $\pm e_i$.
All six $\pm(e_1 + e_2 - e_j)$, with $j \geq 3$.
All twelve $\pm(2e_i - e_j)$, with $i \leq 2, j \geq 3$.
A total of 48 values.

Proof. Exactly as in the proof of Proposition 19. □ □

Note that the number of numerically different component values in a faithful F_4-orbit can be less than forty-eight. In fact, the e-vector recommended below produces twenty-two different values only.

Proposition 55. *Simple vectors satisfying the faithfulness criteria are*

F_4: ○—○$\overset{4}{—}$○—○ , $e = (7, 8, 9, 10, 11)$

$\tilde{F}_4$: ○—○$\overset{4}{—}$○—○—○ , $e = (1, 2, 3, 4, 5, 6)$

Proof. We only have to check a few criteria, namely those appearing in proposition 53. □ □

Conjecture 56. These representations of F_4 and $\tilde{F}_4$ involve exactly those integer vectors that satisfy the inequalities of Proposition 53 and give the quadratic form and the linear invariant in Proposition 52 the same value as the e-vector does.

The computer has been good enough to verify the conjecture for the finite group. Using the periodic structure of the orbit in the affine case, one can prove or disprove the statement for the affine group by a finite computer effort, but this remains to be done.

9.3. **The length function for F_4 and $\tilde{F}_4$.** Recall that the descent set $D(w)$ of a group element w consists of those generators s that shorten w. If the e-vector is chosen appropriately, the descent set will be directly visible in the w-vector.

Proposition 57. *For an F-type Coxeter group* $\overset{1}{○}$—…—$\overset{m}{○}\overset{4}{—}\overset{m+1}{○}$—…—$\overset{n}{○}$ *, represented as in Proposition 51 with increasing e-vector, the descent set corresponding to a vector z consists of all s_i such that $z_i > z_{i+1}$.*

Proof. In a numbers game starting with negative values on all nodes, the descent set consists of the nodes that carry positive numbers. The node numbers in our representations are the differences $z_i > z_{i+1}$. □ □

For $w_o \in F_4$, the descent set consists of all s_i, so the vector representing w_o must be decreasing. In Proposition 54, we found that the component values come in $\pm$pairs if $c = 0$, so in that case, w_o will correspond to $-e$. The general case is almost as simple.

Proposition 58. *If the identity in F_4 is represented by the vector $(e_1, e_2, e_3, e_4, e_5)$, then the element of maximal length will be $(-e_1-2c, \ldots, -e_5-2c)$, where $c = -2e_1 - 2e_2 + e_3 + e_4 + e_5$.*

Proof. A translation $(z_1', \ldots, z_5') = (z_1 + c, \ldots, z_5 + c)$ will make the new $c' = 0$, and since $w_o' = (-e_1', \ldots, -e_5') = (-e_1-c, \ldots, -e_5-c)$, we obtain $w_o = (-e_1-2c, \ldots, -e_5-2c)$. □ □

According to the table in Humphreys, $l(w_o) = 24$ in F_4 and we can check that result with the following length formula.

Proposition 59. *Let $(x_1, x_2, y_1, y_2, \ldots) = (z_1, z_2, z_3, z_4, \ldots)$ represent a group element w and assume that the e-vector has been chosen with increasing components.*
In F_4, the length of w is given by

$$l(w) = \#(\textit{inversions}) + \#(z_i < 0) + \#(x_1 + x_2 < y_j) + \#(2x_i < y_j) \ .$$

In $\tilde{F}_4$, the length of w is given by

$$l(w) = \#(\textit{inversions}) + \sum \left\lfloor \frac{|z_{i_1} - z_{i_2}|}{c} \right\rfloor + \sum\nolimits_+ \left\lceil \frac{x_1 + x_2 - y_{j_1} - y_{j_2}}{c} \right\rceil + \sum\nolimits_+ \left\lceil \frac{2x_i - y_{j_1} - y_{j_2}}{c} \right\rceil ,$$

where $\sum_+$ denotes summation of all positive terms.

Proof. The F_4-formula evidently counts the number of hyperplanes separating e and w, as described in Proposition 53, so the discussion in section 6.5 gives the proof. The reason for summing positive terms only in the $\tilde{F}_4$-formula is the following. The plane $x_1 + x_2 - y_1 - y_2 = 3c$ is the same as $x_1 + x_2 - y_3 - y_4 = -4c$. By neglecting the negative terms, we take each plane into account only once. □ □

9.4. **Other edge labels or extra edges.** The analysis of F-type Coxeter graphs carries over with little effort to $\overset{1}{\circ}\cdots\overset{m}{\circ}\overset{p}{—}\overset{m+1}{\circ}\cdots\overset{n}{\circ}$, with $p > 4$. The numbers game modelling the group has an edge weight $4\cos^2\frac{\pi}{p}$ on $\overset{m}{\circ}\!\rightarrow\!\overset{m+1}{\circ}$, as explained on page 8. The numbers on the nodes are the differences, $x_1 - x_2$ etc, so the three nodes $\overset{m}{\circ}—\overset{m+1}{\circ}—\overset{m+2}{\circ}$, carry the values $(x_{m-1}-x_m)$, (x_m-y_1) and (y_1-y_2) respectively. After firing the middle node, we get $(x_{m-1}-y_1)$, $(-x_m + y_1)$ and $y_1 - y_2 + 4\cos^2\frac{\pi}{p}(x_m - y_1)$ and these differences belong to a permutation vector $(\ldots, x_{m-1}+\gamma, y_1+\gamma, y_1+\lambda, y_2, y_3, \ldots)$, where $\gamma = (4\cos^2\frac{\pi}{p}-1)(x_m-y_1)$ and $\lambda = 4\cos^2\frac{\pi}{p}(x_m-y_1)$

Two special cases may be worth noting. When $p = 6$, the s_m-action simplifies to
$(\ldots, x_{m-1}, x_m, y_1, y_2, \ldots) \overset{s_m}{\mapsto} (\ldots, x_{m-1} + 2(x_m - y_1), y_1 + 2(x_m - y_1), y_1 + 3(x_m - y_1), y_2, \ldots)$,
and when $p = \infty$ it becomes
$(\ldots, x_{m-1}, x_m, y_1, y_2, \ldots) \overset{s_m}{\mapsto} (\ldots, x_{m-1} + 3(x_m - y_1), y_1 + 3(x_m - y_1), y_1 + 4(x_m - y_1), y_2, \ldots)$.
These are the only cases where orbits can have all integer components.

Proposition 60. *All Coxeter groups of type $\overset{1}{\circ}\cdots\overset{m}{\circ}\overset{4}{—}\overset{m+1}{\circ}\cdots\overset{n}{\circ}$ have faithful representations on $\mathbf{R}^{n+1}$ in which $s_1, \ldots, s_{m-1}$ and $s_{m+1}, \ldots, s_n$ act as transpositions of adjacent components while s_m acts by adding $\beta(x_m - x_{m+1})$ to the first $m-1$ components, $(\beta-1)(x_m - x_{m+1})$ to the mth component and $(\beta+1)(x_m - x_{m+1})$ to component $m+1$. Here, $\beta = (4\cos^2\frac{\pi}{p} - 1)$ is integer only in the crystallographic cases*

p	3	4	6	∞
β	0	1	2	3

.

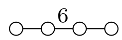
The following quadratic expression

$$Q = (\beta + 1)\sum x_i^2 - \frac{\beta(\beta + 1)}{m\beta - 1}\left(\sum x_i\right)^2 + \sum y_j^2$$

is invariant, as is the linear expression

$$c = y_1 + \cdots + y_{n-m+1} - \frac{\beta+1}{m\beta-1}\sum x_i \ .$$

Proof. A straightforward generalization of the F-type propositions 51 and 52, but without the affine complication. □ □

Example. The hyperbolic group with graph 

has an integer representation with $e = (0,1,2,3,4)$ and

$$(x_1, x_2, y_1, y_2, y_3) \overset{s_2}{\mapsto} (x_1 + 2(x_2 - y_1), x_2 + (x_2 - y_1), y_1 + 3(x_2 - y_1), y_2, y_3) \ .$$

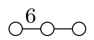
The quadratic invariant is

$$3\sum x_i^2 - 2\left(\sum x_i\right)^2 + \sum y_j^2 = 14 \ ,$$

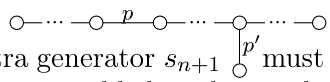
and the linear invariant

$$y_1 + y_2 + y_3 - x_1 - x_2 = 8 \ .$$

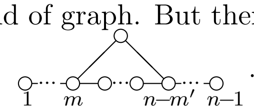
The eigenvalues of the quadratic form are $-1, 1, 1, 1, 3$, so the orbit lives on a hyperboloid surface.

The parabolic subgroup isomorphic to $\tilde{G}_2$, 

, keeps $2x_2 - x_1 = 2$ fixed, and because of the linear invariant, we can compute $y_1 + 3(x_2 - y_1) = -y_1 + y_2 + y_3 - 6$. So we get a representation of $\tilde{G}_2$ using vectors (y_1, y_2, y_3), with $e = (2,3,4)$ and, in fact, exactly the representation studied in section 8.3.

The existence of a linear invariant can be used to slim down the representation by erasing the last component, as we have seen before. Or, we can keep the fat vector and add an extra node and edge, as in 

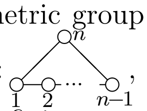
with no change in the representation. Of course, the action of the extra generator s_{n+1} must be as described in Proposition 34, but mirrored. That is, the quantity $-\gamma$ is added to the numbers to the right of the vertical edge.

The possibility of two vertical edges with two new nodes was investigated in section 8.4 and it was shown that the affine groups are the only instances. As might be expected, no new instances exist with the present, more general kind of graph. But there is always the possibility of adding two edges and one new node, such as 

. The action of the new node adds a quantity γ to the first m components and subtracts the same quantity γ from the last m' components. It takes no great effort to generalize the formula in section 6.1 to

$$\gamma = \frac{m+m'-2}{m+m'}\sum_1^n x_i - \sum_1^m x_i + \sum_1^{m'} x_{n-i} - c \ .$$

The only group of this type worthy of any deeper study is of course $\tilde{A}_{n-1}$.

9.5. **A special study of $\tilde{A}_{n-1}$.** We are back at the starting point, the representation of the symmetric group $S_n = A_{n-1}$ by permutations of $(1, \ldots, n)$. Obviously, A_{n-1} is a subgroup of $\tilde{A}_{n-1}$: 

, and, as always, we look for an extension of the permutation representation to this infinite group. We just found a linear representation on $\mathbf{R}^n$, and it turns out that the set of permutations of $(1, \ldots, n)$ is part of an orbit, that models $\tilde{A}_{n-1}$ faithfully.

In this case, we do not even bother to give legality criteria for other e-vectors than $(1, \ldots, n)$. Nor do we consider any other c-value than $c = n$.

Proposition 61. *The group $\tilde{A}_{n-1}$ can be represented as the set of integer vectors $(x_1, \ldots, x_n)$ satisfying*

- $x_1 + \cdots + x_n = n(n+1)/2$.
- *No two x_i are congruent modulo n.*

The generators $s_1, \dots, s_{n-1}$ act as transpositions of adjacent components and the last generator as $s_n : (x_1, x_2, \dots, x_n) \mapsto (x_n - n, x_2, \dots, x_{n-1}, x_1 + n)$. If the identity element is represented by the vector $e = (1, \dots, n)$, the length of the element $w = (x_1, \dots, x_n)$ is given by

$$l(w) = \#(\text{inversions}) + \sum \left\lfloor \frac{|x_{i_1} - x_{i_2}|}{n} \right\rfloor .$$

The descent set $D(w)$ contains s_i if $x_i > x_{i+1}$ and s_n if $x_n > x_1 + n$.

Proof. The faithfulness of the representation is immediate from the fact that the corresponding numbers game starts with -1 on each node.

Both criteria are true for $(1, \dots, n)$ and are invariant under the s_i. Now, assume that there are vectors satisfying the criteria, but not appearing in the representation. Among such vectors, select those with *minimal span*, i.e. minimal $(\max x_i - \min x_i)$ and choose one such *nondecreasing* vector $x = (x_1, \dots, x_n)$. We claim that the span $(x_n - x_1)$ is exactly $n-1$. It cannot be less, for it must accommodate n different congruence classes modulo n. And it cannot be more, for then $x' = (x_n - n, x_2, \dots, x_{n-1}, x_1 + n)$ would have a smaller span, and therefore both x' and $x = s_n(x')$ belong to the representation. So $x = (k+1, \dots, k+n)$, and the first criterion forces $k = 0$. But $x = e$ certainly belongs to the representation, so the contradiction completes the proof.

The length function counts the number of separating hyperplanes, and using our usual methods we can establish that all these are given by $x_i \equiv x_j (\text{mod } n)$. The descent set follows from the numbers game. □ □

This representation is also known from the work of Lusztig, see comment after Prop. 63. The length formula appears in Shi [70]. A forthcoming book [10] by A.Björner and F.Brenti has a more elegant one-term version of the length formula, namely

$$l(w) = \sum_{1 \le i < j \le n} \left| \left\lfloor \frac{x_i - x_j}{n} \right\rfloor \right| .$$

10. The affine groups as infinite permutations

A *permutation representation* of a finite group G is an embedding $G \subseteq S_n$. By Cayley's theorem, such an embedding always exists with n equal to $|G|$. We have been using the term *permutational representation* when one generator has a linear action which is not a permutation.

In order to convert our permutational representations to permutation representations proper, we have computed the values appearing as vector components and described them as linear expressions in the start numbers e_j. Every group element $(x_1, \dots, x_n)$ has a natural action as a permutation of this set of expressions, namely that given by $\sum \lambda_i e_i \mapsto \sum \lambda_i x_i$, for each x_i is also such a linear expression in the e_j.

In B_n, the set of values is $\{\pm e_i\}$ and the group element $(x_1, \dots, x_n)$ permutes them like $\pm e_i \mapsto \pm x_i$. (Of course, every x_i is of the type $\pm e_j$.)

The algorithm used to find the set of values was described on page 19. It is only a special case of a general method of finding a permutation representation of a matrix group W, namely the following procedure. Let $\{u_1, \ldots, u_n\}$ be a basis for $\mathbf{R}^n$ and let $\mathcal{U} = \{Au_i \mid A \in W\}$. Evidently, W is now represented as permutations of $\mathcal{U}$, and the representation is faithful, since the u_i span $\mathbf{R}^n$. In general, $\mathcal{U}$ is an infinite set, and we should speak about *bijections* rather than permutations.

There is of course a dual procedure using row vectors (or transposed matrices) and our set of values algorithm is of that kind. Another variation is the use of an affine representation instead of a linear one. The linear expressions in the e_i acquire a constant term, but that is the only modification necessary.

If integer values are given to $e_1, \ldots, e_n$, such that all our different expressions take different integer values, then the group elements act as bijections of $\mathbf{Z}$. One can also express this situation as an embedding into S_∞, the countably infinite symmetric group. The theme of this section is to try and find such embeddings for the affine groups.

The natural choice, $e = (1, \ldots, n)$, does indeed supply such an embedding $\tilde{A}_{n-1} \subseteq S_\infty$. The permutational vector $(x_1, \ldots, x_n)$ is extended infinitely in both directions by the simple formula $x_{i+n} = x_i + n$. Conversely, every such n-translative infinite permutation with $(x_1 + \cdots + x_n) = n(n{+}1)/2$ belongs to the representation.

Proposition 62. *In the defined embedding of $\tilde{A}_{n-1}$ into S_∞, an infinite permutation vector $(\ldots, x_{-1}, x_0, x_1, \ldots) \in S_\infty$ represents an element in $\tilde{A}_{n-1}$, if two conditions are satisfied*

(1) $x_{i+n} = x_i + n$ *for all* i
(2) $x_1 + \cdots + x_n = n(n{+}1)/2$

Proof. A consequence of Proposition 61. Note that a vector may satisfy both conditions without being a permutation! The criterion for that is that the numbers $x_1, \ldots, x_n$ belong to different congruence classes modulo n. □ □

A simple reflection s_i maps onto a permutation of order two, that is with fixed points and two-cycles only. Such a permutation is conveniently illustrated as in the figures below. For $\tilde{A}_n$, the structure is periodic with period n, for the two-cycles of s_i are of the form $(i{+}kn, i{+}1{+}kn)$.

A natural mechanical model for this structure is a pile of n rulers, each with a protruding pin at every nth mark. The pinheads are round and so large that when a ruler is put on top of another, the pins must occupy different positions. In the complete ruler pile, the only movement possible is switching two neighbour pins by sliding their rulers one unit relative to the pile. The pinheads of ruler 1 are marked $\ldots, 1{-}2n, 1{-}n, 1, 1{+}n, 1{+}2n, \ldots$ etc, so a consecutive sequence of n pinhead numbers has got all congruence classes modulo n in it. Only one such sequence has the sum $n(n{+}1)/2$, and that is the permutational vector.

Before we move on to the $\tilde{B}$-, $\tilde{C}$- and $\tilde{D}$-type groups, let us consider the corresponding finite Coxeter groups as permutations of the set of integers $[-n, \ldots, n]$. The symmetric group action can be envisioned as a mirror at $x = 0$ and therefore, the ordinary transposition by s_3 has a mirror action on the negative side. The figure also shows the two different actions of the special node s_n.

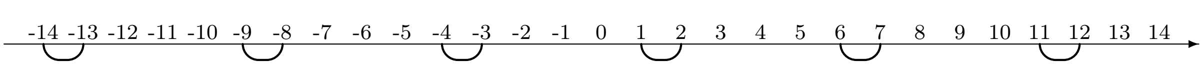


FIGURE 3. The action of $s_1 \in \tilde{A}_4$ as transpositions on $\mathbf{Z}$.

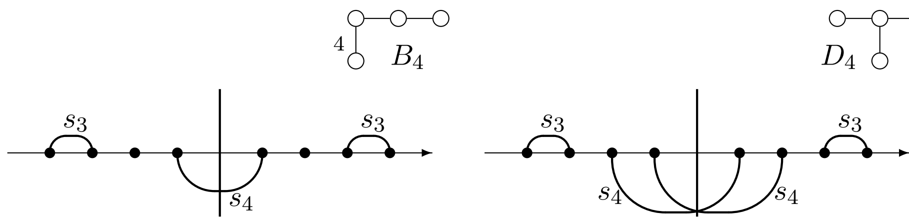


FIGURE 4. The actions of s_3 and s_4 in B_4 and D_4

The corresponding affine groups have an extra node, s_{n+1} and in the previous sections, we have established that its action must be

$$(\ldots, x_{n-1}, x_n) \overset{s_{n+1}}{\mapsto} \begin{cases} (\ldots, c' - x_n, c' - x_{n-1}) & \text{for } \tilde{B}_n \text{ and } \tilde{D}_n \\ (\ldots, x_{n-1}, c' - x_n) & \text{for } \tilde{C}_n \end{cases} .$$

The value $c' = 2n + 2$ can be used in all cases and has a direct optical significance: a second mirror is erected at $x = n + 1$.

We get embeddings into S_∞ by extending the permutational vector $(x_1, \ldots, x_n)$ infinitely in both directions by the mirror formulas

$$\begin{aligned} x_{-i} &= -x_i \ , \\ x_{2n+2-i} &= 2n{+}2 - x_i \ . \end{aligned}$$

Note that as a consequence of these two mirror relations, the $(2n + 2)$-translative property $x_{2n+2+i} = 2n{+}2 + x_i$ holds!

It is now possible to characterize the elements of S_∞ that belong to these representations.

Proposition 63. *An infinite permutation vector $(\ldots, x_{-1}, x_0, x_1, \ldots) \in S_\infty$ represents an element in the defined embeddings of $\tilde{B}_n, \tilde{C}_n, \tilde{D}_n$ into S_∞, if two conditions are satisfied (three for $\tilde{B}_n$ and $\tilde{D}_n$), namely*

1. *$x_{-i} = -x_i$ for all i.*
2. *$x_{2n+2-i} = 2n{+}2 - x_i$ for all i.*
3. *Among $x_1, \ldots, x_n$, an even number have odd $\left\lfloor \frac{x_i}{2n+2} \right\rfloor$. ($\tilde{B}_n$ only)*
4. *Among $x_1, \ldots, x_n$, an even number have odd $\left\lfloor \frac{x_i}{n+1} \right\rfloor$. ($\tilde{D}_n$ only)*

Proof. Exactly as in the $\tilde{A}_n$-case, we first check that the conditions are invariant, then assume that there are vectors outside the representation and satisfying the conditions, select such a vector with minimal $(x_1, \ldots, x_n)$-span and derive a contradiction. □ □

Remark 64. It is clear that $\tilde{B}_n$ is $\tilde{C}_n$-like at one end and $\tilde{D}_n$-like at the other. Depending on which end goes to 0 and which goes to $n{+}1$, we get different representations. The reader should have no difficulty in finding big-endian versions of the little-endian ones given here. For instance, in the third condition above, the fraction is modified to $\left\lfloor \frac{x_i+n+1}{2n+2} \right\rfloor$.

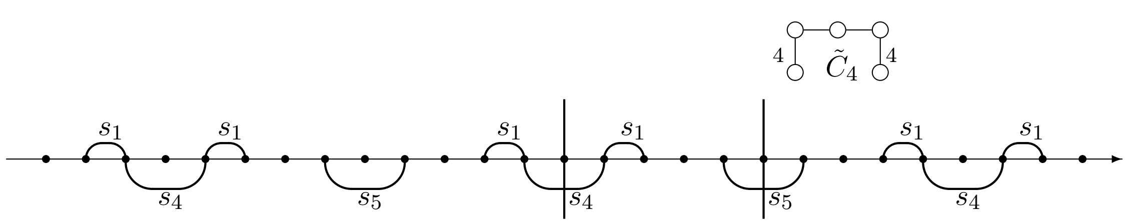


FIGURE 5. The actions of $s_1, s_4, s_5 \in \tilde{C}_4$ as transpositions on $\mathbf{Z}$.

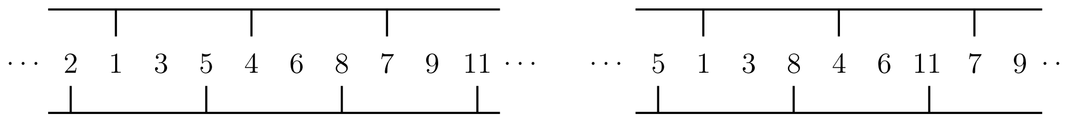


FIGURE 6. Single and double class inversion in $\tilde{A}_3$.

Remark 65. The infinite permutation representation of $\tilde{A}_n$ and its description in Prop. 62 was stated (without proof) by Lusztig [64]. Representations for $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, similar to the ones put forward here, were also known to Lusztig (private communication). A more detailed treatment will appear in the forthcoming book by Björner and Brenti [10].

Although these infinite permutations are fully determined by the components $x_1, \ldots, x_n$ and really only flamboyant versions of our old permutational representations, they are so intuitively appealing that we shall restate some simple facts in the language of transpositions and mirrors.

The ordinary descents $x_i > x_{i+1}$ in the fundamental segment $x_1, \ldots, x_n$ determine which s_i belong to the *descent set*. That holds true also for s_n in the $\tilde{A}_{n-1}$-case, but for $\tilde{B}_n, \tilde{C}_n$, the relevant descent for s_n is $x_{-1} > x_1$, and for $\tilde{D}_n$ it is $x_{-2} > x_1$. This is quite natural, for those are the pairs transposed by s_n. For s_{n+1}, the relevant descent is $x_n > x_{n+2}$ in $\tilde{C}_n$ and $x_n > x_{n+3}$ in $\tilde{B}_n, \tilde{D}_n$.

For an ordinary permutation, the *length* is the same as the number of inversions. Counting inversions in the infinite permutations does not make sense, however, for if $x_i > x_j, i < j$ is an inversion pair, so are infinitely many other pairs, namely those generated by n-translations in the $\tilde{A}_{n-1}$-case and those generated by mirror reflections in the other cases. If we count such an infinite set of translated or mirrored pairs as one *class inversion*, the length function will again be the inversion count! Note that a pair of classes may contribute more than once to the inversion count, as illustrated above. In the second case, $(5,1)$ and $(5,4)$ are two different class inversions.

There are many other ways of expressing the same concept. Counting class inversions in the $\tilde{A}_{n-1}$-case, is the same thing as counting inversions, where the first x_i is in the fundamental segment $(x_1, \ldots, x_n)$, or as counting inversions where the first value is in $\{1, \ldots, n\}$ etc.

For $\tilde{B}_n$ and $\tilde{C}_n$, s_n creates an inversion *within* a class. Instead of counting these, we can clearly count inversions *between* that class and the artificial class $\ldots, -c', 0, c', 2c', \ldots$.

Proposition 66. *The length of an infinite permutation is the number of class inversions. By translations or mirror reflections, $x_1, \ldots, x_n$ define one class each, and these are the classes used for $\tilde{A}_{n-1}$ and $\tilde{D}_n$. For $\tilde{B}_n$ and $\tilde{C}_n$, the class generated by 0 should also be considered in counting class inversions and for $\tilde{C}_n$ also the class generated by $n+1$.*

Proof. We only have to translate our old length formulas in propositions 32, 48, 61 into the new language, and we start with $\tilde{A}_{n-1}$. Let x_i, x_j be two class representatives in the fundamental segment and assume $x_i > x_j, x_j+n, \ldots, x_j+kn$, but $x_i < x_j+(k+1)n$. This means k class inversions if $i>j$ or $k+1$ if $i<j$, and these counts add up to our old formula

$$l(w) = \#(\text{inversions}) + \sum \left\lfloor \frac{|x_i - x_j|}{n} \right\rfloor .$$

Again, in the $\tilde{D}_n$-case, let x_i, x_j be two class representatives in the fundamental segment and assume $x_i > x_j, x_j+c', \ldots, x_j+kc'$, k maximal. Here we have k class inversions if $i > j$, or $k+1$ if $i < j$, and these add up to $\#(\text{inversions}) + \sum \left\lfloor \frac{|x_i - x_j|}{c'} \right\rfloor$ in our old formula. But there are more mirror images of x_j that may give inversions with x_i. If $x_i > -x_j$, assume $x_i > c'-x_j, 2c'-x_j, \ldots, kc'-x_j$, k maximal. This means k more inversions, where $k = \left\lfloor \frac{|x_i + x_j|}{c'} \right\rfloor$.

Finally, if $x_i < -x_j$, some mirror images of $x - j$ to the left of x_i give inversions, namely $-x_j - kc', \ldots, -x_j - c', -x_j > x_i$, k maximal, that is $k+1$ more inversions. Together with the previous term, it all adds up to $\#(x_i + x_j < 0) + \sum \left\lfloor \frac{|x_i + x_j|}{c'} \right\rfloor$.

For $\tilde{B}_n$, we must also count inversions represented by x_i and 0, where x_i is assumed to be in the fundamental segment. Depending on whether x_i is negative or positive, class inversions are given either by $0 > x_i, x_i+c', \ldots, x_i+kc'$, with k maximal, or by $x_i > c', \ldots, kc'$, with k maximal, summarized in $\#(x_i < 0) + \sum \left\lfloor \frac{|x_i|}{c'} \right\rfloor$ as expected.

For $\tilde{C}_n$, we also consider class inversions represented by $\frac{c'}{2}$ and x_i. Their contribution is 1 if $\frac{c'}{2} < |x_i| < \frac{3c'}{2}$, it is 2 if $\frac{3c'}{2} < |x_i| < \frac{5c'}{2}$ etc and so they nicely fill out the last term to give $\ldots \sum \left\lfloor \frac{|2x_i|}{c'} \right\rfloor$. □ □

A *reflection* is a conjugate wsw^{-1} of a simple reflection s. For ordinary permutations, reflection means transposition and simple reflection means adjacent transposition. For the infinite permutations, reflection still means transposition, *class transposition* that is, for by translation or mirror images, a transposition pair defines a transposition of classes. Of course, only those transpositions that comply with the conditions of Proposition 63 can be considered. The truth of this statement is evident if one thinks of wsw^{-1} as "permute, transpose, unpermute".

Example. What are the legal transposition partners of x_1 in the interval $(-c', c')$? (We let $c' = n$ in $\tilde{A}_{n-1}$ and $c' = 2n + 2$ in the mirror groups.) In all cases, the translate $x_{1-c'}$ is illegal. In the mirror groups, the fixed points $-\frac{c'}{2}, 0, \frac{c'}{2}$ are always illegal. The point $c' - 1$ is illegal in $\tilde{B}_n$ and $\tilde{D}_n$ and the point -1 in $\tilde{D}_n$.

The intuitive appeal of these S_∞-embeddings prompts the following question. What about the sporadic affine groups, $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$ and $\tilde{A}_1$, and embeddings into S_∞? We have looked into this question and can answer it affirmatively.

To start with the trivial case, $\tilde{A}_1$ is the affine group $\circ \overset{\infty}{\text{—}} \circ$, also denoted by $I_2(\infty)$. Its elements are of three types, words starting with s_1, like $s_1 s_2 s_1 s_2 s_1 \ldots$, words starting with s_2, like $s_2 s_1 s_2 s_1 s_2 \ldots$ and the identity e. The simplest representation of an element is of course the integer value $\pm l(w)$, where a plus sign denotes the first type and a minus sign the second type. Using an infinite permutation would indeed be overkill.

There are embeddings of $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ into S_∞, obtainable from the permutational representations by computing the occurring component values as expressions in the e_i. We have already found these expressions for the corresponding finite groups in Proposition 19 and further computation shows that the affine groups have the same set of expressions, but translated by an arbitrary multiple of c, the constant in the affine transformation.

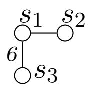
Next, we need some start vector such that all different expressions have different numerical values. After an extensive computer search, one finds that the following suggestions are best possible in some sense.

$\tilde{E}_6$: $e = (3, 6, 11, 13, 19, 20)$, $c = 42$
$\tilde{E}_7$: $e = (7, 14, 32, 34, 35, 36, 40)$, $c = 91$
$\tilde{E}_8$: $e = (12, 17, 31, 39, 46, 47, 58, 72)$, $c = 150$

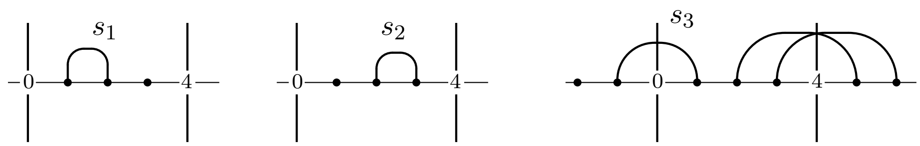
Since the s_i permute value expressions and value expressions are faithfully mapped onto $\mathbf{Z}$, we now have infinite permutations representing these groups. They seem exceptionally unattractive.

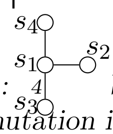
For $\tilde{G}_2$, this road is not even practicable, for in remark 44 we noted that no integer e-vector can give all expressions different values. However, there is a much simpler way of constructing permutation representations of Coxeter groups. The last section of this chapter will present the Cox Box, a construction kit for crackerjack embedders, but right now we are content to admire the finished construction for $\tilde{G}_2$.

Proposition 67. *The affine group $\tilde{G}_2$ can be embedded into S_∞ in the following way.*

The fundamental segment for $\tilde{G}_2$: 

is (x_1, x_2, x_3) with mirrors at 0 and 4 and s_i-actions



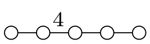
The embedding can be factored via $\tilde{B}_3$: 

by letting $s_3 \in \tilde{G}_2$ correspond to $s_3 s_4 \in \tilde{B}_3$, while s_1, s_2 are unchanged. An infinite permutation in $\tilde{B}_3$ is in $\tilde{G}_2$ if the sum $x_1 + x_2 + x_3$ equals 6 or 10.

Proof. If we throw away everything but the fundamental segment, we have something very similar to the permutational representation derived in section 8.3, the only difference being that the action of s_3 on (x_1, x_2, x_3) used to be $(-x_1 + x_2 + x_3 - 8, x_2, x_3)$, but now is $(-x_1, 8 - x_3, 8 - x_2)$.

Recall that the orbits used to live on a paraboloid surface but now reside in two parallel planes (coinciding if we use an e-vector with $e_1 + e_2 + e_3 = 8$). By projecting the orbit onto the plane, one obtains our new representation. We leave the simple details as an exercise. □ □

Proposition 68. *The remaining affine groups, $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4$, also have embeddings into S_∞, more complicated, but similar to the others.*

Proof. For $\tilde{F}_4$: 

, the fundamental segment is $(x_1, \ldots, x_{18})$, with mirrors at 0 and 19. The transpositional actions are

s_1 : $3 \leftrightarrow 5$, $4 \leftrightarrow 6$, $9 \leftrightarrow 11$, $10 \leftrightarrow 12$, $15 \leftrightarrow 17$, $16 \leftrightarrow 18$,
s_2 : $1 \leftrightarrow 3$, $2 \leftrightarrow 4$, $7 \leftrightarrow 9$, $8 \leftrightarrow 10$, $13 \leftrightarrow 15$, $14 \leftrightarrow 16$,
s_3 : $-1 \leftrightarrow 1$, $-2 \leftrightarrow 2$, $7 \leftrightarrow 8$, $(9 \ldots 12) \leftrightarrow (15 \ldots 18)$,
s_4 : $(1 \ldots 6) \leftrightarrow (7 \ldots 12)$, [short for $1 \leftrightarrow 7$, $2 \leftrightarrow 8, \ldots$]
s_5 : $(7 \ldots 12) \leftrightarrow (25 \ldots 20)$ [short for $7 \leftrightarrow 25$, $8 \leftrightarrow 24, \ldots$].

For $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, the permutations of values in the previous page is already of the type sought, except that there are many unused integers. By moving values closer to the origin, one can fill out the holes and get something like the other infinite permutations. □ □

Remark 69. The fact that all affine groups can be represented by infinite periodic permutations is a consequence of the periodic structure of their root systems. Figures 11 and 12 illustrate the fact that the root system of an affine group can be obtained by iterated translation of the finite set of roots of the correponding tilde-less subgroup. In principle, one can arbitrarily assign a tilde-less root to each integer in an interval of $\mathbf{Z}$ and let all other roots be assigned values obtained by iterated translation by the length of that interval. Since group elements permute the roots, one does indeed obtain a representation by infinite permutations in this way.

11. The Cox Box for building permutation representations

Most important algorithms of computational group theory, such as coset enumeration, use generators and relations as antithetic forces – the generators are constructive and produce new elements by multiplication, the relations are destructive and make elements disappear or coincide. In the case of Coxeter groups, all generators have order two, $s^2 = e$, and so they are, so to speak, less productive than in the general situation. What we are going to present here is perhaps more of a game than a systematic approach to the alternative strategy of *using the relations as building blocks.* [THIS SECTION INTENTIONALLY LEFT VAGUE!]

11.1. **The Cox Box blocks.** Let us open the Cox Box and look at some of its contents. The box has several compartments, one for each Coxeter graph edge, such as $\overset{a}{\circ}\overset{4}{—}\overset{b}{\circ}$. Letters are used for node numbering to avoid confusion. All edges have their compartments, even the invisible ones (meaning edge label 2). Here are some of the Cox Box blocks to be found in these compartments.

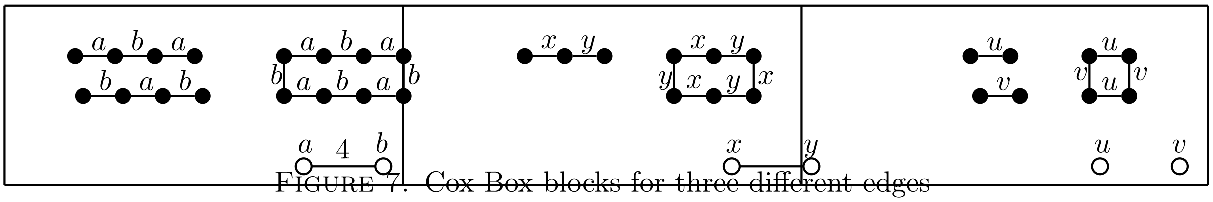
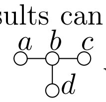
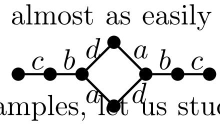
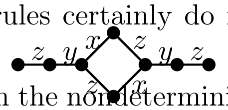


FIGURE 7. Cox Box blocks for three different edges

Edges that bear the same label stick together, and that is how you construct with Cox Box blocks. Example: $\bullet\overset{x}{—}\bullet\overset{y}{—}\bullet + \bullet\overset{y}{—}\bullet\overset{z}{—}\bullet = \bullet\overset{x}{—}\bullet\overset{y}{—}\bullet\overset{z}{—}\bullet$. Assuming that our Coxeter graph is B_3: $\underset{x}{\circ}—\underset{y}{\circ}\overset{4}{—}\underset{z}{\circ}$, we can use the blocks $\bullet\overset{x}{—}\bullet\overset{y}{—}\bullet$, $\bullet\overset{y}{—}\bullet\overset{z}{—}\bullet\overset{y}{—}\bullet$ and $\bullet\overset{y}{—}\bullet\overset{x}{—}\bullet$ to build $\bullet\overset{x}{—}\bullet\overset{y}{—}\bullet\overset{z}{—}\bullet\overset{y}{—}\bullet\overset{x}{—}\bullet$. This construction defines a certain subgroup of S_6, in the following way. The six black dots are elements of the permuted six-set, and each letter is a permutation of order two. For instance, z transposes the two middle elements, x transposes the first two and also the last two elements etc. But not only do we have the relations $x^2 = e, y^2 = e, z^2 = e$, we also have $xyx = yxy$ and $yzyz = zyzy$, as can be laboriously checked for each of the six points. Why is that? Because the Cox Box contains only blocks with these particular properties!

If we have another look at the resulting embedding into S_6, we may recognize the signed permutations model of B_3! That useful model materializes with no effort from our side, and similar results can be obtained almost as easily for almost all finite and affine Coxeter groups. From D_4:  we can build , easily recognizable as signed permutations.

Before we go on to other examples, let us study the rules of the game in some greater depth and reconsider B_3: $\underset{x}{\circ}—\underset{y}{\circ}\overset{4}{—}\underset{z}{\circ}$. How do we know when our construction is completed? Why, for example, can we not stop after $\bullet\overset{x}{—}\bullet\overset{y}{—}\bullet\overset{z}{—}\bullet\overset{y}{—}\bullet$? Well, if we do, we have constructed an embedding into S_5, in which $xyx = yxy$ is false. The right endpoint is taken one step to the left by xyx, but it is left fixed by yxy. In fact, a y-edge must always be connected to an x-edge and vice versa. The general rule is that an odd edge label, e.g. $\overset{x}{\circ}—\overset{y}{\circ}$ or $\overset{x}{\circ}\overset{5}{—}\overset{y}{\circ}$, forces x and y to appear together in the construction, never one without the other. A more evident rule is that each edge in the Coxeter graph must have at least one building block in the construction.

These rules certainly do not make the construction unique. Another solution for B_3 is the following: , and now we see an embedding $B_3 \subset D_4$, rather unexpected! But worse than the nondeterminism of the construction process is the sad fact that the resulting representation need not be faithful. It is possible to follow the construction rules and yet introduce new and unwanted relations. The moral is that representations built from the Cox Box should be verified by some other means.

11.2. **Examples of Cox Box constructions.** We have collected the simplest Cox Box constructions for some finite and affine groups in the figure in the next page. Note in particular, that all constructions can be carried out on $\mathbf{Z}$ and that we retrieve the infinite permutation representations that we have already made aquaintance with. As long as we are satisfied with permutations and do not insist on linear representations, noncrystallographic groups as H_3 involve no complications. The embedding $H_3 \subset S_{12}$ given here has a geometric interpretation: Let S_{12} denote permutations of the twelve vertices of an icosahedron, then H_3 is the subgroup generated by rigid rotations and reflections, i.e. the symmetry group of the regular solid.

It is evident from the diagrams that $H_3 \subset D_5$, a result published by Sekiguchi and Yano, [69]. The assertion in Humphreys [56] p 48, that $H_3 \subset D_3$ is a misprint.

Remark 70. We have not touched upon the question of which blocks to put into the Cox Box. In all the examples that we have tried, two kinds of blocks is enough in each compartment $\overset{a}{\circ}\underset{p}{\text{—}}\overset{b}{\circ}$. First, there is the $(p-1)$-path with alternating labellings and starting with either a or b. Second, there is the $2p$-circuit with alternating labels. Simple verification shows that these blocks do belong in this compartment, experience shows that no other blocks are needed.

Note. The Cox Box terminology was inspired by a Gilbert and Sullivan opera [48].

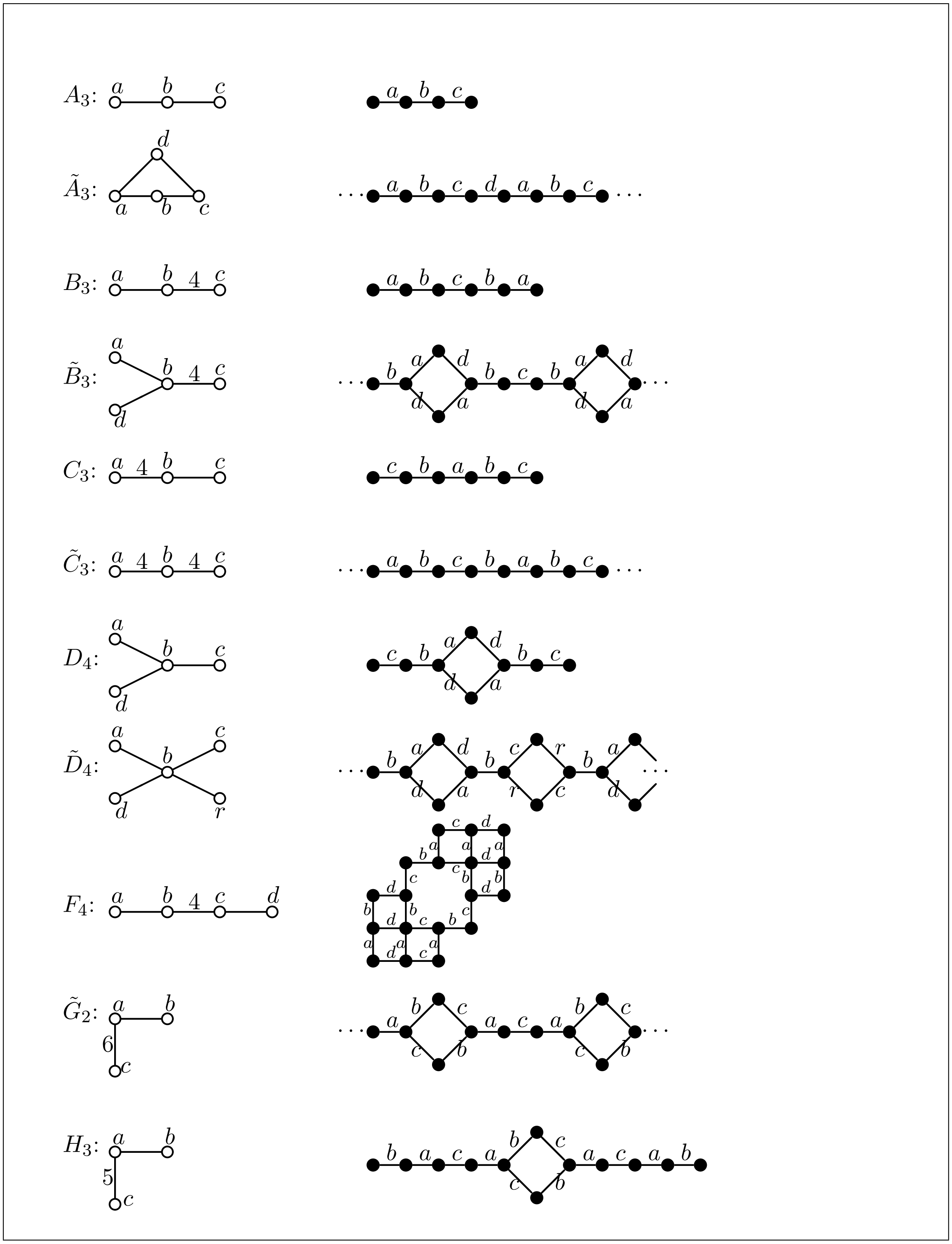


FIGURE 8. Permutation representations with Cox Box blocks

12. The Bruhat order for permutation representations

In the concluding section of this chapter, we are going to present a generalization of the tableau criterion for the Bruhat order in the symmetric group to most groups with permutation representations like those in figure 8. As we have seen, all finite and all affine groups have such representations, so this is a useful result. The criterion uses the embedding in S_k of S_∞, but for each group, it can be interpreted in terms of the permutational vector and result in formulas, much like the length formulas we have seen in so many versions. Our plans are to collect all these results in a forthcoming paper, and the sketchy presentation in this short section is just a preview of coming attractions.

Let us review some facts about the weak order and the Bruhat order for ordinary permutations. The Bruhat ordering is an extension of the weak ordering, in the following sense. A permutation π precedes a permutation σ in the weak ordering if there is a sequence of adjacent transpositions transforming π into σ and such that each transposition creates an inversion. Similarly, π precedes σ in the Bruhat ordering if there is a sequence of not-necessarily-adjacent transpositions transforming π into σ and such that each transposition creates an inversion.

Example. Let $\pi = (2,1,3,4)$ and $\sigma = (3,1,4,2)$. We have $l(\pi) = 1$ and $l(\sigma) = 3$ and a transposition chain $(2,1,3,4) \mapsto (3,1,2,4) \mapsto (3,1,4,2)$ demonstrating that $\pi < \sigma$ in Bruhat order. But there is no such chain using adjacent transpositions, so no weak order relation exists.

The *tableau criterion* (see page 10) involves sorting all initial segments and comparing them: $(2) \le (3)$, $(1,2) \le (1,3)$, $(1,2,3) \le (1,3,4)$, $(1,2,3,4) \le (1,2,3,4)$. The conclusion is that $\pi < \sigma$. The *dual tableau criterion* is equivalent, it sorts final segments instead: $(4) \ge (2)$, $(3,4) \ge (2,4)$, $(1,3,4) \ge (1,2,4)$, $(1,2,3,4) \ge (1,2,3,4)$.

We now want to generalize these things to the permutation representations of the previous sections. First note that some of the s_i are no longer adjacent transpositions, so it is not evident that the statement "s creates an inversion" has any sense. If, for example, s is a double transposition, it seems possible that one of them could create an inversion and the other one resolve an inversion. However, the symmetries of the groups, whether they be translations or mirror reflections, guarantee that all transpositions in s create inversions if one of them do. Just to take one example, in D_n, the two transpositions $x_{-2} \leftrightarrow x_1$ and $x_{-1} \leftrightarrow x_2$ together constitute s_n. But since $x_{-i} = -x_i$, inversions are created in neither pair or in both.

If we use *class inversion* for an inversion together with its translated and reflected images, and if every s_i creates or resolves exactly one class inversion (note the special treatment of the B, C-cases in Prop. 66), a general length formula is $l(w) = \#(\text{class inversions})$.

In all cases, a reflection element $t = wsw^{-1}$ is a not-necessarily-adjacent transposition, together with its symmetric transpositions. This is clear, as the action is "permute, transpose, unpermute". So the Bruhat order can be described combinatorially easily enough. But is there a generalization also of the tableau criterion? Yes, there is, and for the finite groups, there is absolutely no change from the symmetric group!

Proposition 71. *For a finite Coxeter group of type B, C, D, G or H, represented as permutations of $-n, \dots, n$, the Bruhat relation $\pi < \sigma$ holds when the following criterion is satisfied. Any initial segment $(\pi_{-n}, \dots, \pi_i)$, sorted in increasing order must be componentwise less than or equal to the corresponding sorted initial segment of σ.*

Proof. A transposition that creates a class inversion changes some of the sorted segments and always by replacing a number by a larger number. Therefore, the criterion is necessary.

To show sufficiency, it is enough to show the existence of a transposition τ such that $l(\pi) \le l(\sigma\tau) \le l(\sigma)$ and such that π and $\sigma\tau$ satisfy the criterion. Iteration will then give the full chain. We shall define τ as a transposition $\sigma_j \leftrightarrow \sigma_k$, together with $-\sigma_k \leftrightarrow -\sigma_j$. To find j, do like this. Ignore all positions where $\pi_i = \sigma_i$, these can be left fixed and might as well not exist. Let $a = \sigma_j = \pi_{j'}$ be the largest number left. The criterion implies $j < j'$. Among $\sigma_{j+1}, \dots, \sigma_{j'}$, let

$b = \sigma_k$ be the largest. Thus, all numbers $(b+1), \ldots, (a-1)$ are either to the left of σ_j or to the right of $\sigma_{j'}$. By the dual of the ordinary tableau criterion, those to the right are covered by the corresponding π-segment, so in the π-segment to the left of π_j, there are not sufficiently many elements greater than b for it to make a difference when b and a are interchanged in σ.

There is also the interchange of $-b$ and $-a$, but because of complete symmetry the same argument can be used. □ □

We would like to extend the result to the infinite permutations, but there seem to be complications. Is it possible to sort an infinite interval? Yes, it is! Assuming that the **Z**-axis has ben cut in two between x_0 and x_1, the right half-axis is sorted by putting its smallest element in x_1, its next smallest in x_2 etc. And the left half-axis sorts its largest element into x_0, its next largest into x_{-1} etc. So we do not have to change the criterion at all, just state it in a suitable way.

Proposition 72. *For an affine Coxeter group of type $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{F}$ or $\tilde{G}$, represented as infinite permutations of* $\mathbf{Z}$*, the Bruhat relation $\pi < \sigma$ holds when the following criterion is satisfied. Any initial half-infinite segment $(\ldots, \pi_i)$, sorted in increasing order must be componentwise less than or equal to the corresponding sorted initial segment of σ.*

Proof sketch. The necessity is simple, for a transposition that creates inversions replaces some numbers by greater numbers in some of the initial segments. The sufficiency is proved in the same fashion as before. First, we remove all positions where $\pi_i = \sigma_i$. Then, we find a suitable transposition pair and check that the criterion is still satisfied with π and $\sigma\tau$. □

Note. For other versions of tableau criteria for B_n and D_n, see Proctor [66]. Combinatorial Bruhat order criteria for some of the affine groups will appear in Björner and Brenti [10].

Chapter 3

Reduced word representations

13. Introduction

For any group defined by generators and relations, the natural thing is to represent group elements as *words in the alphabet* S, where S is the set of generators. The main advantages are evident, namely

- the multiplication algorithm is simply concatenation of words,
- a computer implementation can use available data structures, such as arrays or linked lists.

But there are also disadvantages, and these often outweigh the advantages.

- Many words correspond to the same group element, and there is, in general, no finite algorithm that solves the *word problem*, i.e. determines whether two different words are equivalent.
- In general, no finite algorithm exists that will solve the *reduction problem*, i.e. reduce any word to an equivalent shortest possible word.
- In general, not even the *recognition problem* can be solved for the language of reduced words, i.e. there may exist reduced words that cannot be proved to be such. (All nonreduced words w can of course be proved such by a finite word sequence $w = w_1, w_2, \ldots, w_n$, in which adjacent words are equivalent by one of the relations in the group presentation and the last word w_n is shorter than w.)

Coxeter groups are far better than the average group in these respects. The word problem is indeed solvable for any Coxeter group. Two words w, u are equivalent if and only if wu^{-1} is equivalent to the empty word, so one can apply the Word theorem of Tits (see [76]) to the product wu^{-1}. According to this theorem, if the reduction is at all possible, then it can be brought about using only the elementary substitutions $s_i s_i \to e$ and $s_i s_j \cdots \to s_j s_i \cdots$. As these never increase the length of the word, only finitely many words can be generated in this way, so after a finite number of steps it is known whether the empty word is among these.

The computational complexity of this method seems to increase exponentially with the word length $l(wu^{-1})$, so a better algorithm would be welcome. As an alternative to Tits's Word Theorem, we could refer to the faithful matrix representation, known to exist for any Coxeter group ([56], p 108ff). The word problem can be solved by simply performing the matrix multiplications corresponding to the two words and checking that the result is the same in both cases. However, the amount of computation can be reduced enormously if the matrix representation is replaced by a faithful orbit. For the affine Coxeter groups, the permutation representations of the previous chapter are such orbits and for the general infinite Coxeter group, the numbers game representation is of this kind. With any of these methods, the computational complexity of the word problem seems to be proportional to the word length, but this is not strictly true. The number of operations (additions and subtractions) is proportional to the word length, but the size of the numbers also grows linearly, so the *bit complexity* is proportional to $n \log n$.

The reduction and recognition problems are, in a sense, solved if the word problem is solved. For there are only finitely many words u of equal or smaller length than a given word w and, in principle, the word problem can be solved for every such pair (u, w). That algorithm would run in exponential time, even if the word problems are solved in linear time, so a better approach is clearly desirable. The numbers game and the permutational representations obviously provide $n \log n$–time recognition of reduced words, since the concept of a legal move sequence corresponds exactly to the reduced word concept. By the results of the next section, there is even a recognition algorithm with linear bit complexity.

It is less obvious how to find a reduction of a given nonreduced word by using numbers games or permutation representations, but the following algorithm solves this problem too in $n \log n$–time.

(1) Perform the move sequence corresponding to the given word w.
(2) Make any illegal move, i.e. corresponding to an s such that $l(ws) < l(w)$.
(3) Repeat step 2 until a position with no illegal moves appears. This must then be the starting position.
(4) The sequence of illegal moves in reverse order will correspond to a reduced word for w.

FIGURE 9. $n \log n$–time reduction algorithm for words in any Coxeter group

14. Language automata

Consider words on a finite alphabet $A = \{a, b, \ldots\}$. A *language* $L \subset A^*$ is a set of words. Languages are classified according to their complexity, finite languages being the simplest and regular languages coming next. A *regular language* can be defined by a *regular expression* (frequently used by computer programmers). As an example, the language defined by the regular expression

$$abba + c(aa)*$$

contains the words $abba, c, caa, caaaa, caaaaaa, \ldots$. In other words, the set of regular languages is the closure of the set of finite languages under unions $(+)$, concatenation and iteration$(*)$. Kleene (1956) characterized regular languages as those languages that can be recognized by a *finite automaton* (i.e. an ordinary computer with finite memory).

A closely related concept is that of a *rational language*. It is defined by Berstel and Reutenauer (1988) as a formal power series in the variables of the alphabet A

$$\sum_{w \in A*} \alpha_w w,$$

with coefficients in $\mathbf{Z}$ or $\mathbf{R}$ (or any semiring) and satisfying a certain rationality condition. For a one letter alphabet, rationality means that the series is the power series expansion of a rational function $P(a)/Q(a)$. For several variables, a rational series can be expressed using polynomials, addition, multiplication and inversion. It is known that the *support* of a rational language, that is the set of words with nonzero coefficients, is a regular language and, conversely, that

$$\sum_{w \in L} w$$

is rational for any regular language L (see Berstel and Reutenauer [6])

Our language L is going to be *the language of reduced words* in the alphabet S of a Coxeter group (W, S). The recognition algorithm based upon a permutational representation or the numbers game representation of the group does not correspond to a finite automaton, as the numbers involved are unbounded. This is clear already for $\tilde{A}_2$, considered as permutations of the set of all integers. In fact, no representation of the group elements can be used in the recognition automaton, for there is only a finite number of states available.

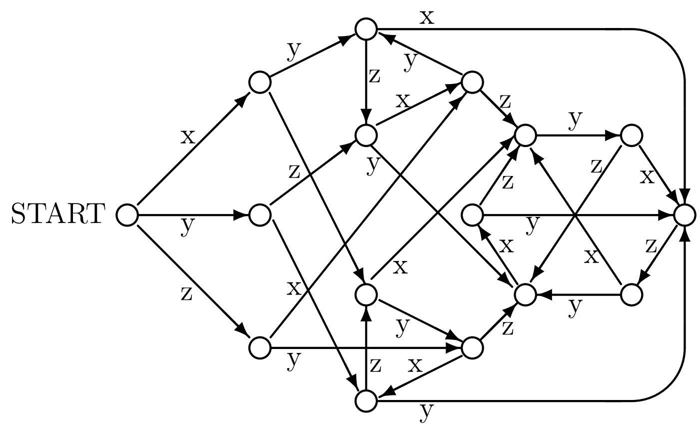


FIGURE 10. A finite automaton that recognizes reduced words in $\tilde{A}_2$

We can represent the automaton as a directed graph with edge labels in S, no two edges from one node bearing the same label, and with one node designated as the starting state. The automaton is supposed to enter a permanent error state when it reads a symbol for which there is no edge leading from the current state.

For a finite group, the Cayley graph itself is the required automaton. There is a node for each element and an arrow $w \xrightarrow{s} ws$ if $l(ws) = l(w) + 1$.

For an infinite group like $\tilde{A}_2$, some states must correspond to infinitely many different group elements. Elements sharing the same state must have the same descent set, $D(w)$, the same descent sets $D(ws)$ for each $s \notin D(w)$ and so on. The problem is to show that a finite amount of information is sufficient to determine all these descent sets. Knowledge of all reflections t, such that $l(wt) < l(w)$, i.e. the *reflection descent set* $\mathcal{D}(w)$ constitutes sufficient information, for (recall that reflections are palindromic words)

$$\mathcal{D}(ws) = \{s\} \cup \{sts \mid t \in \mathcal{D}(w)\} .$$

However, for the infinite groups there is no bound for the cardinality of the reflection descent sets, in fact $|\mathcal{D}(w)| = l(w)$. The solution is to be found in restricting one's attention to *small reflections*, of which there are only finitely many. A successful definition of small reflection must satisfy

$$\mathcal{D}_{\text{small}}(ws) = \{s\} \cup \{sts \mid t \in \mathcal{D}_{\text{small}}(w) \text{ and } sts \text{ is small}\} .$$

Our definition of small reflection has this property, and as we shall see, other possible definitions must include our small reflections. For this reason, the constructed automaton will be smallest possible. In our example $\tilde{A}_2$, there are six small reflections: x, y, z, xyx, yzy, zxz and the sixteen states correspond to the sixteen subsets that can occur as $\mathcal{D}_{\text{small}}(w)$. The starting state is the empty subset, the following three states are one element subsets $\{x\}, \{y\}$ and $\{z\}$, the six states in the two triangles are the two-element subsets $\{x, xyx\}$ etc and the hexagon states are three-element subsets $\{x, y, xyx\}$ etc and $\{x, xyx, xzx\}$ etc. It is easily seen that no other subsets can occur as $\mathcal{D}_{\text{small}}(w)$.

The automaton for $\tilde{A}_2$ was invented by Anders Björner in 1990 (unpublished). It was generalized by the present author ([39]) to $\tilde{A}_3$ and some other groups and by Richard Stanley to all $\tilde{A}_n$ (unpublished). Patrick Headley found automata for the other affine groups [52] and the detailed analysis by Kimmo Eriksson of the geometry of affine Coxeter complexes ([43]) also implies the regularity of all the affine Coxeter languages.

Gabor Moussong ([65]) made an important contribution by proving that all hyperbolic Coxeter groups have regular languages of reduced words (see [56] for a comprehensive list of hyperbolic

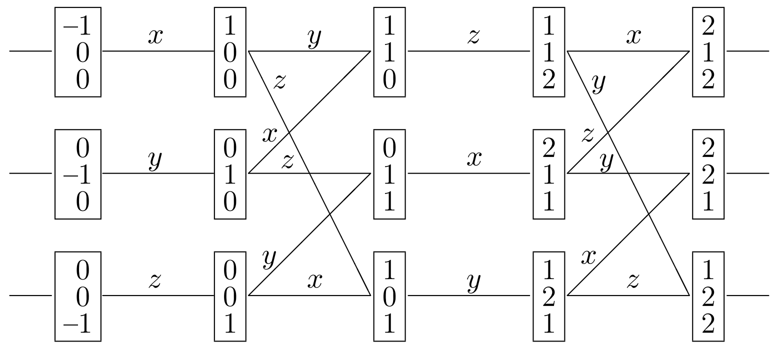


FIGURE 11. The edge labelled root poset of $\tilde{A}_2$.

Coxeter groups). In 1991, a preprint by Davis and Shapiro [34] stated the theorem that all Coxeter groups have regular languages; however, their proof remains incomplete. In 1993 Brigitte Brink and Robert B. Howlett ([17]) established that Coxeter groups are automatic. Their finite automaton recognizes the smaller language of *lexicographically minimal reduced words*, and it is not generally true that automatic groups have finite automata for the language of reduced words (see [20] for a counterexample). However, Brink and Howlett state (without proof) that their theorem 2.8 implies the parallel wall property conjectured by Davis and Shapiro, thus providing the missing link in their proof. For this reason, we believe that the following result should be attributed to Brink, Howlett, Davis and Shapiro.

Theorem 73. *For any Coxeter group, the language of reduced expressions is regular.*

Our next section is devoted to a proof of this theorem. Although it is much simpler than Brink's and Howlett's, the reader should be aware of the fact that the end product – the automaton – is very similar to the one in [17]. Another proof will appear in Patrick Headley's thesis [53].

15. A FINITE AUTOMATON FOR REDUCED EXPRESSIONS

Rather than working with reflections as palindromic words in the s_i, we shall invoke the standard geometric realization and deal with the corresponding *roots*. Recall from page 6 that the matrix $B_{i,j} = 2\cos\frac{\pi}{m_{ij}}$ defines a symmetric bilinear form, negative definite for the finite groups, negative semidefinite for the affine groups and indefinite for all other groups. To each primitive reflection s_i there is a *primitive root* $\boldsymbol{e_i}$ and a (not necessarily orthogonal) reflection matrix $\boldsymbol{S_i} = \boldsymbol{I} + \boldsymbol{e_i}\boldsymbol{e_i}^\top\boldsymbol{B}$, so in particular we have $\boldsymbol{S_i}\boldsymbol{e_i} = -\boldsymbol{e_i}$. A reflection $t = s_{i_k}\cdots s_{i_1}\cdots s_{i_k}$ corresponds to the matrix $\boldsymbol{T} = \boldsymbol{S_{i_k}}\cdots\boldsymbol{S_{i_1}}\cdots\boldsymbol{S_{i_k}}$, also a reflection with $\boldsymbol{T\alpha} = -\boldsymbol{\alpha}$ for the *root* $\boldsymbol{\alpha} = \boldsymbol{S_{i_k}}\cdots\boldsymbol{S_{i_2}}\boldsymbol{e_{i_1}}$. One of the roots $\boldsymbol{\alpha}$ and $-\boldsymbol{\alpha}$ has all components positive, the other one all components negative. The matrix $\boldsymbol{S_i}$ maps positive roots onto positive roots, with one exception: $\boldsymbol{e_i}$ maps to $-\boldsymbol{e_i}$. These properties are well-known, see Humphrey's book [56], chapter 5.

Recall that a position in the numbers game is a vector $\boldsymbol{p}$ and that the new position after playing node i is $\boldsymbol{S_i}^\top\boldsymbol{p}$. The simplest way to generate the roots is by playing *the dual of the numbers game*. A position in the dual game is a root $\boldsymbol{\alpha}$ and playing i leads from $\boldsymbol{\alpha}$ to $\boldsymbol{S_i\alpha}$. As $\boldsymbol{S_i} = \boldsymbol{I} + \boldsymbol{e_i}\boldsymbol{e_i}^\top\boldsymbol{B}$, only the i-th component of $\boldsymbol{\alpha}$ is affected, namely by adding to α_i the i-th component of $\boldsymbol{B\alpha}$. The dual game interpretation demonstrates that the roots constitute a symmetric graded graph (see figure 11), with each covering edge labelled by some s_i.

In connection with the numbers game, the roots have a particularly nice interpretation. The game starts with certain numbers $p_1, p_2, \ldots$ on the nodes, but in the course of the game, new numbers appear. All of these are certain linear combinations of the p_i, namely $\boldsymbol{\alpha{\cdot}p}$ for some root $\boldsymbol{\alpha}$. The primitive root $\begin{pmatrix}1\\0\\0\end{pmatrix}$ has the value p_1, the root $\begin{pmatrix}1\\1\\0\end{pmatrix}$ the value $p_1 + p_2$ (that appears on node one or two after one move in the numbers game) etc. A move from $\boldsymbol{p}$ to $\boldsymbol{S_i}^\top \boldsymbol{p}$ moves the value $\boldsymbol{\alpha}\cdot\boldsymbol{p}$ along an edge labelled s_i (if there is one, otherwise it stays), and it is clear that the only interchange between a positive and a negative root is between $\boldsymbol{e_i}$ and $-\boldsymbol{e_i}$. Usually, one starts with –1 on all nodes, so all positive roots have negative values at this stage (and vice versa). A legal play sequence is a reduced word $w = s_1 \cdots s_n$ and each move brings in exactly one positive value; therefore the number of positive roots with positive values equals $l(w)$.

All our statements about the numbers game and its dual game are proved in [42], so we move on to the heart of the matter: the states of the finite automaton for $\tilde{A}_2$.

15.1. An automaton for $\tilde{A}_2$.

Proposition 74. *Let $\mathcal{D}_{small}(w)$ be a subset of the six roots*

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},$$

namely those that have positive values after a numbers game starting with -1 on all nodes and with a play sequence corresponding to w. Then $\mathcal{D}_{small}(w)$ completely determines $\mathcal{D}_{small}(ws)$ for every $s \notin D(w)$. There are sixteen different $\mathcal{D}_{small}(w)$, and so there is a recognizing automaton with sixteen states. This is the minimal number of states for any recognizer of reduced words in $\tilde{A}_2$.

Proof. To fix ideas, we take an example. The reduced word $w = xyzy$ corresponds to the numbers game $(-1,-1,-1) \xrightarrow{x} (1,-2,-2) \xrightarrow{y} (-1,2,-4) \xrightarrow{z} (-5,-2,4) \xrightarrow{y} (-7,2,2)$, and the linear combinations $-7\alpha_1 + 2\alpha_2 + 2\alpha_3$ are positive for α in $\mathcal{D}_{\text{small}}(w) = \{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\}$. Now, we can determine $\mathcal{D}_{\text{small}}(wx)$ as follows. The positive values on $\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ move along their x-edges to $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\1\end{pmatrix}$. The positive value on $\begin{pmatrix}0\\1\\1\end{pmatrix}$ disappears from the set of small roots. A new positive value enters the primitive root $\begin{pmatrix}1\\0\\0\end{pmatrix}$. We can deduce all this from $\mathcal{D}_{\text{small}}(w)$ without knowing w. An apparent uncertainty concerns the value that enters $\begin{pmatrix}0\\1\\1\end{pmatrix}$ from $\begin{pmatrix}2\\1\\1\end{pmatrix}$. But that value must be negative, for $\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix}$, and we know that the fired number p_1 is negative and that $p_1 + p_2 + p_3 = -3$ is an invariant in this numbers game! This argument works for any w (and, in fact, for any affine group).

Only sixteen of the sixty-four subsets appear as $\mathcal{D}_{\text{small}}(w)$. There are restrictions of two kinds: if the values $\boldsymbol{e_i}\cdot\boldsymbol{p}$ and $\boldsymbol{e_j}\cdot\boldsymbol{p}$ on any two primitive roots are either both positive or both negative, then $\boldsymbol{(e_i + e_j)}\cdot\boldsymbol{p}$ has the same sign; all three values $\boldsymbol{e_1}\cdot\boldsymbol{p}$, $\boldsymbol{e_2}\cdot\boldsymbol{p}$, $\boldsymbol{e_3}\cdot\boldsymbol{p}$ cannot be positive, for $p_1 + p_2 + p_3 = -3$.

The constructed automaton is minimal if and only if for any two different states $\mathcal{D}_{\text{small}}(w_1)$ and $\mathcal{D}_{\text{small}}(w_2)$, there is a play continuation which is legal for one and illegal for the other. This is easily verified for the sixteen $\tilde{A}_2$–states but it is also a special case of theorem 80 below. □ □

15.2. Automata for affine groups. The $\tilde{A}_2$–automaton generalizes to all affine groups. The common feature of these groups is the linear invariant in the numbers game, which is a consequence of the fact that the matrix $\boldsymbol{B}$ is singular for the affine groups (see [42], p 64). By the

Group	# small roots	# states
$\tilde{A}_n$	$n(n+1)$	$(n+2)^n$
$\tilde{B}_n$	$2n^2$	$(2n+1)^n$
$\tilde{C}_n$	$2n^2$	$(2n+1)^n$
$\tilde{D}_n$	$2n(n-1)$	$(2n-1)^n$
$\tilde{E}_6$	72	13^6
$\tilde{E}_7$	126	19^7
$\tilde{E}_8$	240	31^8
$\tilde{F}_4$	48	13^4
$\tilde{G}_2$	12	7^2

TABLE 3. Small roots and states in the affine Coxeter groups

Perron–Frobenius theorem, the nullspace is spanned by a vector $\boldsymbol{\lambda}$ with nonnegative components and $\boldsymbol{\lambda}{\cdot}\boldsymbol{p}$ is the invariant. The invariant value is of course $-\sum \lambda_i$. The dual game is an extremely efficient algorithm for generating the roots recursively, but if we are satisfied with the small roots, the situation is even better: the small roots come out first and there are only finitely many.

Definition. The *small roots* are defined recursively as follows. All primitive roots $\boldsymbol{e_i}$ are small. If $\boldsymbol{\alpha}$ is a small root with an s_i-labelled edge to $\boldsymbol{\beta}$ and if $\alpha_i < \beta_i < \alpha_i + 2$, then $\boldsymbol{\beta}$ is small too.

The first part of our next proposition can be found as a remark, tucked away in a remote corner of Kimmo Eriksson's thesis [42], p 71.

Proposition 75. *The number of small roots in an affine group is equal to the total number of roots in the corresponding finite group (see table 3). Recognizing automata for the language of reduced words can use the subsets of type $\mathcal{D}_{small}(w)$ as states.*

Proof. Assisted by two computers, Jonas (human) and Balzac (electronic), we have constructed the small roots case by case. The correctness of the automata can be proved by the argument of the previous proposition. □ □

The computer also found the number of states in many small cases. The formulas in the table were, originally, verified by the computer only for $n \leq 8$. When they were presented at the Discrete Math Seminar at KTH, Ulf Berggren conjectured the following result, which, at that time, we were unable to prove. Patrick Headley later told us that a Shi [71] contained the required enumerative result. The following proposition also appears in [52].

Proposition 76. *For an affine group of type $\tilde{X}_n$, the number of states in the automaton is $(\frac{p}{n}+1)^n$, where p is the number of roots in the finite group X_n.*

Proof. The set of small roots defines a *hyperplane arrangement* $\boldsymbol{\mathcal{H}}$, namely the set of hyperplanes that are fixed by some small root reflection. The hyperplanes divide $\boldsymbol{R^{n+1}}$ into *compartments*, i.e. connected components of $\boldsymbol{R^{n+1}} - \boldsymbol{\mathcal{H}}$. A numbers game position $\boldsymbol{p}$ gives positive value to a certain root if it is on the positive side of the corresponding hyperplane. Therefore, the sets $\mathcal{D}_{\mathrm{small}}(w)$ correspond to the compartments, so to know the number of states in our automaton, we must count these compartments.

This is exactly what Shi [71] has done (although with different terminology) and his result is $(h+1)^n$, where h is the *Coxeter number* of the finite group. That number is defined as the order of a *Coxeter element* (see chapter 5 for definitions and some results) but it is known that $h = \frac{p}{n}$ (see page 79 in Humphreys [56]), so the formula of the proposition is also valid. □ □

15.3. **Automata for nonaffine infinite groups.** For each of the affine groups, we could compute the small roots and verify that there are only finitely many. The first step in the general case is proving this finiteness property. This result is equivalent to Brink's and Howlett's Theorem 2.8, and according to these authors, their theorem is equivalent to the Parallel Wall Theorem of [34]. The second step is proving that no positive value enters a small root from a neighbouring big root. In the affine case, the negative linear invariant sprang to our aid, but there is no such thing in the general Coxeter group. The final step is to check whether we have the minimal number of states.

Theorem 77. *For any Coxeter group, the small roots are only finitely many.*

Proof. Let $\boldsymbol{\alpha}$ be a small root and let $\boldsymbol{c} = \boldsymbol{B\alpha}$. For every $0 < c_i < 2$, there is an s_i-labelled edge to another small root, $\boldsymbol{\beta}$ in which α_i has been replaced with $\alpha_i + c_i$.

Let $\boldsymbol{T_\alpha}$ and $\boldsymbol{S_i}$ be the reflection matrices corresponding to $\boldsymbol{\alpha}$ and $\boldsymbol{e_i}$. A straightforward calculation shows that

$$\boldsymbol{T_\alpha}\,\boldsymbol{\alpha} = -\boldsymbol{\alpha} \quad , \quad \boldsymbol{T_\alpha}\,\boldsymbol{e_i} = \boldsymbol{e_i} + c_i\boldsymbol{\alpha}$$
$$\boldsymbol{S_i}\,\boldsymbol{e_i} = -\boldsymbol{e_i} \quad , \quad \boldsymbol{S_i}\,\boldsymbol{\alpha} = \boldsymbol{\alpha} + c_i\boldsymbol{e_i}$$

and that is the very definition of the dual game on a two-node graph with multiplier c_i in both directions. It is easy to check ([42], p 32) that the game is infinite if $c_i \geq 2$ and that otherwise (as in our case), we must have $c_i = 2\cos\frac{\pi}{m}$ for some integer m. Thus we conclude that $\boldsymbol{T_\alpha}$ and $\boldsymbol{S_i}$ generate a finite dihedral subgroup of W and that the order of $\boldsymbol{T_\alpha S_i}$ is m.

Bourbaki [15], Exercise V, §4, 2d), tells us that this dihedral group, or a conjugate of it, is contained in a finite parabolic subgroup of W. Now, there are only finitely many finite parabolic subgroups and each of them has only finitely many angles $\frac{\pi}{m}$ between reflecting hyperplanes. It follows that there are only finitely many *small change types*, i.e. $\boldsymbol{B\alpha}$-vectors where small components $-2 < c_i < 2$ are kept but all large components are replaced by either $+\infty$ or $-\infty$. If there were infinitely many small roots, then there would be some infinite chain of small roots $\boldsymbol{\alpha_0} < \boldsymbol{\alpha_1} < \boldsymbol{\alpha_2} < \ldots$, and some of these, say $\boldsymbol{\alpha_j}$ and $\boldsymbol{\alpha_k}$, would necessarily have the same small change type. Forget about $\boldsymbol{\alpha_0}, \ldots, \boldsymbol{\alpha_{j-1}}$, since we are going to redefine them presently!

Let $\boldsymbol{c} = \boldsymbol{B\alpha_j}$ and suppose that some $c_r \geq 2$. Then, the r-th component of $\boldsymbol{B\alpha_{j+1}}$ will be at least as big, for it increases by c_iB_{ir}, where c_i and B_{ir} are nonnegative. Iterating the argument we see that s_r will never be legal, so the r-th component is constant for $\boldsymbol{\alpha_j}, \ldots, \boldsymbol{\alpha_k}$. Analogously, if the r-th component of $\boldsymbol{B\alpha_k}$ is ≤ -2, the r-th component of $\boldsymbol{B\alpha_{k-1}}$ will be at least as negative etc. So the only edge labels that occur in the chain $\boldsymbol{\alpha_j}, \ldots, \boldsymbol{\alpha_k}$ are those that correspond to small components in $\boldsymbol{B\alpha_j}$ and $\boldsymbol{B\alpha_k}$. We are going to construct a chain continuation $\ldots, \boldsymbol{\alpha_{j-1}}, \boldsymbol{\alpha_j}$ using only these edge labels, so we can restrict our attention to these components of the $\boldsymbol{B\alpha}$-vectors.

Let the edge from $\boldsymbol{\alpha_{k-1}}$ to $\boldsymbol{\alpha_k}$ be labelled s_i. By assumption then, there is an s_i-labelled edge from $\boldsymbol{\alpha_j}$ to some $\boldsymbol{\alpha_{j-1}}$. The relevant components of $\boldsymbol{B\alpha_{j-1}}$ and $\boldsymbol{B\alpha_{k-1}}$ are identical, for both are $\boldsymbol{B\alpha_j} + c_i\boldsymbol{Be_i}$, so the chain can be continued by small changes. After j steps, we must reach a primitive root $\boldsymbol{e_r}$ and after one more step $-\boldsymbol{e_r}$. But here the change is exactly 2, so it is not small and we have a contradiction. □ □

Theorem 78. *Let $\boldsymbol{\alpha}$ be a small root connected to a big root $\boldsymbol{\beta}$ by an edge labelled s_i, then the value entering $\boldsymbol{\alpha}$ from $\boldsymbol{\beta}$, when node i is played, is negative.*

Proof. After playing node i, the root $\boldsymbol{e_i}$ certainly carries a positive value. If the value on $\boldsymbol{\alpha}$ is also positive, a contradiction ensues in the following way. As before, $\boldsymbol{T_\alpha}$ and $\boldsymbol{S_i}$ generate a subgroup, and since $c_i \geq 2$, it is the infinite dihedral group. Thus, there are infinitely many roots of the type $(\boldsymbol{T_\alpha S_i})^{\boldsymbol{k}}\,\boldsymbol{\alpha}$, and being positive linear combinations of $\boldsymbol{\alpha}$ and $\boldsymbol{e_i}$, they all carry positive values. But the total number of positive values on positive roots is finite, in fact equal to the length $l(w)$ of the element w corresponding to $\boldsymbol{p}$. □ □

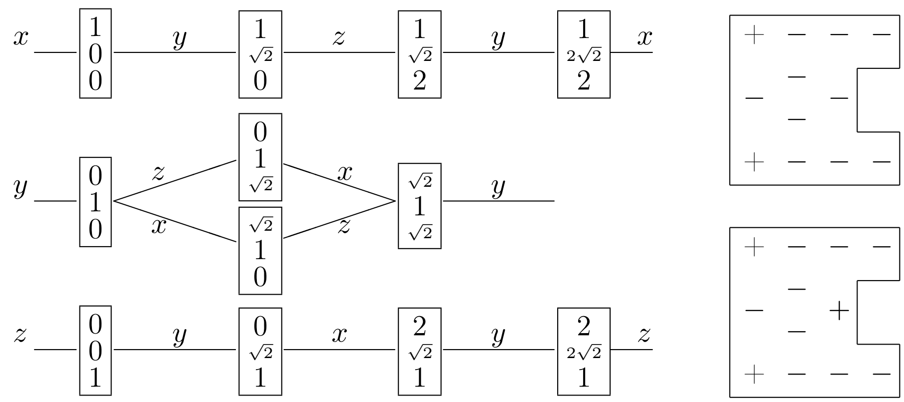


FIGURE 12. Small roots for $\tilde{C}_2$ and two states that should be identified.

Remark 79. The constructed automata are not necessarily minimal. For all affine groups except $\tilde{A}_{n-1}$, there are a few superfluous states. The simplest example is $\tilde{C}_2$ (see Figure 12). The two states shown lead to the same state, after one move.

Theorem 80. *The recognizing automaton constructed from the small roots is minimal for $\tilde{A}_n$.*

Proof. The structure of $\tilde{A}_{n-1}$ is evident in Figure 11. On each level there are n roots, and for $k < n$, the roots on level k consist of k ones in consecutive positions with wraparound.

We want to show that if two reduced words, w_1 and w_2, in $\tilde{A}_n$ have different states $\mathcal{M}_1 \neq \mathcal{M}_2$, then there is some continuation $s_{i_1}s_{i_2}\cdots s_{i_n}$ that is legal from one of the states and illegal from the other. The $\mathcal{M}$-sets consist of small roots that have positive values in the numbers game position $\boldsymbol{p}$ corresponding to the word w.

Assume that some small root $\boldsymbol{\alpha}$ is in $\mathcal{M}_1$ but not in $\mathcal{M}_2$. If $\boldsymbol{\alpha}$ is a primitive root $\boldsymbol{e_i}$, then s_i is a legal continuation in $\mathcal{M}_2$ but not in $\mathcal{M}_1$. If $\boldsymbol{\alpha}$ is a root on level two, say $\boldsymbol{\alpha = e_i + e_{i+1}}$ we know that $p_i + p_{i+1} < 0$ for the numbers game position corresponding to w_2, so either s_i or s_{i+1} must be legal. If this continuation is illegal for w_1, we have found what we are looking for. Otherwise, w_1s and w_2s are two new reduced words with states that differ already on level one, and we covered that case three sentences ago.

In general, for a root on level $k+1$, say $\boldsymbol{e_i + \cdots + e_{i+k}}$ with $p_i + \cdots + p_{i+k} < 0$ for w_2, at least one of $s_i, \ldots, s_{i+k}$ must be a legal continuation. If this s is either s_i or s_{i+k}, we see that w_1s and w_2s have states differing on level k and induction can be applied. If s is one of the intermediate s_j, the continuation leaves the differing values on the same root, so we can go for some time, choosing a legal s among the $s_i, \ldots, s_{i+k}$ etc. But this path corresponds to a finite subgroup, so this cannot go on for ever. In the end, the values are transferred downstairs and we can carry on by induction. □ □

From the $\tilde{C}_2$–example, it is clear that labelled edges in the Coxeter graph produce noninteger $2\cos\frac{\pi}{m_{ij}}$ and that these values appear in the roots. From the computational point of view, this is a major disadvantage. A mathematician with some Coxeter group experience would guess that the so called *crystallographic* groups could be handled with integer arithmetic, and this is indeed the case. In this connection, crystallographic just means that all edge labels are either 3, 4 or 6. The computer results for $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{F}_4$ and $\tilde{G}_2$ were obtained using the following method.

Instead of using the matrix $\boldsymbol{B}$, with symmetric entries $\sqrt{2}$ and $\sqrt{3}$, we can use either of the matrices $\boldsymbol{B', B''}$, defined as follows.

$$B_{ij} = B_{ji} = \sqrt{2} \;\Rightarrow\; B'_{ij} = 1,\; B'_{ji} = 2,\quad B''_{ij} = 2,\; B''_{ji} = 1$$

$$B_{ij} = B_{ji} = \sqrt{3} \;\Rightarrow\; B'_{ij} = 1,\; B'_{ji} = 3,\quad B''_{ij} = 3,\; B''_{ji} = 1$$

As proved in [42], $\boldsymbol{B'}$ or $\boldsymbol{B''}$ can replace $\boldsymbol{B}$ as weight matrices for the numbers game and the dual game. The numbers in the components will change, of course, but the signs of these numbers as well as the overall structure will remain intact. The problem is how to define small change and big change with these other weights. It turns out that the correct definition is the following one.

Definition. Let $\boldsymbol{c'} = \boldsymbol{B'\alpha'}$ be the change vector of a root in the dual game using the nonsymmetric, integer matrix $\boldsymbol{B'}$ and let $\boldsymbol{c''} = \boldsymbol{B''\alpha''}$ be the change vector of the corresponding root in the dual game using the transposed matrix $\boldsymbol{B''}$. Playing node i changes the i-th component of $\boldsymbol{\alpha'}$ as well as of $\boldsymbol{\alpha''}$, and it is called *small change* if $c'_i c''_i < 4$.

A root is *small* if it is accessible from a primitive root by a sequence of positive small changes.

Proposition 81. *The labelled graph of small roots according to the definition above is isomorphic to the graph of small roots as defined in section 15.2, but for crystallograhic groups, its computation requires integer arithmetic only.*

Proof. In one of the parallel dual games we use $S'_i = I + e_i e_i^\top B'$, in the other one $S''_i = I + e_i e_i^\top B''$. Thus, $\alpha' = S'_{i_k} \cdots S'_{i_2} e_{i_1}$ and $\alpha'' = S''_{i_k} \cdots S''_{i_2} e_{i_1}$. Formerly, we had the formula $T_\alpha = I + \alpha\alpha^\top B$, but this is now replaced by $T_{\alpha'} = I + \alpha'\alpha''^\top B$. We sketch a proof of this formula by induction. It is true for $\alpha' = e_i$, assuming that it holds for some α', it can be deduced for $\beta' = S'_i\alpha'$, using $T_{\beta'} = S'_i T_{\alpha'} S'_i$ and the relation $(B'S'_i)^\top = B''S''_i$.

We get
$$T_{\alpha'}\,\alpha' = -\alpha' \quad , \quad T_{\alpha'}\,e_i = e_i + c''_i\alpha'$$
$$S'_i\,e_i = -e_i \quad , \quad S'_i\,\alpha' = \alpha' + c'_i e_i$$

and as shown in [42], p 32, this is the nonsymmetric dual game of a finite dihedral group if $c'_i c''_i = 4\cos^2\frac{\pi}{m} < 4$. But we have shown earlier that this finiteness is also equivalent to small change. ◻ ◻

16. Normal forms for reduced words

We can now return to the main theme of this chapter — the computational aspects of representing Coxeter group elements by words in the generators. As in many other areas of computer algebra, notably the computations with polynomial ideals using Gröbner bases, the key to success is finding the optimal *normal form* of the algebraic object. For any group element w, there are many reduced words. Which one should represent w in the computer?

As a simple analogy, consider representing rational numbers $\frac{m}{n}$ by a data structure containing two integers, the second of which must be positive. The pairs $(3,6)$, $(1,2)$ and $(2,4)$ represent the same rational number, and everyone would pick $(1,2)$ as the normal form. Most mathematicians would use relative primeness to define this normal form, possibly with extra rules for the zero case, but there is a better definition. If we introduce an *ordering* of all pairs, we can always choose the *first* candidate as the normal form. In this case, the *backwards lexicographic order* corresponds to the relatively prime normal form.

We are going to define similar orderings of words in the s_i and show that the corresponding normal form is well suited for computation, particularly so for the finite groups. There is a large overlap between our work in this area and the exhaustive investigations of F. du Cloux [37], so we are going to be very brief.

16.1. **Orderings of words in Coxeter groups.** Any ordering of the generators s_i extends to a lexicographic order on the set of words. Note that this is not a well-ordering of the set of all words! In fact, no word has a lexicographic predecessor (for *bv* is preceded by *buzzz*...), so in computer science the length-then-lex ordering is more common. Since all reduced words for w have the same length, this is no problem in our case.

For the purpose of ordering words in finitely presented groups, the *transposed lex order* is often superior. It is not mentioned by Knuth [61] and we have not been able to find a reference to it, but, in all probability, it must have been used before. The transposition refers to a matrix Q of yes/no-questions, question Q_{ij} being “Is the i-th letter of the word an s_j?”. A comparison of two words involves posing these questions, one after the other, until one word answers "yes" and the other word "no". The yes-word is then proclaimed greater than the no-word.

Depending on the order in which the questions are put, we get eight different word orderings. The description is simplified if we assume the alphabetic order $s_1 > s_2 > \ldots > s_n$, as in the example below.

$$Q = \begin{array}{c c} & \begin{array}{ccc} c & b & a \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \end{array}$$

The ordinary lex comparison starts in the upper left corner and runs through the rows, one by one. The *reverse lex* order starts in the upper right corner, the *backward lex* order in the lower left and the *backward reverse lex* order in the lower right; all proceed row by row. These orders are all in common use. The *transposed* lex order starts in the upper left corner and runs through the columns, one by one, the *transposed reverse* lex order in the upper right corner etc. In the face of transposed backward reverse lexicographic order, chaos may seem preferable!

Example. $acb < bac$ in lex order but $acb > bac$ in transposed lex order. That is because acb will answer yes to Q_{21} while bac will answer yes to Q_{12}.

It is a trivial but important observation, that the normal forms corresponding to these orderings are all *hereditary*, i.e. any consecutive subword of a normal form is still a normal form.

To the transposed lex normal form is associated a factorization, obtained by making n–1 cuts, the i-th cut being made after the longest initial segment consisting only of the first i letters of the alphabet. For instance, $ababdbc = a, bab, \emptyset, dbc$. (Empty words may be included, so that there will always be n factors.) The following property is true for many groups, but we state it only for Coxeter groups.

Proposition 82. *Every element w factors uniquely as $w = w_1 w_2 \ldots w_n$, with $l(w) = l(w_1) + \cdots + l(w_n)$ and where the transposed lex normal form of w_i is either the empty word or a word starting with the i-th letter of the alphabet and not containing the $(i+1)$-st or later letters.*

Proof. The transposed lex normal form for w gives such a factorization. To prove that it is unique, we just have to note the characterization of w_n as the minimal length element of $W_J w$, where J is the parabolic subgroup generated by the first $n-1$ generators, and use induction. □ □

As we shall see in the next section, this factorization is useful in describing and recognizing as well as in computing with normal forms.

Up to now, the setting has been any finitely presented group, but now that we specialize to Coxeter groups, we are in for a big surprise!

Proposition 83. *For reduced words in a Coxeter group, the normal forms corresponding to lex order and transposed lex order coincide.*

Proof. Let w' be the lex normal form and w'' the transposed lex normal form of the same group element w, assume that $w' \neq w''$ and that this is the shortest such counter-example. Let s be the last alphabet letter occurring in w' and let $w' = w_1' s w_2'$, with s-less w_1'. The subword sw_2' is in lex normal form, so all equivalent reduced words must start with s too. Now, we construct a numbers game position on the nodes of the Coxeter graph, such that the reduced words for w correspond exactly to the legal terminating games. An obvious way is to take a terminal position (positive values on all nodes) and play backwards, following w', right to left. If, from that start position, we play w_1', we must reach a position with only node s playable.

Return to the start position, remove node s and start playing. The w_1'-game terminates in $l(w_1')$ moves, and therefore all games terminate after that number of moves. The same thing can be stated without reference to the numbers game: no reduced word for w can have its first

s-occurrence later than w'. Therefore, w'' has its first s in that same position (not earlier, for w'' is first in transposed lex order). But we also know that the terminal position is the same for all play sequences, so if $w'' = w_1''sw_2''$, we see that w_1' and w_1'' provide a shorter counter-example. Too short, maybe, if both are empty, but in that case w_2' and w_2'' will do. The contradiction proves the proposition. □ □

If lex normal form and transposed lex normal form are the same, why all the fuss about transposed lex order? The point is, of course, that we now have the factorization of Proposition 82, and all the good things that come with it.

16.2. **The normal form for finite Coxeter groups.** Let us order the generators s_i in any way and let the corresponding lexicographic order on words define a normal form for reduced words, as above. As our first example we take A_3: $\overset{s_1}{\circ}\!-\!\overset{s_2}{\circ}\!-\!\overset{s_3}{\circ}$ with its natural ordering. The reduced word s_3s_1 is not a normal form, for it can be rewritten as s_1s_3. There are 24 normal forms, one for each of the 4! elements and these can be displayed in the following way.

$$\left\{\begin{matrix}\emptyset\\ s_1\end{matrix}\right\}\left\{\begin{matrix}\emptyset\\ s_2\\ s_2s_1\end{matrix}\right\}\left\{\begin{matrix}\emptyset\\ s_3\\ s_3s_2\\ s_3s_2s_1\end{matrix}\right\}$$

Pick one alternative from each of the boxes, concatenate them in order, and you have a normal form! (We use $\emptyset$ to denote the empty word.)

Similar constructions can be written down for all finite Coxeter groups and all s_i-orderings. The paper by du Cloux lists the factorizations obtained by choosing the best s_i-ordering for each group. The bestness criterion is not explicitly stated, but seems to be of aesthetical character.

Assuming that $w = w_1w_2 \dots w_n$ is the factorization of Proposition 82, what can we say about the normal form of ws, where s is a generator letter? It turns out that only one factor is affected, and the change is a single deletion or insertion.

Proposition 84. [du Cloux] *The normal form for ws is obtained from the normal form for w by insertion or deletion of one letter.*

Proof. The proposition can be proved by repeated application of the exchange condition (see p 9), and this is essentially what du Cloux does in [37]. We shall only indicate the two essential steps in the proof.

If w_ns is reducible, it will still be an n-th normal form factor after the reduction, that is to say, all reduced words will still start with s_n. Otherwise we would have $w_ns = s'w'$, with $s' \neq s_n$. But then $w_n = s'w's$, contradicting the assumption that all reduced words start with s_n.

If w_ns is reduced, it will either still be an n-th normal form factor or, as above, we have $w_ns = s'w'$. Now $l(w_n) = l(w')$, so $s'w's$ is a reducible expression for w_n. Two letters must be deleted, and since we know that w_n does not begin with s' and does not end with s, these two letters are to be deleted. Therefore $w_ns = s'w_n$, so w_n is left intact and passes the extra letter on to w_{n-1}. □ □

The algorithm can be regarded (and programmed) as an array of n automata connected in series. A letter is input to the w_n-automaton, it is either used to change the state of the automaton, or it is transformed and output while the automaton keeps its state. Note also the recursive nature of the automata: The w_n-automaton consists of a similar array of n automata etc. The recursion is potentially infinite, but for all the finite groups it ends very quickly.

16.3. **The normal form for infinite Coxeter groups.** When the group is infinite, the lex order still gives a normal form, which factorizes as before. The difference is that some brackets now contain infinitely many words. For the affine groups, all brackets except the last can be made finite. As a simple example, we take $\tilde{A}_2$.

$$\left\{\begin{array}{c}\emptyset\\ s_1\end{array}\right\}\left\{\begin{array}{c}\emptyset\\ s_2\\ s_2s_1\end{array}\right\}\left\{\begin{array}{c}\emptyset\\ s_3\alpha^*\beta^*\\ s_3\alpha^*s_1s_2\gamma^*\end{array}\right\}, \text{ where }\left\{\begin{array}{l}\alpha = s_1s_2s_1s_3\\ \beta = s_2s_1s_3\\ \gamma = s_3s_1s_2\end{array}\right.$$

The star means zero or more repeats. It is understood that initial segments of the two regular expressions are implicitly listed in the brackets. In this case, it was possible to express an infinite set of words by a finite set of regular expressions and we state without proof that this will in fact be the case for every infinite Coxeter group. The construction by Brink and Howlett [17] of a finite automaton for normal forms is equivalent to this statement, by Kleene's characterization of regular languages. Some minor details need to be filled in, but that is all.

Proposition 84 is true for any Coxeter group as well, but not very useful, for everything happens in the last normal factor. One needs a transition table, listing how regular expressions are tranformed into each other, and we have not found simple descriptions even for $\tilde{A}_n$. The iterated expressions (α, β, γ in the example) behave differently depending on the iteration exponent (or rather its residue class modulo n).

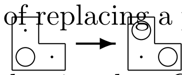
He who does not fear complications can attempt to formulate these transition rules. We prefer to withhold our formulas, considering that F. du Cloux states that he has in preparation a new paper on the normal forms for infinite groups ([37], p 4).

Part 2. Games and Coxeter groups

Chapter 4

Pebblings

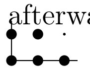
17. The pebbling game

The pebbling game of Kontsevich is played on the grid points of the first quadrant. One starts with a single pebble on the origin and a legal move consists of replacing a pebble with two pebbles, one above and one to the right of the vanishing pebble:  .

In the original game, only one pebble was allowed on each grid point, but Chung, Graham, Morrison and Odlyzko [22] introduced a stacking version, where pebbles are allowed to accumulate, at least temporarily. The stacking game is equivalent to the original game, in the following sense. If a position with at most one pebble per point is reachable in the stacking version, then it is reachable in the original pebbling game.

The original problem, posed by Kontsevich in 1981, was to show that the ten grid-points closest to the origin, $\{(i,j) \mid i+j \leq 3\}$, form an *unavoidable set*, meaning that every game position has at least one pebble in this set. The intended proof was the following.

To a pebble at (i,j) assign the weight 2^{-i-j}. That makes the total weight of the pebbles equal to 1 in all positions, for each move splits a pebble into two, half as heavy, and the total weight was 1 to start with. With at most one pebble on each point, all grid points outside the ten-point triangle can carry at most $\sum_{i,j\geq 0} 2^{-i-j} - (1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8}) = \frac{3}{4}$, so some pebble must be left in the triangle.

Shortly afterwards, A. Khodulev [58] made the surprising observation, that already the five-point set  is unavoidable. The first complete proof appeared eleven years later in [22], together with new enumerative results.

The purpose of this paper is to extend these results to the higher dimension analogues of the pebbling game and to a more general poset version.

18. Pebbling in $\mathbf{Z}^n$

The n-dimensional version of the game, suggested by Paul Vaderlind [79], uses the integer grid points of the first orthant. One starts with a single pebble on the origin and a legal move replaces a pebble by n pebbles, each one step away in the n coordinate directions.

The *weight* of a pebble with coordinates $(x_1, \ldots, x_n)$ is $n^{-x_1-\cdots-x_n}$ and it is obvious that the total weight of all pebbles is unchanged by a move in the pebbling game. If there were a pebble on each point in the first orthant, the total weight would be

$$\sum_{x_i\geq 0} n^{-x_1-x_2-\cdots} = \left(\sum_{i=0}^{\infty} n^{-i}\right)^n = (1-\frac{1}{n})^{-n} \to e \quad \text{when } n \to \infty$$

Weight calculations can be used to prove that a certain point set is unavoidable, for example the seven-point set in $\mathbb{Z}^6$ consisting of the origin and its six neighbours in the positive orthant. The total weight that can be carried on all other orthant grid points is

$$(1-\frac{1}{6})^{-6} - 1 - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} = 0.98598\ldots\ ,$$

but the total pebble weight is one.

Dimension six is the lowest dimension in which this kind of proof will work, but in fact the following is true.

Proposition 85. *The four-point set in $\mathbb{Z}^n$, $n \geq 3$ consisting of the origin and three neighbour points is unavoidable.*

Proof. Consider the unit cube defined by the four-point set. When the four points have been emptied, three other points on the cube will have received two pebbles each, one of which must be sent along to the $(1, 1, 1)$-point. But in a legal game, no point will receive more than two pebbles, as shown in the proof of Proposition 88 below. □ □

Proofs of this kind are greatly facilitated if several pebbles are allowed to occupy the same point, at least temporarily. The following result appears as Lemma 3 in [22] in the two-dimensional case and will be proved in much greater generality in our next section, so let us just state it as a fact.

Fact 86. If a configuration of pebbles with at most one pebble per point is reachable by moves which allow stacking of pebbles, then it is also reachable by moves which do not allow stacking.

The *level* of an orthant point is the sum of its coordinates. Thus, level zero contains the origin only, level one has n points, level two $n(n+1)/2$ points etc. The following *level trimming* procedure for determining whether or not a set of points, X, is unavoidable is given in [22]. Starting at level zero and proceeding one level at a time, perform the moves required to remove all pebbles from a point in X or all but one pebble from a point not in X. Stacking of pebbles is allowed.

The following fact is not completely obvious, but again, a much more general statement will be proved in the next section.

Fact 87. The configuration after trimming levels 0 through k is independent of the order in which the moves are performed. The set X is unavoidable if and only if the trimming procedure can go on for ever without running out of pebbles.

Level trimming supplies a polynomial time algorithm to determine whether or not a given set X is unavoidable, as stated in the next proposition. Let us warn the reader that a much better result will appear in Theorem 96.

Proposition 88. *Let level k be the last one containing a point from X and consider the configurations after trimming levels $1, \ldots, k, k+1, \ldots, nk$. The set X is unavoidable if and only if none of these contains a point with three or more pebbles on it at any stage.*

Proof. In the one-dimensional case, there are no unavoidable sets and no three pebble points. The two-dimensional case is a consequence of Theorem 1 in [22], so we can assume $n \geq 3$. If there are three or more pebbles on a point, $x = (x_1, x_2, x_3, \ldots)$, at least two of these must propagate to each of the points $(x_1+1, x_2, x_3, \ldots)$, $(x_1, x_2+1, x_3, \ldots)$ and $(x_1, x_2, x_3+1, \ldots)$ (an equilateral triangle on the next level). On the next level, each point in the triangle $(x_1+1, x_2+1, x_3, \ldots)$, $(x_1, x_2+1, x_3+1, \ldots)$ and $(x_1+1, x_2, x_3+1, \ldots)$ receives at least two pebbles from the triangle below, one of which must be sent on to the point $x' = (x_1+1, x_2+1, x_3+1, \ldots)$. So now there are at least three pebbles on x' and the game goes on forever.

Now, assume that there are no three-pebble points on levels $k, \ldots, nk$. This implies that each point sends at most one pebble to its neighbours on the next level. Level $k+1$ intersects the coordinate axes in n points, none of which can have more than one pebble, so they send no pebbles to their neighbours. Therefore, on level $k+2$ the axis points have zero pebbles and the axis neighbours at most one pebble. Iterating the argument, we find that on level $k+m$, all points with distance less than m from an axis have at most one pebble. The center point on level nk is $k, k, \ldots, k$ and with $m = (n-1)k$, we see that all other points have at most one pebble. Therefore, when level nk has been trimmed, the game is over. □ □

A byproduct of the proof is a polynomial bound for the length of the game corresponding to an *avoidable set* X. Again, a much better bound will appear in Theorem 96. Let us now consider the connection between an avoidable set X and the end position after the game. Points in X that are never touched by a pebble are of no consequence, but modulo these uninteresting points, the correspondence is in fact bijective.

Definition. The *voidance set* of a (finite) pebbling game consists of all points in $\mathbb{Z}^n$ that at some stage were pebble points but end up empty.

As defined, the voidance set seems to depend on the particular sequence of moves leading to the final position, but in our next section, (Proposition 103), we shall show that games leading to the same position have the same voidance set.

Fact 89. A reachable game position is completely specified by its voidance set.

As combinatorial objects, voidance sets are more tractable than reachable positions, not to mention pebbling games. A hundred-point voidance set in $\mathbb{Z}^2$ might correspond to a position with a thousand pebbles, which in turn may arise from 10^{100} games.

It turns out that the points that are played in a two-dimensional pebbling game form a characteristic configuration bounded by two lattice paths. A corresponding voidance set is the set of left and lower boundary points on these paths.

Definition. A *polyominoid set* in $\mathbb{Z}^2$ consists of all points on or between two lattice paths with common start point and common endpoint. As demonstrated in Fig.13, the paths may be partially or totally coincident, but without loss of generality, we may assume that they are not strictly crossing. We call (x, y) a *left boundary point* if $(x-1, y)$ is not in the polyominoid and a *lower boundary point* if $(x, y-1)$ is not in the polyominoid.

Observation 90. Polyominoid sets correspond bijectively to parallelogram polyominoes in the sense of M-P.Delest and X.Viennot [35]. If the left path is translated one step upwards, the lower path one step to the right, and the endpoints are rejoined in the obvious way, we get a polyomino of the parallelogram type.

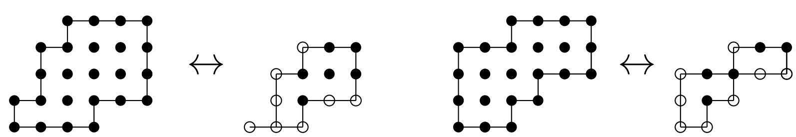


FIGURE 13. Two polyominoes with polyominoids and left-lower boundary points.

Not counting the left lower point (the origin of coordinates), a polyominoid set with height h and width w has h left and w lower boundary points, so the cardinality of the voidance set is $w + h + 1$, one more than the length of each path.

The following enumeration result is classical in the context of polyominoes and noncrossing lattice paths, see [35] and [47]. Still, for convenience we give the proof in the polyominoid case.

Proposition 91. *The number of polyominoid sets with lattice paths of length k, i.e. with $k+1$ left and lower boundary points, is the Catalan number*

$$C_{k+1} = \frac{1}{k+2}\binom{2k+2}{k+1} = \binom{2k}{k} - \binom{2k}{k-2}$$

Proof. A lattice path of length k can be represented as a binary k-vector. A pair of paths with common endpoints means two binary vectors, $\mathbf{u}$, $\mathbf{v}$, with the same number of ones. Complementing the second vector and concatenating it to the first vector, one gets a $2k$-vector with k ones, and there are $\binom{2k}{k}$ of these.

The polyominoid (weakly noncrossing) condition is $\sum_1^r u_i \leq \sum_1^r v_i$ for all $1 \leq r \leq k$. Otherwise, let r' be the first index for which $\sum_1^{r'} u_i = 1 + \sum_1^{r'} v_i$ and let us switch the $(k-r')$-tails between **u** and **v**. Now, there are two more ones in the first vector, $\mathbf{u}'$, than in the second, $\mathbf{v}'$, and as for every such pair, r' can be defined as above, the correspondence is bijective. Finally, the complemented concatenation trick shows that these nonpolyominoid pairs are $\binom{2k}{k-2}$. $\square$ $\square$

Every pebbling game in $\mathbb{Z}^2$ defines a polyominoid, viz. the set of all points that have been played. In $\mathbb{Z}^n$, the play may of course use all n dimensions, but the set of points that have been played still form a polyominoid set, although folded and meandering through the dimensions.

Definition. A *folded polyominoid set* in $\mathbb{Z}^n$ is defined by a consistent labelling of the edges of a polyominoid set with coordinate directions. Consistency means that for each square in the polyominoid, adjacent sides have different labels but opposite sides have the same label. Thus, for a polyominoid with height h and width w, it is sufficient to specify $h+w$ labels, for example on the left and lower edges.

$\mathbf{u} = (y, y, 0, 0, z, 0, 0)$

$\mathbf{v} = (x, 0, z, 0, x, y, 0)$

FIGURE 14. A folded polyominoid with left-lower labels and label vectors.

Labelling of left and lower edges may be seen as distribution of k labels over $2k$ places, namely the pair of k-vectors **u** and **v** defining the boundary paths. Unlabelled places contain zeroes. There are of course compatibility restrictions on this distribution and these can be stated concisely if we introduce the notation $|\mathbf{u}_{\dots r}|$ for the number of labels in the initial r-segment of **u**. Thus, moving r steps along the left boundary path, we go $|\mathbf{u}_{\dots r}|$ steps upward and $r - |\mathbf{u}_{\dots r}|$ steps to the right. And moving r steps along the lower boundary path, we go $|\mathbf{v}_{\dots r}|$ steps to the right and $r - |\mathbf{v}_{\dots r}|$ steps upwards.

Theorem 92. *For pebbling in $\mathbb{Z}^n$ with $n \geq 3$, the following combinatorial objects correspond bijectively to each other.*

1. *Reachable positions with the highest pebble on level $k+1$.*
2. *Voidance sets of cardinality $k+1$.*
3. *Folded polyominoids with boundary path lengths k.*
4. *Pairs of integer k-vectors,* **u** *and* **v**, *with a total of k nonzero elements (labels) in $\{1, \dots, n\}$, such that*
   (a) *if for any $0 \leq r < k$, $|\mathbf{u}_{\dots r}| + |\mathbf{v}_{\dots r}| = r$ then $u_{r+1} \leq v_{r+1}$,*
   (b) $|\mathbf{u}_{\dots r}| + |\mathbf{v}_{\dots r}| \geq r$ *for all $1 \leq r \leq k$,*
   (c) *if the same label occurs in u_i and v_j, then $|\mathbf{u}_{\dots i}| + |\mathbf{v}_{\dots j}| \leq \max(i, j)$.*

Proof. The folded polyominoid characterization of reachable positions in $\mathbb{Z}^n$ will emerge as a corollary of Proposition 110. In dimension three and higher, no node is played twice (this will be proved in Proposition 107), so the voidance set consists of all left and lower boundary points of the folded polyominoid, and we have already noted that their cardinality is $k+1$.

A folded polyominoid may be unfolded in the xy-plane in at least two ways (xy-reflections), more if there are intermediate singleton levels, but condition (a) uniquely defines the left and lower boundary vectors **u** and **v**. For both paths reach the same point in r steps if and only if $|\mathbf{u}_{\dots r}| + |\mathbf{v}_{\dots r}| = r$.

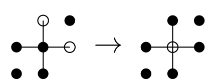
Similarly, condition (b) expresses the fact that the left path should keep to the left of the lower path. (The binary vectors in the proof of Proposition 91 correspond to our vectors in a somewhat confusing way: the nonzero labels in **u** mean binary zeroes and zero labels mean binary ones, in **v** it is the other way around.)

Condition (c) means that the horisontal strip to the right of the vertical segment u_i must not intersect the vertical strip above the horizontal segment v_j. Either v_j is to the left of u_i, which implies $|\mathbf{v}_{\dots j}| \leq i - |\mathbf{u}_{\dots i}|$, or u_i is below v_j, which means $|\mathbf{u}_{\dots i}| \leq j - |\mathbf{v}_{\dots j}|$. $\square$ $\square$

The beautiful enumeration result for twodimensional polyominoids makes one hope for some similar formula for folded polyominoids. If one exists, it has eluded this author so far. The conditions in the theorem makes computer calculations of these numbers easy and we include a table of them. Note the Catalan numbers in the second column! The row $k = 2$ is $n(3n-1)/2$ and it can be proved that row k is a k-th degree polynomial in n.

$f_{k,n}$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$k=0$	1	1	1	1	1	1
$k=1$	1	2	3	4	5	6
$k=2$	1	5	12	22	35	51
$k=3$	1	14	57	148	305	546
$k=4$	1	42	300	1126	3045	6756
$k=5$	1	132	1680	9220	32985	91236

FIGURE 15. Number of folded polyominoes in $\mathbb{Z}^n$ with circumference $2k$.

The theorem does not apply to the two-dimensional case, for the same polyominoid may correspond to several voidance sets. The reason is that some points in the polyominoid may be played twice. Which points? First, they must receive two pebbles, so their left and lower neighbours are in the polyominoid. Second, their right and upper neighbours must be emptied, so these must be a left and a lower boundary point. It follows that a twice played point must be a singleton on its level. The result of the second play is that two old voidance points are replaced by one new voidance point.  . Let us call this configuration a *crossing*.

Theorem 93. *For pebbling in $\mathbb{Z}^2$, reachable positions with the highest pebble on level $k+1$ correspond bijectively to folded polyominoids with boundary pathlengths k and with any subset of the crossings marked as voidance points. The generating function for the number of such reachable positions is*

$$g(x) = \frac{1 - 6x + 4x^2 + 4x^3 + \sqrt{1-4x}}{2(1 - 6x + 8x^2 - 4x^4)} = 1 + 2x + 5x^2 + 14x^3 + 43x^4 + 140x^5 + \cdots$$

with asymptotic behaviour $g_k \sim const \cdot G^k$, where $G = 4.112\ldots$.

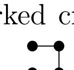
Voidance sets of cardinality $k+1$ correspond bijectively to folded polyominoids with boundary lengths $k+t$ and with t crossings marked as voidance points, $t \geq 0$. The generating function for the number of such voidance sets is

$$h(x) = \frac{2 - 11x + 12x^2 + x\sqrt{1-4x}}{2(1 - 7x + 14x^2 - 9x^3)} = 1 + 2x + 5x^2 + 15x^3 + 51x^4 + 187x^5 + \cdots$$

with asymptotic behaviour $h_k \sim const \cdot G^k$, where $G = 4.147\ldots$.

Proof. Knowing that the number of unmarked polyominoids is C_{k+1}, we can write down a recursion

$$g_k = C_{k+1} + \sum_{r=2}^{k-2} (C_{r+1} - 2\,C_r)(g_{k-r} - 2\,g_{k-r-1})$$

with the following interpretation. Let r be the level where the first marked crossing appears. Then $C_{r+1} - 2\,C_r$ is the number of unmarked polyominoids ending with a  on level r, for we have to subtract polyominoids ending with or . By the same reasoning, the number of marked polyominoids starting with and reaching $k-r$ levels is $g_{k-r} - 2\,g_{k-r-1}$.

It is well-known that the Catalan numbers have the generating function $(1+\sqrt{1-4x})/2x$. Standard manipulations and hard work (thanks, Maple!) produce the expression for $g(x)$.

The recursion for h_k is derived analogously, the only difference being that each marked crossing reduces the number of voidance points and has to be compensated for:

$$h_k = C_{k+1} + \sum_{r=2}^{k-2} (C_{r+1} - 2\,C_r)(h_{k-r+2} - 2\,h_{k-r+1}) \ .$$

The roots of the denominators determine the asymptotic exponents (see the chapter by Odlyzko in [50]) as 1/(smallest root). Exponents greater than four were to be expected, since $C_k \sim \text{const} \cdot 4^k$. □ □

An important part of the paper by Chung, Graham, Morrison and Odlyzko is the enumeration of minimal unavoidable sets in $\mathbb{Z}^2$. The asymptotic expression found is const $\cdot\, \gamma^k$ with $\gamma = 4.147\ldots$, exactly our result when counting voidance sets! The agreement is hardly a coincidence, for in Proposition 104 of next section, we prove that *a minimal unavoidable set is a voidance set with an extra point*. The extra point must be chosen such that level trimming becomes infinite and such that deletion of any other point makes level trimming finite again.

Theorem 94. *Every minimal unavoidable set in $\mathbb{Z}^2$ can be constructed from the left and lower boundary points of a marked polyominoid by adding a polyominoid point on the second highest level.*

The generating function for the number of minimal unavoidable sets of cardinality k is

$$m(x) = x^3\,\frac{(1-3x+x^2)\sqrt{1-4x} - 1 + 5x - x^2 - 6x^3}{1 - 7x + 14x^2 - 9x^3} = 4x^5 + 22x^6 + 98x^7 + 412x^8 + 1700x^9 + \cdots$$

with asymptotic behaviour $m_k \sim \text{const} \cdot H^k$, where $H = 4.147\ldots$.

Proof. Let us look at all possibilities of adding an extra unavoidable point to a marked polyominoid ending like , i.e. a point that cannot be emptied by further play. Positions outside the polyominoid can be emptied, for adding such a point means building a larger polyominoid, e.g. . The corner point can also be emptied by building a new marked crossing . Other polyominoid points, however, really are unavoidable (for emptying them would stack three pebbles somewhere), but only the corner neighbours produce *minimal* unavoidable sets. In, for example, , the last lower boundary point could be deleted.

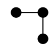
So either of the corner neighbours in , produces a minimal unavoidable set together with the left and lower boundary points. The same reasoning for the special cases et cetera shows that the configurations and always produce one minimal unavoidable set each, and those are all there are.

Counting marked polyominoids ending in is a simple matter. There are $h_{k-2} - 2h_{k-3}$ of type and from these we subtract $h_{k-3} - h_{k-4}$ of type , so the result is $h_{k-2} - 3h_{k-3} + h_{k-4}$. The symmetric case gives a factor two and the final expression is $m_k = 2(h_{k-2} - 3h_{k-3} + h_{k-4})$.

The generating function $m(x)$ and the asymptotic expression for m_k follow immediately from the corresponding results for $h(x)$. □ □

There are four four-point unavoidable sets in $\mathbb{Z}^3$. The origin and its three neighbours was proved unavoidable in Proposition 85. The square in any coordinate plane is also unavoidable, for no point may be fired twice in dimension $n \geq 3$. These two examples of minimal unavoidable sets in $\mathbb{Z}^n$ are, in fact, generic.

Theorem 95. *Every minimal unavoidable set in $\mathbb{Z}^n$ can be constructed from the left and lower boundary points of a folded polyominoid by adding one of the following points.*

- *The highest point of the folded polyominoid (unless it is already a left or lower boundary point).*
- *A point that forms an isoceles triangle with the two corner neighbours (provided that the folded polyominoid ends with a*  *).*

Proof. Simpler than the two-dimensional case. One just has to test all positions of an extra point. □ □

Now, at last, we can state the optimal version of Proposition 88.

Theorem 96. *Let level k in $\mathbb{Z}^n$ be the last one containing a point from X and consider the configurations after trimming levels $1, \dots, k+1$. The set X is unavoidable if no three-pebble point occurs during this game. If X is avoidable, level trimming will come to an end no later than after level $2k$.*

Proof. The truth of the statement for our minimal unavoidable sets can be established easily enough by a direct check and since each unavoidable set contains a minimal one, the rest is clear. The analogous method applied to the voidance sets proves the other half of the statement. □ □

19. Pebbling a poset

The pebbling game generalizes immediately to any digraph, but to preserve its essential features we restrict ourselves to infinite but locally finite posets with $\hat{0}$ and without maximal elements. The game board is the Hasse diagram, one starts with a single pebble on $\hat{0}$ and a move consists of removing a pebble from any node x and adding a pebble to each node covering x.

Following Björner, Lovász and Shor [13], we say that node x is *fired.* It is illegal to fire a node unless all of its covering nodes are empty, but we also consider a stacking variant of the game in which pebbles are allowed to accumulate, at least temporarily.

The *shot count* is a record of the number of times each node has been fired during a game, so it is a function from nodes to nonnegative integers.

Proposition 97. *Different move sequences lead to the same position if and only if they have the same shot count.*

Proof. A node x that was fired $f(x)$ times has got $\sum f(y) - f(x)$ pebbles, where the sum is taken over all nodes y covered by x. □ □

The bijective correspondence between reachable positions and shot counts is useful, for shot counts are less complex combinatorial objects. A simple characterization of shot counts comes next.

Proposition 98. *A finite distribution f of nonnegative integers over the nodes is a shot count of a legal game if and only if there is a 0 or a 1 on $\hat{0}$ and for every other node x, the difference $\sum f(y) - f(x) = 0$ or 1, where the sum is taken over all nodes y covered by x.*

Proof. A game with shot count f is defined by the following rule. Always fire a maximal node in the subset of nodes x that have a pebble and have not been fired $f(x)$ times yet. Simple verifications. □ □

Proposition 99. *If a configuration of pebbles with at most one pebble per node is reachable by moves which allow stacking of pebbles, then it is also reachable by moves which do not allow stacking.*

Proof. A consequence of the previous proposition. □ □

The last three propositions have the flavour of *strong convergence*, a concept introduced by Anders Björner and developed in [42]. A game is strongly convergent if either

- every possible game has the same length and ends in the same terminal position, or
- every game goes on for ever.

Since the posets we are interested in are infinite, there are no terminal positions unless we restrict legal moves in the following way. Choose an arbitrary subset of nodes X as *nodes to be emptied* and call a pebble *obstructing* if it is in X or (recursive definition!) covers an obstructing pebble.

Proposition 100. *With the new rule that only obstructing pebbles may be moved, pebbling is strongly convergent for every poset and for every set X of nodes to be emptied.*

Proof. As shown by Kimmo Eriksson [62], strong convergence of a game is equivalent to the *polygon property*, i.e. from any position where two different plays, x and y, are possible, either there are two play sequences of equal length, one starting with x, the other with y and leading to the same position, or there are two infinite play sequences, one starting with x and the other with y.

In a pebbling game position where two nodes, x and y, can be fired, there are two cases. Either there is a finite play sequence in which both nodes are fired, assume that its shot count is f. One can play either x or y first, and the algorithm in the proof of Proposition 98 then defines the rest of a sequence leading to the same position. Or, there is an infinite sequence, for as long as one of the obstructing pebbles x, y is unplayed, there is certainly some playable node left.
So the pebbling game has the polygon property and is therefore strongly convergent. □ □

In the corresponding stacking version of the game, more moves are allowed, but the terminal position will be the same. A pebble is obstructing if it is stacked or in X or covers an obstructing pebble.

Proposition 101. *The stacking version of the game in which only obstructing pebbles may be moved is strongly convergent.*

Proof. A consequence of the previous three propositions. Note that the shot count determines the length of the game. □ □

To find out whether a set X is unavoidable, one can play the game level after level, allowing temporary stacking of pebbles. The *level* of a node x is the length of the shortest path from $\hat{0}$ to x. Starting at level zero and proceeding one level at a time, one fires all obstructing pebbles on that level. This is called *level trimming*.

Proposition 102. *The set X is unavoidable if and only if the level trimming procedure can go on forever, without running out of obstructing pebbles.*

Proof. Suppose that X can be emptied in a finite game with shot count f. It is obvious that the trimming procedure has a shot count g with $g \leq f$, componentwise, and the proposition follows from this. □ □

Definition. The *voidance set* of a game consists of all points that at some stage were visited by a pebble but are empty by the end of the game.

Proposition 103. *There is a one-to-one-to-one correpondence between reachable positions, shot counts and voidance sets.*

Proof. There is only one small thing left to prove, namely that the shot count f can be reconstructed from its voidance set. It is evident that level trimming produces one candidate, say f^*, with only absolutely necessary firings, so $f^* \leq f$ componentwise. Suppose that there are nodes x for which $f^*(x) < f(x)$, and choose such a node on the lowest possible level. Level trimming must leave a pebble on x since it can be fired once more, therefore x is not in the voidance set of f^*, but it is in the voidance set of f, contrary to our assumption. □ □

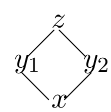
A set X is *minimal unavoidable* if all strict subsets of X are avoidable. A characterization of minimal unavoidable sets is easy as soon as all voidance sets are known, for we have the following result.

Proposition 104. *Let X be a minimal unavoidable set and x a node on the highest level in X. Then $X - \{x\}$ is a voidance set.*

Proof. Because of the minimality, $X - \{x\}$ is avoidable, so level trimming is a finite procedure. Any uninteresting point $y \in X$, untouched in this level trimming, would be as uninteresting in the continued infinite level trimming of X, for the influence of x is noticeable only on higher levels. Therefore, $X - \{y\}$ would still be unavoidable, contradicting minimality. □ □

In the sequel, we shall concentrate on shot counts, as characterized by Proposition 98. The *support* of a shot count is the subposet of nodes that have been fired, i.e. with nonzero shot count. In the $\mathbb{Z}^n$ case, this subposet has very nice properties, and the reason for this turns out to be that the poset of points in the first orthant has V-completion.

Definition. A poset P has *V-completion* if whenever y_1 and y_2 cover x, there is a node z covering both y_1 and y_2. (Birkhoff [7] says that P is *upper semimodular.*)



Proposition 105. *For a poset P with V-completion, the support of any shot count is a* ranked poset *(even if P is not) and has a* unique maximal element *(even though P has not).*

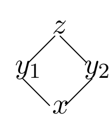
Proof. The support $\operatorname{supp} f$ also has V-completion, for if x, y_1, y_2 are in $\operatorname{supp} f$ (see illustration above), by Proposition 98 we get $f(z) \geq f(y_1) + f(y_2) - 1 \geq 1$, so z is also in the support. V-completion is the simplest case of the polygon property, so there is strong convergence and a unique terminal, i.e. maximal node. All paths to this terminal have equal length and this provides the ranking. □ □

Corollary 106. *For a pebbling game in $\mathbb{Z}^n$, let k be the highest level on which pebbles have been fired. Then, exactly one firing took place on level k.*

Unexpectedly, the characterization of reachable pebble positions is somewhat more difficult in the plane than in higher dimensions. The reason is that in higher $\mathbb{Z}^n$, *every node is covered by at least three nodes.*

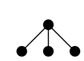
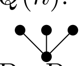
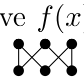
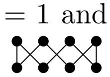
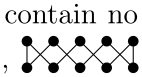
Such a poset must be infinite, so the triple-cover property never applies to the subset $\operatorname{supp} f$ for any shot count f. However, a more interesting property follows, namely the dual of V-completion.

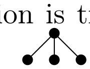
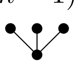
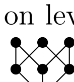
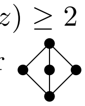
Definition. A poset P has Λ-*completion* if whenever z covers y_1 and y_2, there is a node x covered by both y_1 and y_2. (P is called lower semimodular in [7]).



Proposition 107. *If a poset P has V-completion and if every node is covered by at least three nodes, then $f(x) = 1$ for any shot count f and any node x in $\operatorname{supp} f$. Further, $\operatorname{supp} f$ has Λ-completion.*

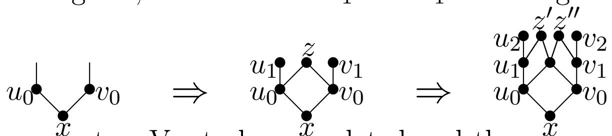
Proof. In order to carry out a (finite) induction proof of the statement $f(x) = 1$, we shall need a stronger induction assumption.

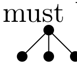
$Q(n)$: Level n and all higher levels of $\operatorname{supp} f$ have $f(x) = 1$ and contain no tridents  or , nor any of the following zig-zag shapes: , , , ...

By Proposition 105, the assumption is true for the very highest level. From $Q(n)$, one can infer $Q(n-1)$, as follows. A trident  on level $n-1$ would mean $f(z) \geq 2$ on level n. A trident  can be V-completed to either , a forbidden zig-zag, or , a forbidden trident.

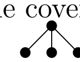
A node x on level $n-1$ with $f(x) \geq 2$ will have at least three covering nodes y_1, y_2, y_3 in P and since $f(y_i) \geq 1$, that means a forbidden trident. Finally, if there is a zig-zag on level $n-1$, V-completion gives either a zig-zag or a trident on level n.

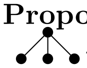
Thus, we have proved $Q(0)$ and the first part of the proposition. We now assume that there is some Λ-shape that cannot be completed to a quadrangle. Let the shortest completing polygon have bottom element x. We know that such a polygon exists, for x may be $\hat{0}$. Completing the bottom V we get z, distinct from u_1 and v_1 in the figure (or there would be a shorter polygon).

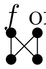
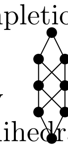
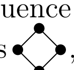


Now, there are two V-s to be completed and the same argument shows that z' and z'' must be distinct from u_2 and v_2, but also distinct from each other, for we cannot have a trident  .

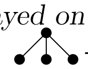
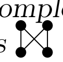
Iterating, we move up level for level and the final Λ-shape ($u_n = v_n$) must produce a zig-zag, contradicting the Q-property just proved. □ □

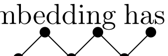
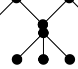
Scrutinizing the proof, one finds that the assumption about triple covers in P is used only to prove $f(x) = 1$, which in turn is used only to prove nonexistence of -tridents. Thus, there is a weaker form of the proposition with the advantage that it can be applied to $\mathbb{Z}^2$.

Proposition 108. *If a poset P has V-completion and f is a shot count such that* $\operatorname{supp} f$ *has no* *-tridents, then* $\operatorname{supp} f$ *has Λ-completion.*

Remark 109. The subposets $\operatorname{supp} f$ of the last two propositions contain no tridents or zig-zags, but they may contain the X-shape  . By repeated V-completion upwards and Λ-completion downwards, one finds that an X-shape must be part of a *dihedral interval*, e.g. this one:  (This is the Bruhat poset of the dihedral group $I_2(4)$.) The poset P may continue below the V and above the Λ, but on the levels in-between there are no nodes outside the dihedral interval. This is an easy consequence of Proposition 105. Therefore, all dihedral intervals, if any, may be replaced by quadrangles , while maintaining the shot count properties. Conversely, if a shot count poset has some level with only two nodes on it, then the quadrangle containg these nodes may be expanded into a dihedral interval. Therefore, a characterization of shot count posets may as well assume that there are no X-shapes.

The following characterization is valid for all $\mathbb{Z}^n$ but its main result is that everything may be considered as taking place in $\mathbb{Z}^2$. The pebbling game may meander through all n dimensions, but the poset structure of the shot count is planar.

Proposition 110. *If a pebbling game is played on a poset P with V-completion and the nodes that have been fired form a subposet without* *-tridents or X-shapes*  *, then this subposet is isomorphic to a polyominoid subset of $\mathbb{Z}^2$.*

Proof. Assume that the embedding has been constructed for levels zero through k, so the last two levels look something like  . V-completion forces an extension to level $k+1$ and Λ-extension justifies it. Since -tridents cannot occur, everything is specified except whether the boundary points on level k have single or double covers. In both cases, the embedding is straight-forward. □

20. Pebbling a Coxeter group

The first orthant of $\mathbb{Z}^n$ is a poset with the origin as minimal element $\hat{0}$, but it is also a monoid, generated by $s_1, s_2, \ldots, s_n$ modulo commutation relations $s_i s_j = s_j s_i$. Here, s_i may be interpreted as a unit translation in the i-th coordinate direction, and a point with coordinates $(x_1, \ldots, x_n)$ may be identified with the monoid element $s_1^{x_1} \cdots s_n^{x_n}$. As we have seen, a pebbling position has a shot set (the set of played points) with a unique highest point, w say, and every lattice path from the origin to w defines a *word* in the s_i or, equivalently, a *reduced expression* for w in the monoid. (Reduced means shortest possible, so superfluous 1-factors like in $s_3 1 s_1$ are outruled.) Some of these paths visit only nodes that have been played, i.e. points inside the polyominoid, and we name these paths *polyominoid paths*. So the left and lower boundary paths belong to these and all other polyominoid paths interpolate between them.

A more appealing definition of a polyominoid path from the origin to w is the following. Start with a special blue-coloured pebble on the origin. When the blue pebble is fired, it reappears as one of the new pebbles on the next level. The first pebble to be fired on each level should be the blue one. As a consequence of these rules, the blue pebble will always be the only pebble fired on the highest level – at node w – and the path travelled by the blue pebble will be a polyominoid path to w.

The left example in Figure 16 is a folded polyominoid with five polyominoid paths, namely $w = yuxz = yxuz = yxzu = xyuz = xyzu$. The equivalence of these expressions follows from three commutation identities, corresponding to the three squares in the polyominoid. Now, consider the monoid generated by $\{x, y, z, u\}$ modulo these three relations. In this monoid, there are exactly five reduced expressions for w, namely the five polyominoid paths. In fact, the polyominoid is the interval under w in the *right factor order* on the monoid (where 1 is the minimal element and α covers β if $\alpha = \beta s$ for some generator s).

$$\begin{array}{ccccc} \bullet & \mathrm{x} & \bullet & \mathrm{z} & \bullet \\ \mathrm{u} & & \mathrm{u} & & \mathrm{u} \\ \bullet & \mathrm{x} & \bullet & \mathrm{z} & \bullet \\ \mathrm{y} & & \mathrm{y} & & \\ \bullet & \mathrm{x} & \bullet & & \end{array} \Rightarrow \begin{array}{ll} ux = xu, & zx \neq xz \\ uz = zu, & uy \neq yu \\ yx = xy, & yz \neq zy \end{array} \qquad \begin{array}{ccccc} \bullet & \mathrm{x} & \bullet & \mathrm{z} & \bullet \\ \mathrm{y} & & \mathrm{y} & & \mathrm{y} \\ \bullet & \mathrm{x} & \bullet & \mathrm{z} & \bullet \\ \mathrm{y} & & \mathrm{y} & & \\ \bullet & \mathrm{x} & \bullet & & \end{array} \Rightarrow \begin{array}{ll} yz = zy\,, & yz \neq zy \\ yx = xy, & zx \neq xz \end{array}$$

FIGURE 16. The left folded polyominoid defines commutation relations, the right one does not.

There are two different interpretations of this fact. Either the pebbling took place in $\mathbb{Z}^4$, and afterwards we noted the isomorphism of the polyominoid with the interval under w in a certain monoid with generators x, y, z, u. Or else the pebbled poset was that of the monoid, all from the start!

The trouble with the first interpretation is that some folded polyominoids do not define consistent commutation relations. In the right-hand example of Figure 16, the upper right square shows that y and z commute, but at the same time the nonexistence of a lower right square indicates that y and z do not commute.

The trouble with the second interpretation, that we are pebbling the monoid poset rather than $\mathbb{Z}^4$, is that the monoid poset does not have the V-completion property. Since x and z do not commute, the three-move pebbling game in which the origin, x and z are fired does not even have a unique highest shot point w and so cannot be an interval.

The first difficulty can be taken care of by not allowing repeated labels on the left (or lower) boundary path. The polyominoid must be folded in every crease, and in a new direction each time. Our next proposition states that this can always be done, and in $n!$ different ways.

Definition. A pebbling game in $\mathbb{Z}^k$ is *cubic* if the highest played point is $(1, 1, 1, \ldots)$. The corresponding folded polyominoid set of fired points is called a *cubic polyominoid set*.

The cubic polyominoid defines a number of paths from the origin to $(1, 1, 1, \ldots)$, and it is obvious that every such path must consist of exactly k segments, one in each coordinate direction. In particular, all firings in the cubic game occurred within the unit cube.

Proposition 111. *The number of pebbling positions in $\mathbb{Z}^k$ resulting from cubic games is $n!$ times the number of polyominoid sets with boundary path lengths k.*

Proof. Assign any permutation of the labels $s_1, s_2, \ldots, s_k$ to the segments of the left boundary path. Application of the rule that opposite sides of a square have the same label gives the labelling of the lower boundary path, and as there are no repeated labels, consistency of labelling is automatic. □ □

The rule about no repeated labels means that the monoid may as well be replaced by a Coxeter group. A reduced expression for an element of a Coxeter group certainly never contains $\cdots s_i s_i \cdots$, for all generators are involutions, $s_i^2 = 1$. The Coxeter graph contains all information about commutation relations – an edge between two vertices meaning that the corresponding s_i and s_j do not commute – and it is possible to associate a Coxeter graph to every cubic polyominoid set. Finally, the concept of a product of all generators in any order is already established in Coxeter group theory and called a *Coxeter element*.

Proposition 112. *To every cubic polyominoid set in $\mathbb{Z}^k$, one can associate an unlabelled Coxeter graph with vertices $s_1, s_2, \ldots, s_k$ and a Coxeter element in the corresponding group, such that the cubic polyominoid set translates into the lower weak order interval of the Coxeter element.*

Proof. Unfold the polyominoid! Any polyominoid path from the origin to $(1, 1, 1, \ldots)$ consists of vertical segments, say $s'_1, \ldots, s'_m$, and horizontal segments, say $\bar{s}_1, \ldots, \bar{s}_p$ (where $m+p = k$). The relative order among the s'_i never changes, nor does the relative order between the $\bar{s}_j$. Therefore, the Coxeter graph construction starts with two paths, one with m nodes labelled by the s'_i and the other with p nodes labelled by the $\bar{s}_j$.

If the polyominoid is an $m \times p$ rectangle, the Coxeter graph will be these two disjoint graphs and the Coxeter element will be $s'_1, \ldots, s'_m \bar{s}_1, \ldots, \bar{s}_p$, i.e. the expression corresponding to the left boundary path of the rectangle. The $m \cdot p$ squares of the polyominoid mean that each s'_i commutes with each $\bar{s}_j$, and this is exactly what is signified by the lack of edges between the two paths in the Coxeter graph.

A nonrectangular polyominoid has got at least one *concave corner*, and for every such corner, there should be an edge in the Coxeter graph between the s'_i and the $\bar{s}_j$ labelling the vertical and horizontal sides of the concave corner. The polyominoid paths will now trace every reduced expression of some element in this Coxeter group, for according to Tits's Word theorem (cf. Bourbaki [15]), every such expression can be obtained by repeated application of commutation relations. But an obvious continuity argument proves that a nonpolyominoid path (in the full rectangle) can be reached only by commuting a concave corner pair of generators, and this is stopped by an edge in our Coxeter graph. □ □

Figure 17 demonstrates the construction for the eight different Coxeter elements of A_4. As proved in the next chapter, Coxeter elements correspond to acyclic edge orientations on the Coxeter graph, so their number must be $2 \cdot 2 \cdot 2 = 8$.

Let us look more closely at the fourth row, that is the element $s_1 s_4 s_3 s_2$ with three reduced expressions. To construct the polyominoid, we note that s_1 and s_4 occur in the first position, therefore the lower left corner must have edges labelled 1 and 4. Our conventions in Theorem 92 assign 1 to the vertical and 4 to the horizontal segment. But s_1 also switches position with s_3, so we add a 1,3-square to the right. Finally, we append a 2-segment, for s_2 always keeps the last position. Our conventions make such segments horizontal.

Now that the polyominoid is constructed, the Coxeter graph follows easily. The upper path consists of the vertical labels, the lower path of the horizontal labels and the edge between them comes from the only concave corner.

There exist more polyominoids of height four than those occurring in Table 17, all corresponding to Coxeter elements in other groups with graphs of this particular type: two paths with some connecting edges. The limitation is not as severe as it may seem – it includes all finite, affine, compact hyperbolic and fifty-two of the fifty-eight noncompact hyperbolic groups. Table 18 gives examples of other height four polyominoids with groups other than A_4.

Remark 113. In general, edges of a Coxeter graph can carry integer labels. These are irrelevant in reduced expressions for Coxeter elements, which is why we have chosen to ignore them. Table

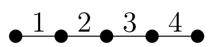
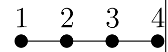
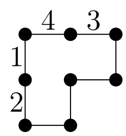
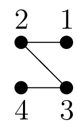
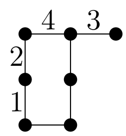
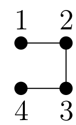
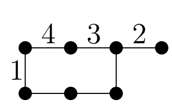
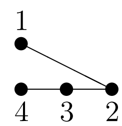
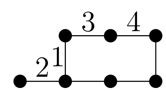
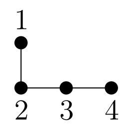
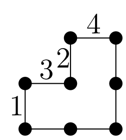
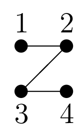
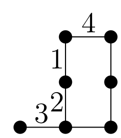
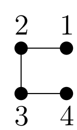
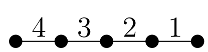
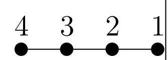
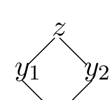
Reduced expressions	Cubic polyominoid	Coxeter graph	Acyclic edge orientation
$s_1s_2s_3s_4$			1→2→3→4
$s_2s_1s_4s_3$ $s_2s_4s_1s_3$ $s_2s_4s_3s_1$ $s_4s_2s_1s_3$ $s_4s_2s_3s_1$			1←2→3←4
$s_1s_2s_4s_3$ $s_1s_4s_2s_3$ $s_4s_1s_2s_3$			1→2→3←4
$s_1s_4s_3s_2$ $s_4s_1s_3s_2$ $s_4s_3s_1s_2$			1→2←3←4
$s_2s_1s_3s_4$ $s_2s_3s_1s_4$ $s_2s_3s_4s_1$			1←2→3→4
$s_1s_3s_2s_4$ $s_3s_1s_2s_4$ $s_1s_3s_4s_2$ $s_3s_1s_4s_2$ $s_3s_4s_1s_2$			1→2←3→4
$s_3s_2s_1s_4$ $s_3s_2s_4s_1$ $s_3s_4s_2s_1$			1←2←3→4
$s_4s_3s_2s_1$			1←2←3←4

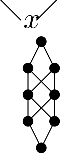
FIGURE 17. Coxeter elements in A_4 and their cubic polyominoids

17 could equally well be interpreted as listing Coxeter elements in F_4, with the same graph as A_4 except for labels.

Remark 114. For every cubic polyominoid there is a corresponding Coxeter element, but not the other way around! If a reduced expression can start with three different s_i, there can be no corresponding polyominoid.



Let us now turn to the other possibility, which is pebbling the Coxeter group itself, considered as a poset. The main objection is that the weak order has no V-completing z in general – it seems that we need some extra edges. To our help springs the Bruhat order!



Trivially, every V can be completed to a polygon in the weak order, and the Bruhat order

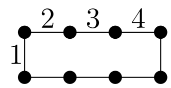
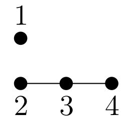
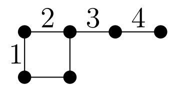
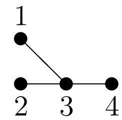
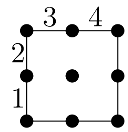
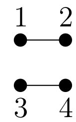
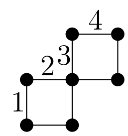
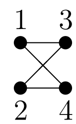
Reduced expressions	Cubic polyominoid	Coxeter graph	Name of group
$s_1s_2s_3s_4$ $s_2s_1s_3s_4$ $s_2s_3s_1s_4$ $s_2s_3s_4s_1$			$A_1 \times A_3$
$s_1s_2s_3s_4$ $s_2s_1s_3s_4$			D_4
$s_1s_2s_3s_4$ $s_1s_3s_2s_4$ $s_3s_1s_2s_4$ $s_1s_3s_4s_2$ $s_3s_1s_4s_2$ $s_3s_4s_1s_2$			$A_2 \times A_2$
$s_1s_2s_3s_4$ $s_2s_1s_3s_4$ $s_1s_2s_4s_3$ $s_2s_1s_4s_3$			$\tilde{A}_3$

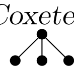
FIGURE 18. Other cubic polyominoids of height 4

inserts criss-crossing edges like in our illustration. The definition of y covering x in Bruhat order is that a reduced expression for x can be obtained by deleting some s_i from a reduced expression for y. Deleting s_1 from $\cdots s_1s_2$ we find the lowest criss-crossing Bruhat edge and the others have analogous explanations. There are other Bruhat edges than these criss-crosses within a polygon, but as we shall see, they are irrelevant in our pebbling games.

[Our statement above that every V can be completed to a polygon is not correct, for if ss' has infinite order, the polygon will extend to infinity, never getting a hat. But the criss-crossing Bruhat edges will still be there.]

There is still one minor impediment to using this poset for pebbling, and that is the possibility of the group being finite, contrary to our general pebbling assumption that the poset has no maximal elements. As an ad hoc solution, we add an infinite path above the unique maximal element w_o for the finite groups. These new nodes will never be played anyway.

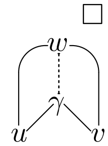
The main improvement, compared to $\mathbb{Z}^k$, is that we are not restricted to Coxeter elements any more. The only remaining restriction is the no-tridents condition from Proposition 108.

Proposition 115. *For any Coxeter group (W, S) and any element w, such that the lower weak order interval under w has no*  *-tridents even when Bruhat edges are considered, this interval is the set of played points in some pebbling game on the Bruhat order of (W, S).*

Proof. Clearly, a pebbling game can be played in which all points in the interval are fired once (and no other points are fired) at least if stacking of pebbles is allowed. We have to prove that, after this play sequence, no point has more than one pebble on it.

Certainly, no point in the interval has more than one pebble, for in that case it must have received at least three pebbles (one of which was fired away), contradicting the no-tridents assumption. We must show that no point γ outside the interval can Bruhat-cover two points u, v within the interval. This will be proved as a separate lemma. □ □

Lemma 116. *Let γ, u, v, w be Coxeter group elements such that*

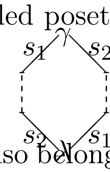


- *$u \neq v$,*
- *$u < w$ and $v < w$ in the weak order,*
- *γ covers both u and v in the Bruhat order.*

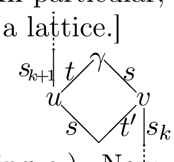
Then $\gamma \leq w$ in the weak order.

Proof. By definition of Bruhat order we have $\gamma = ut_1 = vt_2$, where the elements t_1 and t_2 are *reflections* (i.e. of type gs_ig^{-1}), and our proof splits into three cases, depending on whether any of them is a *simple reflection* s_i.

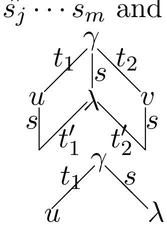
Both reflections simple: Here we can use Kimmo Eriksson's notion of a *polygon poset* (see [42], sec 9.4). Eriksson characterized lower intervals in the weak order as finite edge labelled posets, satisfying four conditions, one of which (the "hat property") states the following. If two edges labelled s_1 and s_2 go down from an element γ, then this hat can be completed to a polygon, alternatingly labelled by s_1 and s_2. Applied to the hat in $[1, \gamma]$, this property produces a polygon as in the illustration, the lower part of which also belongs to the interval $[1, w]$. Now, the dual property (the "cup property") can be applied to the two edges leading up from λ, and we conclude that the whole polygon is in $[1, w]$. In particular, we have $\gamma \leq w$. [We could also have referred to the known fact that weak order is a lattice.]



One nonsimple reflection: Here we can use the *Strong Exchange Condition* (see Humphreys [56], p 117). Let $v = s_1 \cdots s_k$ be a reduced expression and note that $u = s_1 \cdots s_k\, st$ has length k only. Then, according to the Strong Exchange Condition, there is an index i, such that $u = s_1 \cdots \hat{s_i} \cdots s_k\, s$ (omitting s_i). Now, let m be the length of w. Since $u < w$ and $v < w$, there are expressions $w = uu' = vv'$, with u' and v' of length $m - k$, e.g. $u' = s_{k+1} \cdots s_m$. Once more, we use the Strong Exchange Condition, but now in its left-handed version and with the reflexion $t' = sts$. Note that $v' = t's\, s_{k+1} \cdots s_m$ has length $m - k$ only, therefore $v' = s\, s_{k+1} \cdots \hat{s_j} \cdots s_m$. Thus, $w = vv' = \gamma s_{k+1} \cdots \hat{s_j} \cdots s_m$ and we have proved $\gamma \leq w$.



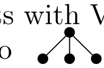
Two nonsimple reflections: Proof by contradiction! Assume γ chosen as the lowest possible counterexample and let $\gamma = s_1 \cdots s_k\, s$ be a reduced expression. As before, we find down-going s-edges from u and v, giving new reflections $t_1' = st_1s$ and $t_2' = st_2s$ on the level below. By our assumption then, $\lambda \leq w$. But now λ can play the role of v in case two above, and we conclude $\gamma \leq w$.

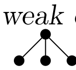


□ □

Remark 117. Lemma 116 is similar to Theorem 3.7 in Björner and Wachs [14], although it can be shown, that neither implies the other.

Remark 118. It is not true that every pebbling game on the Bruhat order corresponds to a lower interval in the weak order. For instance, a game with only three firings, 1, s_1, s_2s_1, is legal, for the last two points are connected by a Bruhat edge. But unless s_1 and s_2 commute, these three points do not form a lower weak order interval. It is, however, possible to change the rules in order to make the correspondence perfect. Just add the rule that a pebble may not be fired if has arrived to its point along a Bruhat edge that is not a weak order edge!

For pebbling games on posets with V-completion, we know by virtue of proposition 108, that if the set of fired points has no -tridents, it must consist of polyominoids connected by dihedral intervals. So we get a corollary that does not involve pebbling at all.

Corollary 119. *Let $[1, w]$ be an interval in the weak order of a Coxeter group such that, even counting Bruhat edges, the interval contains no* *-tridents. Then the interval structure is that of polyominoids connected by dihedral intervals.*

21. Conclusions

The pebbling game is closely related to the *checker jumping game*, where a move looks like $\boxed{\bigcirc\bigcirc\cdot} \longrightarrow \boxed{\cdot\ \cdot\ \bigcirc}$, in any direction. The main difference seems to be that pebbling moves create pebbles while checker jumps annihilate checkers, but that is only a superficial discrepancy. If checkers were small and empty gridpoints large, a natural interpretation of the move $\boxed{\circ\ \circ\ \bullet} \longrightarrow \boxed{\bullet\bullet\circ}$ would be that a black spot is created! Exactly the same invariant weight method can be used in both games, but for some reason, it is much more successful in checker jumping. In fact, the bounds stemming from weight considerations are sharp for the natural reachability problems solved by the present author and Bernt Lindström [41], but far from sharp in pebbling.

Our interpretation of pebbling as a strongly convergent game demonstrates the similarity with the *chip firing* game of Björner, Lovász and Shor [13]. Chips can accumulate on the nodes and when a node is fired, it sends one chip to each neighbour. Some of the chip firing analysis can be applied to pebbling, but the most interesting part focuses on recurrent positions, and that phenomenon cannot occur in pebbling.

No, pebbling has its own special features and most astonishing is the uniform structure of reachable positions, regardless of whether the game is played on $\mathbb{Z}^3$, $\mathbb{Z}^{17}$ or any poset satisfying a few regularity conditions. In all cases, the same combinatorial object emerges: the folded polyominoid. In addition to the geometric and game interpretations there are several others. The set of paths leading from the origin to the highest point of the polyominoid may be seen as a set of words in an n-letter alphabet and this is often, but not always, an equivalence class corresponding to some commutation relations (also called a *trace* or an element of the *commutation monoid* according to Cartier and Foata [21]). Of special interest is the *Coxeter group* case and as we have seen, many folded polyominoids occur as intervals in the Cayley graph of the appropriate Coxeter group. Finally, a *heap of pieces* in the sense of Viennot [80] can be associated to every folded polyominoid in a number of ways.

Chapter 5

Chip-firing and Coxeter elements

22. Introduction

For the theory of finite reflection groups, Coxeter elements play an important role. A Coxeter element is a product $w = s_1 s_2 \cdots s_n$ of all the generating reflections s_i, taken in any order. All Coxeter elements are conjugate and therefore have the same eigenvalues. It turns out that these eigenvalues immediately determine the exponents of the group, and this is probably the simplest way of computing these numbers.

For infinite Coxeter groups, much less is known about the Coxeter elements. A'Campo [1] showed that they have infinite order, Howlett [55] that they have a real eigenvalue ≥ 1. The affine case has been treated in more detail by Steinberg [74] and Berman, Lee and Moody [5].

Our interest in the matter is the enumerative aspect. Surprisingly, the combinatorics turns out to be a special case of the chip-firing game by Björner, Lovász and Shor [13]. The connection is as follows. Given a Coxeter element $w = s_1 s_2 \cdots s_n$, we put a certain number of chips on each vertex s_i in the Coxeter graph, namely the number of neighbours s_j that succeed s_i in w. Every Coxeter element gives rise to a well-defined distribution of chips and the legal play sequences correspond to rotations of the n-letter words. As a second surprise, the reachability relation partitions these game positions precisely according to conjugacy classes.

23. Edge orientations and chip-firing

Let G be a connected, undirected graph. An *acyclic edge orientation* is an assignment of directions to all edges, such that the resulting digraph is acyclic. This is always possible. A simple observation is that the resulting digraph contains at least one *sink*, i.e. a vertex with no out-going edges.

We go on to explain the connection with the *chip-firing game* of Björner, Lovász and Shor, introduced in [13]. If each arrow-head is detached and pronounced a chip, we get a distribution of chips on the vertices. This distribution contains all information, as stated by the following result.

Proposition 120. *An acyclic edge orientation can be retrieved from its distribution of chips, i.e., the in-degrees determine all edge directions.*

Proof. It is well-known that an acyclic digraph must have a sink, so for some vertex, the number of chips equals the degree. That reveals the orientation of all edges at that vertex. But after removing these edges and their chips, we still have a distribution corresponding to an acyclic edge orientation, so the procedure can be continued until all edge orientations have been revealed. □

□

A legal move in the game consists of choosing a vertex with at least as many chips as the degree and then moving one chip to each neighbour. Translated into edge orientations, a legal move means choosing a sink and reversing its edges. Since neither sinks nor sources belong to any cycles, the graph will still be acyclic and contain a sink, so the game goes on forever. The following fact is crucial.

Proposition 121. *Let u and v be two acyclic edge orientations. Then there is a legal game from u to v if and only if there is a legal game from v to u.*

Proof. If a single move can be inverted, so can a sequence of moves. Thus, it is sufficient to consider the case when v is the result of firing a single vertex in position u. In order to prove this by induction, we first strengthen the property like this: For every vertex x that can be fired in position u, there is a continuation in which *every other vertex is fired exactly once.* Clearly, such a firing sequence leads back to u.

The statement is true for the case $\bullet\!\leftarrow\!\bullet$ and induction over the number of vertices proves the proposition: Fire some vertex x, to reach some position u_1. Let $u_1 - x$ be the acyclic edge orientation obtained by deleting x and its edges. By the induction assumption, it is clear that there exists a firing sequence from $u_1 - x$ in which all vertices are fired exactly once, and this is still legal after reinserting x and all its edges, since these edges are directed out from x. $\square$ $\square$

Remark 122. According to this result, *reachability* constitutes an equivalence relation that partitions acyclic edge orientations into *reachability classes.*

Remark 123. The proposition is not generally true for chip-firing games. The simplest counterexample is $u = \overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}$ and $v = \overset{1}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}$. The position u can never reappear, although the game is infinite.

By Theorem 3.3 in [13], the total number of chips in an infinite game must be at least equal to the total number of edges. The distributions considered by us have exactly one chip for each edge, so they are minimal among infinite game positions. This minimality, together with the recurrence property in the last proposition, characterizes these positions.

Definition. A position is *recurrent* if there is some game in which it occurs twice. It is *minimal recurrent* if no chip can be removed without destroying the recurrency.

Proposition 124. *Minimal recurrent chip-firing positions are precisely positions corresponding to acyclic edge orientations.*

Proof. By Theorem 4.1 in [12], for any recurrent position u, there is a recurrent game from u to u such that each vertex is fired exactly once. So along each edge a chip is fired in each direction. Let us always use the same chip on the return route! After the game, remove all chips that were not used. The result is a position corresponding to an edge orientation. Further, it must be acyclic, for all vertices are fired and vertices in a circuit can never be fired. $\square$ $\square$

For many graphs, it is now a rather simple matter to enumerate acyclic edge orientations and reachability classes. Two basic cases are covered by our next proposition.

Proposition 125. *For a tree with n nodes, there are 2^{n-1} acyclic edge orientations but only one reachability class. For an n-cycle, there are $2^n - 2$ acyclic edge orientations and $n-1$ reachability classes of sizes $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$.*

Proof. An n-vertex tree has got $n-1$ edges with no restrictions on orientations. The statement that all are reachable from each other is obvious for a two-vertex tree $\bullet\!\leftarrow\!\bullet$. Assume that it is true for all n-vertex trees and consider an $(n+1)$-vertex tree $T_{n+1} = \mathrm{x}\text{—}T_n$ (where x is a leaf vertex) and two acyclic edge orientations on T_{n+1}, u and v. By assumption, their restrictions to T_n can be connected by a game and if x is fired whenever possible, this also defines a game on T_{n+1}, say from u' to v'. Now, either $u' = u$ or u' is the result of firing x in u. The same argument for v' confirms that u and v are in the same reachability class.

For an n-cycle, exactly two orientations are forbidden, namely all n clockwise or all n anti-clockwise. Consider the $\binom{n}{k}$ orientations with k anti-clockwise edges. Firing a node may be seen as moving the anti-clockwise arrow one step forward, e.g. $\bullet\!\leftarrow\!\bullet\!\rightarrow\!\bullet\!\leftarrow\!\bullet\!\leftarrow\!\bullet$ to $\bullet\!\leftarrow\!\bullet\!\leftarrow\!\bullet\!\rightarrow\!\bullet\!\leftarrow\!\bullet$

It is obvious that any position with k anti-clockwise arrows can be reached in this way. $\square$ $\square$

24. Coxeter elements

Let (W, S) be an irreducible Coxeter group, defined by a connected, edge labelled graph G with vertex set S and labels in $\{3, 4, \ldots\}$.

A product of all n generators, in any order, is called a *Coxeter element.* Two permutations of $s_1, \ldots, s_n$ define the same Coxeter element if and only if one can be transformed into the other by repeated application of the commutation rule $s_i s_j = s_j s_i$ for nonconnected vertices. This is a consequence of Tits's Word Theorem, see Tits [76], 1969. Because of this, the edge labels are not so important in our line of investigation. In most cases, we shall not even mention them in our statements.

Every permutation of $s_1, \ldots, s_n$ induces an *acyclic edge orientation* on G by directing the edge $s_i \leftarrow s_j$ if s_i precedes s_j. By the above, we have the following simple result.

Proposition 126. *There is a bijective correspondence between Coxeter elements and acyclic edge orientations of the Coxeter graph.*

We can choose a slightly different outlook and regard the acyclic edge orientation corresponding to a certain Coxeter element w as a *poset.* A specific n-letter word in the s_i representing w can be viewed as a *linear extension* of the partial order and there is of course a wealth of enumerative results to be applied. We confine ourselves to the following useful observation.

Proposition 127. *The sinks of the acyclic edge orientation corresponding to a Coxeter element w are the s_i that appear as the first letter of some n-letter word representing w. The number of such words starting with s_i can be expressed as*

$$\binom{n-1}{n_1 \ldots n_k} e(G_1)e(G_2)\cdots e(G_k),$$

where the G_j are the components of $G - s_i$, $n_j = |G_j|$ and $e(G_j)$ denotes the number of linear extensions of the poset G_j. This formula gives a recursion for the computation of $e(G)$.

Proof. In any n-letter word representing w, each sink s_i has all its vertex neighbours to the right, so it can be freely moved to the left end of the word.

The n_j letters in component G_j may come in $e(G_j)$ different relative orders. The factor $\binom{n-1}{n_1 \ldots n_k}$ reflects the number of ways that the $n-1$ positions after the first letter may be distributed over the G_j. □ □

If the first letter of a Coxeter word is moved last, the correponding vertex obviously changes from sink to source. Conversely, every sink is a first letter of some Coxeter word corresponding to the edge orientation and therefore, any chip-firing play corresponds to rotation of the word.

Proposition 128. *Rotation of Coxeter words induces an equivalence relation on the set of Coxeter elements, that corresponds precisely to the reachability relation on the set of acyclic edge orientations.*

If $w = s_1 s_2 \cdots s_n$, then $s_1 w s_1^{-1} = s_2 \cdots s_n s_1$ so rotation equivalent elements are conjugate. The converse is also true.

Proposition 129. *Coxeter elements belong to the same conjugacy class if and only if they are rotation equivalent.*

Proof. What we have to prove is that acyclic edge orientations in different reachability classes correspond to nonconjugate elements. According to Proposition 125, there is nothing to be proved when the graph is a tree. We refer to [40] for a proof in the general case. The principal idea of this proof appears in the proof of our next proposition. □ □

For an important class of Coxeter groups, including all finite and affine groups, propositions 125, 128 and 129 enumerate conjugacy classes of Coxeter elements. For the tree case, this is an old result (see [56], 8.4) but the cycle case may be new. Recall that an n-cycle with all edges labeled by 3 is the graph of the affine group denoted by $\tilde{A}_{n-1}$.

Proposition 130. *In $\tilde{A}_{n-1}$ (and in all groups with n-cycle Coxeter graphs) the Coxeter elements fall into $n-1$ different conjugacy classes of sizes $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$. A representative of the k-th class is $w_k = s_1 s_2 \cdots s_k s_n s_{n-1} \cdots s_{k+1}$. A Coxeter element $w = s_{i_1} \cdots s_{i_n}$ belongs to class k if exactly k of the indices precede their numerical successors. The numerical successor of i is defined as $i+1$, unless $i = n$ in which case it is 1.*

Proof. In the proof of Proposition 125, class k is characterized as having k anti-clockwise oriented edges $s_i \leftarrow s_{i+1}$, and that is also the number of s_i that precede s_{i+1} in w_k. What remains to be shown is that w_k and $w_{k'}$ are nonconjugate if $k \neq k'$. Instead of referring to the previous proposition, we shall give a direct argument.

If two group elements are conjugate, then so are their m-th powers. Let us study elements of the form $s_i w_k^m s_i$, where $m = n!$ (or at least divisible by $1, 2, \ldots, n-1$). When s_i is a sink or a source, this is of course a one-step rotation of w_k^m. We shall prove that in all other cases, it is equal to w_k^m itself! Then, we can iterate and draw the conclusion that every conjugate of w_k^m is a rotation, whence the statement about w_k can be deduced.

In fact, $s_i w_k = w_k s_{i-1}$ if $3 \leq i \leq k$ and $s_i w_k = w_k s_{i+1}$ if $k+2 \leq i \leq n-1$, as can be verified directly. The double-step relations $s_2 w_k^2 = w_k^2 s_k$ and $s_n w_k^2 = w_k^2 s_{k+2}$ have more complicated, but still trivial, verification. We conclude that, unless s_i is the first or last letter in w_k, we have $s_i w_k^m = w_k^m s_i$ if m is divisible by k and by $n-k$. The remaining details are easy. □ □

Example. Consider $\tilde{A}_5$. In the Coxeter element $w = s_3 s_5 s_1 s_2 s_4$, 1, 3 and 5 precede their numerical successors, so w is conjugate to $w_3 = s_1 s_2 s_3 s_5 s_4$. In fact, we have $uwu^{-1} = w_3$ with $u = s_4 s_5 s_3$.

In $\tilde{A}_{n-1}$ (as in all Coxeter groups with n generators), there are $n!$ Coxeter words and one may ask how many of these that fall into each conjugacy class. We know that rotating the word does not alter its conjugacy class, so it is enough to consider permutations where n comes last. In that case, n does not precede its numerical successor, so the conjugacy class number k is $(n-1)$ minus the number of inversions among numerically adjacent pairs $(1,2), (2,3), \ldots, (n-2, n-1)$. Ignoring the n we get a permutation $\pi \in S_{n-1}$. Now, $r+1$ precedes r in π if and only if r is a *descent* in π^{-1}, so we can write $k = n-1-d(\pi^{-1})$. But it is known that the *Eulerian numbers* count permutations with given number of descents. The definition is

$$A(n-1, m) = |\{\pi \in S_{n-1} : d(\pi) = m-1\}|,$$

see Stanley's book [73]. Putting $m = n-k$ we get the following formula. Recall that there are n rotations of every word obtained above.

Proposition 131. *In $\tilde{A}_{n-1}$ (and in all groups with n-cycle Coxeter graphs), the $n!$ words representing Coxeter elements are partitioned by conjugacy into $n-1$ classes of sizes $n\,A(n-1, n-k)$ for $k = 1, \ldots, n-1$.*

Chapter 6

Stack sorting and flip sequences

25. The flip game and its applications

The flip game is played with an array of n objects. A move consists in reversing a subsegment of the array — for short, we shall call this a *flip*. Often, we are going to allow the reversal of several disjoint segments in one move, a *multiflip*. Note that total reversal is not considered a flip!

$$\begin{aligned} \text{FLIP} &: (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9) \mapsto (1\ 6\ 5\ 4\ 3\ 2\ 7\ 8\ 9) \\ \text{MULTIFLIP} &: (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9) \mapsto (1\ 4\ 3\ 2\ 5\ 7\ 6\ 9\ 8) \\ \text{NOT A FLIP} &: (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9) \mapsto (9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1) \end{aligned}$$

The object of the flip game is to find the shortest sequence leading from one permutation to another permutation. You can try your skill at this one: How many flips are needed to take you from (1 2 3 4 5) to (5 4 3 2 1)? How many multiflips? [2]

An adjacent transposition is a flip, so obviously you can get from any permutation to any other permutation in at most $n(n-1)/2$ flips (this number is the top-to-bottom distance in the Hasse diagram of the weak order for S_n). As we shall see, n flips are enough even in the worst case, which is the top-to-bottom trip. Using multiflips, it is sometimes possible in $n-1$ moves.

As any permutation is a product of flips, flip sequences may be used for representing the group. A rough estimate of the space efficiency of a naïve implementation of this data type is obtained like this: there are $\binom{n}{2}-1$ different flips, the maximal length of a sequence is n, so the representation uses $n!/\binom{n}{2}^n \sim (\frac{2}{n})^n$ of the available possibilities, as compared to $(\frac{1}{e})^n$ for the usual n-vector representation of permutations. Stated that way, it looks like a devastating waste of memory space. But remembering that, in calculations with permutations, n is often a small number, we can state it differently. Say that $n = 100$ so that the usual representation takes 100 cells of computer memory. The flip sequence will take exactly the same one hundred cells (in fact less on the average, if we use linked lists instead of arrays and if the pointer can be accommodated in the two high bytes of the word!), and the fact that the contents now range from 1 to 4950 is of no consequence. It is our opinion, that there may well exist interesting calculations in which the flip sequence is the optimal representation of a permutation.

But there are two other main reasons for studying flip sequences:

- Flip sequences have interesting applications in geometry, computer science and genetics.
- The natural generalization to Coxeter groups has a number of interesting properties.

Before we move on to the combinatorics, a sightseeing tour of the application field leaves straight away. Please join in!

25.1. **Slopes of n points in plane geometry.** The problems of this section are concerned with configurations of n points in the plane, not all on one line. Two such configurations of five points are shown in the figure. Every pair of points determines a line, and all these lines must be drawn too. There are five lines in the left picture but six lines in the right picture. On the

[2][u]Five flips or four multiflips. (Count ten points for each correct answer!)

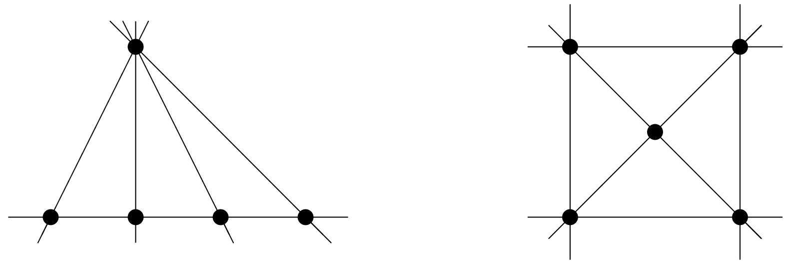


FIGURE 19. Two configuration of five points and the lines generated by them.

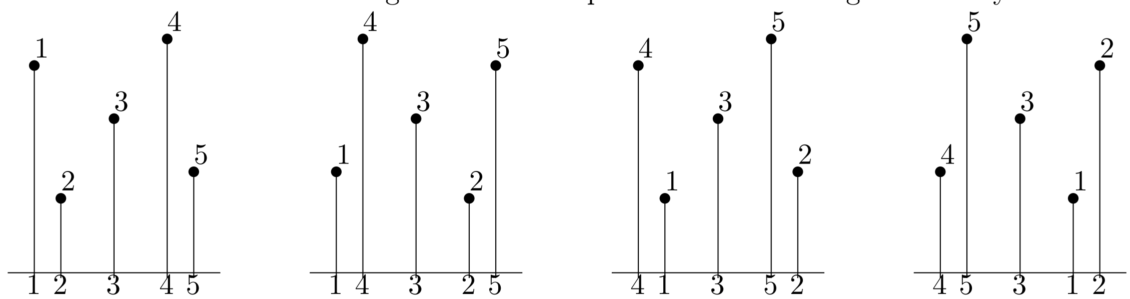


FIGURE 20. A counter-clockwise rotating constellation, defining a flip sequence.

other hand, these lines have five different slopes in the left configuration, but only four different slopes in the right one. Two questions now arise:

(1) What is the minimum number of lines determined by n noncollinear points in the plane?
(2) What is the minimum number of slopes determined by n noncollinear points in the plane?

The first question was posed by by a young Erdös and answered by a middle-aged Erdös. The answer is n, so the left-hand picture illustrates the extreme case. The second question emanates from P.Scott [67] (1970), and was accompanied by a conjecture, proved to be true by Peter Ungar [78] in 1982. The answer is n if n is an even number and also if $n = 3$, but for $n = 5, 7, 9, \ldots$, the answer is $n - 1$. The right-hand figure is the extreme case for these odd numbers — the square just has to be replaced by a regular hexagon, octagon etc. For an even number of points and for $n = 3$, however, nothing can beat the left-hand picture. For an exposition of results of this kind, see Section 1.11 of Oriented Matroids [11].e

Ungar's proof uses a technique developed by Goodman and Pollack [49]. They project the points vertically down onto a horizontal axis and they number the points and their projections by $1, \ldots, n$. Now, the point constellation starts to rotate slowly, around its center of gravity, say. Each time that two or more points happen to have the same x-co-ordinate, a flip takes place in the permutation of projected points.

Every line in the configuration thus corresponds to a flip in the permutation. Lines with the same slope correspond to simultaneous flips, i.e. slopes correspond to multiflips.

When the constellation has rotated 180 degrees counter-clockwise, the permutation is the reverse of the start permutation, for the situation is as if we were to project everything upwards in the start position. To summarize, a flip sequence transforms $1 \ldots n$ to $n \ldots 1$ and we seek a lower bound for the number of flips or multiflips needed to perform this transformation. The transformation is none other than what we have called w_o, the longest element of A_{n-1}.

Ungar's proof makes no mention of points or slopes, just permutations and flips. It shows that when n is even, the number of multiflips needed is at least n, therefore the number of flips needed is also at least n. The extreme cases of the previous figure prove that the bound is sharp. The cases $n = 5, 7, 9 \ldots$ for multiflips are direct consequences of the bounds for $n = 4, 6, 8 \ldots$ and the known extreme case. The same cases for single flips cannot be derived directly from Ungar's result, but will be covered in section 26.2.

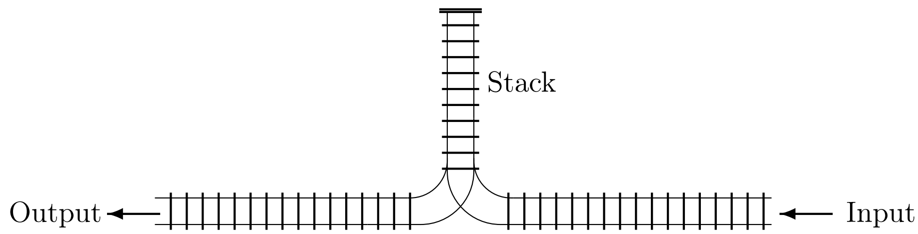


FIGURE 21. The railway network model of stack sorting.

25.2. **Stack sorting in computer science.** Anyone who ever opened a textbook in computer science must have marvelled at the superabundance of sorting methods. Donald Knuth [61] managed to get the state of the art twenty years ago, into one chapter, 388 pages long, and to justify its size he quotes an anonymous computer manufacturer's estimate, that over 25 percent of the running time on their computers was spent on sorting. Obviously, Knuth felt uneasy about passing on this highly dubious piece of information, for a few sentences later, he adds: *But even if sorting were almost useless, there would be plenty of rewarding reasons for studying it anyway!* We suggest that the reader make a mental note of this statement, for we are going to discuss some quite impractical methods.

The railway model of stack sorting was suggested by E. Dijkstra. A train of railroad cars coming in from the right can be permuted by means of a siding. By definition, this siding is a *stack*, for the last car to enter it must be the first car out. The traffic-flow on the main track is always right to left, so this is a *one-pass method*. Assuming that the cars are numbered $1, 2, \ldots, n$ in input order, what output permutations are possible?

Knuth gives a complete analysis of the situation. The only permutations obtainable are those with no occurrence of the pattern 312. Note that 412 has the same pattern, as has 736 etc, so all these triples are forbidden. As one might guess, this forbidden pattern occurs in most permutations. While there is a total of $n!$ permutations, there are only $\binom{2n}{n} - \binom{2n}{n-1}$ that can be stack-sorted in one pass. (Yes, this is the Catalan number, so often present in enumerative formulas.)

The analysis of one-pass stack-sorting appeared as Exercise 2.2.1-2 in Vol. 1 of Knuth's Fundamental algorithms series, an exercise on stacks, not on sorting! When the enormous sorting chapter was published, five years later, many computer scientists had hooked on to the train and worked on diverse modifications and generalizations of stack sorting. So in Exercise 5.2.4-19, we are asked to analyze k-pass sorting. We can think of this as a main track with k dead end sidings, or as a circular main track, permitting the train to pass the same siding k times. The simplest, but also the most interesting result is the following.

Proposition 132. [Knuth, Tarjan] *Any permutation of n elements can be stack-sorted in k passes if $n \leq 2^k$.*

One natural modification of the stack concept is the *all-out stack*. All stacks accept their input item by item. Ordinary stacks also output individual items, but when an all-out stack pops one item, it pops them all. Some existing hardware stacks have this property, deep containers that must be emptied by use of gravity. Note that, as in the railway figure, all items must enter the stack; there is no by-pass.

The workings of an all-out stack is exactly what we have called a multiflip and a k-pass sorting with an all-out stack performs a sequence of multiflips. Therefore, the flip sequence analysis can give complexity results for this kind of sorting.

One closely related problem is called *pancake flipping*. Here, we are only allowed to flip initial segments of the permutation. For this case too, we can consider the shortest possible expression for w_o as a product of pancake flips. The bounds by Gates and Papadimitriou [49] from 1979 were recently improved by Cohen and Blum [24].

25.3. **Flip mutations in genetics.** A DNA-molecule is a linear structure, the significant parts of which are called genes. As is well-known, DNA has the ability to make a perfect copy of itself. At some point in this duplication process, the molecule is very sensitive to disturbances and can break at one or more points. A chemical mending crew steps in and makes prompt repairs. However, when a segment has broken off, it often happens that it is glued back flipped. Such flips are now considered to be the most common source of mutations.

The new techniques for DNA-charting enormously improve the possibilities of finding out how different species are related. The charts of species within the same genus often display the same gene material, but in slightly different order. If a small number of mutation events can explain the differences, the species are considered to be closely related. Combinatorially expressed, the *shortest flip sequence* leading to the observed permutation has to be computed.

Two computer scientists, V. Bafna and P. Pevzner, recently proved some interesting facts, relevant to the genetic problem [4].

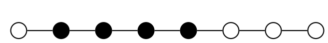
Proposition 133. [Bafna, Pevzner] *If total reversal (w_o) is considered a legal flip, then any permutation can be ordered with at most $n-1$ flips. Only two permutations need the maximum number of flips, namely the* Gollan *cycle* $(1\ 3\ 5 \ldots n \ldots 4\ 2)$ *and its inverse.*

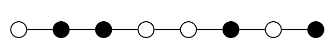
Note that the Gollan cycle is in cycle notation – not a permutation vector, although it may look like one.

The most interesting part of Bafna's and Pevzner's paper may be the observation that not permutations but *signed permutations* provide the most realistic model of the genetic situation. A *gene* is a directed fragment of the DNA-molecule, so a mutation of the original order is given naturally by a signed permutation. There is no mention of Coxeter groups in their work, but every algebraic combinatorialist must ask herself "Can this be done for any Coxeter group?". (As the author became aware of at his first combinatorics conference, this is one standard courtesy question to all speakers. The other one is "Is there a q-analogue?".) The answer is, of course, "Yes". (And that is as far as you will get, before the chairman interrupts with "Let's thank the speaker again!").

26. Analysis of the flip game

Our intention is to formulate the concepts *flip* and *multiflip* in such a way that they make sense for any Coxeter group. The first observation is that flips and multiflips in A_{n-1} (ordinary permutations, that is) are determined by the set of adjacent tranpositions involved. As shown in our example, that amounts to specifying a subset of nodes in the Coxeter graph.

FLIP : (1 6 5 4 3 2 7 8 9) 

MULTIFLIP : (1 4 3 2 5 7 6 9 8) 

It is clear that a flip involving five values is an internal affair of four adjacent nodes in the Coxeter graph of A_{n-1}. A multiflip is specified by any strict subset of nodes. The connected components of the corresponding subgraph define flips. So let $J \subset S$ be a connected subset of the nodes (an interval, to be sure) and W_J be the generated subgroup. We know that the longest element $w_o(W)$ is the total flip (see page 9) and therefore our flip must be $w_o(W_J)$.

The subset J defined by a multiflip consists of several disconnected intervals, $J_1, J_2, \ldots$, and the flips $w_o(J_i)$ are independent, so the multiflip can be written as a product $w_o(W_{J_1})w_o(W_{J_2})\cdots$. But this product must also be $w_o(W_J)$, for the length function of a reducible group is the sum of the lengths of each component. Anyway, the following definition is justified.

Definition. Let (W, S) be a Coxeter system with graph G. An element $w \in W$ is a *multiflip* if $w = w_o(W_J)$ for some strict subset $J \subset S$ such that W_J is finite. If J is connected (as a vertexset of G), the multiflip w is a *flip*.

Without the finiteness condition on W_J, the notation $w_o(W_J)$ would be meaningless, for an infinite group has no longest element. But if W is affine or compact hyperbolic, then all strict *parabolic subgroups* (that is what W_J-subgroups are called) are finite.

We already know about flips and multiflips in A_{n-1}, but what about the signed permutations of B_n and D_n? Just as there are two kinds of s_i-actions in B_n, there are two kinds of flips. If the special node s_n is involved, we flip a symmetric interval $-k, \ldots, k$, if it is not involved, we flip a positive interval and its negative mirror image. The situation is similar in D_n — if both special nodes, s_1 and s_n, are involved, we flip a symmetric interval, otherwise two mirror images.

Comparing this to the situation in the genetic application, we must admit that this flip is not a good model of the DNA-flips. In a genetic flip, all genes in the interval are both inverted and flipped. The correct model of a genetic flip would be a sequence of two B_n-flips, first flipping the symmetric interval $-r, \ldots, r$ to invert all genes, then multiflipping $-k, \ldots, k$ and $(k+1, \ldots, r$. As far as we know, there has been no studies in this direction.

26.1. **Flip sequences and the Möbius function.** The theorem of Peter Ungar states the minimal number of multiflips necessary to get all the way from e to w_o in the symmetric group. In this section, we are going to prove analogous bounds for all other finite Coxeter groups as well. However, we want to start with an account of other interesting aspects of these *allowable sequences*, as they were named by Goodman and Pollack [49].

Goodman and Pollack noted the fact that every configuration of n points in the plane gives rise to an allowable sequence from e to w_o. A natural question to ask is whether every allowable sequences emanates from some point configuration. There are, however, allowable sequences that are not realizable in this way. One has to generalize from point configurations to oriented matroids to get the right correpondence, see [11], page 35–38.

The allowable sequences of a general Coxeter group has another characterization in terms of the Möbius function. For any poset, the Möbius function $\mu(x, y)$ is defined recursively by $\mu(x, x) = 1$ and $\sum_t \mu(x, t) = 0$ if the sum is taken over all t in an interval $[x, y]$ (see [73] for a good introduction to μ). When the poset is the weak order, the following is true ([8], Th. 4).

Proposition 134. [Deodhar [36], 1977] *Suppose that $x \leq y$, then $\mu(x, y) \neq 0$ if and only if $y = xw$, where w is a multiflip.*

An allowable sequence is therefore characterized by the fact that *the Möbius function is nonzero over every link in the chain.* In fact, it is $+1$ if $l(w)$ is even and -1 if $l(w)$ is odd. Closely related to this is the following topological property. It is to be understood in the context of the simplicial *order complex*, that can be defined for every interval $[x, y]$ of any poset. The simplices of the order complex are the chains $x < x_1 < x_2 < \ldots < y$, so the faces are maximal chains, where the length increases by one in each link.

Proposition 135. [Björner [8], 1984] *All weak order intervals, except multiflips, are contractible complexes.*

A multiflip interval is always homotopy equivalent to some sphere, but we must not go further in this direction. If we have already strayed too far, the purpose was to emphasize that multiflip sequences is not just a stupid way of sorting railway cars!

26.2. **Ungarian theorems for finite Coxeter groups.** The main theorem extends from the symmetric group to all other finite Coxeter groups the lower bound for the length of an allowable sequence from e to w_o.

Theorem 136. *For the groups $B_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(p)$, the shortest possible allowable sequence has length $\frac{2}{n} l(w_o)$, where n is the number of nodes in the Coxeter graph and $l(w_o)$ is tabulated on page 10.*

Conjecture 137. The number of shortest possible allowable sequences is $2^n - 2$.

Proof. Before endeavouring a general proof, let us see what the theorem and the conjecture mean in the simplest case, $I_2(p)$. There are two generators, a and b and a relation $ababa\ldots = babab\ldots$. The only multiflips are a and b themselves, so it is evident that the shortest allowable sequences are $ababa\ldots$ and $babab\ldots$. In this case, the formulas correctly state that $\frac{2}{2}p = p$ and $2^2 - 2 = 2$.

The computer was kind enough to check the formulas for the sporadic groups (E, F, H-types), and could confirm the formulas. (But for E_8 only after several weeks of continuous computing!) Anyway, we now feel entitled to concentrate our efforts on B_n and D_n.

B_n: We want to prove that the shortest multiflip sequence from $-n\ldots n$ has $2n$ links. Two pages ago, we found out what the multiflips look like, and it is clear that they can be regarded as multiflips on S_{2n+1}, for $-n, \ldots, n$ are $2n+1$ symbols to be permuted. The e and the w_o are also the same in both groups, so an allowable sequence for B_n is also an allowable sequence for S_{2n+1}. And so, by Ungar's theorem, it has at least $2n$ links.

An example of an allowable sequence for B_n of the minimal length $2n$ is the following. The first link leaves only $-n$ and n fixed and flips the whole interval between them. The next $n-1$ links move $-n$ and n towards the origin, one unit step at a time. The next link *switches* these two numbers and the last $n-1$ moves take them to their final destination.

D_n: We have to carry through a rather complicated argument, adapted from Ungar's original method. We designate positions -2,-1,1,2 as *the middle* and note that $2(n-1)$ values have to *cross the middle* in the course of the game. The only fliptype in which numbers cross the midddle are the central symmetric flips $-k\ldots k$ bringing $2k-4$ numbers across. There are also the flips involving one of the special nodes s_1 and s_n, these do not bring anything across, but they take two numbers into the middle and bring two numbers out of the middle. Two such flips may co-operate to take two numbers across, three may co-operate to take four numbers across etc.

The idea is to show that, on the average, no more than one crossing per move is possible, therefore at least $2(n-1)$ moves are needed to perform $2(n-1)$ crossings. For the central flips, we can take Peter Ungar's argument verbatim. After the flip, leading to $k\ldots-k$, nothing more can happen in the middle for some time, for the numbers are in decreasing order. The next $k-2$ moves must be noncrossing and each move can shorten the decreasing run by at most one unit at both ends. Also, the central k-flip must have been preceded by a step-wise building up of this long increasing run (unless it is the first crossing move and can use the original increasing order). This accounts for another $k-3$ noncrossing flips. The total is $2k-4$ moves needed to achieve $2k-4$ crossings.

Flips bringing two values into the middle and taking two other values out of the middle can never attain even an average of one crossing per move, so these never occur in the optimal allowable sequences.

There is only one disturbing fact left to take care of. Our move-counting arguments did not apply to the very first and very last flip. But there is really nothing special about them, as can be seen like this. Let $u_1u_2\ldots u_k = w_o$ be the sequence. The total flip w_o evidently commutes with everything else, so we can rotate the sequence freely, making u_1 and u_k lose their special status. □

□

The conjectured expression $2^n - 2$ for the number of optimal sequences has a reasonable explanation. The *greedy strategy* in the flip game is to choose the set $J \subset S$, used to compute the multiflip $w_o(W_J)$, as large as possible. That is to say, when at some point we have reached w, we use every playable node for the next multiflip, i.e. choose $J = S - D(w)$. Ungar [78] proved that the greedy strategy will take you from any w to w_o in at most $n-1$ moves. The conjecture is based upon the assumption that the greedy strategy is the *only* strategy that will produce optimal flip sequences from e to w_o. The strategy cannot be applied for the first move as it would choose an illegal move, the total flip. But there are $2^n - 2$ legal subsets of S and any of these may be chose for the first move. From then on, everything is completely determined by the greedy strategy.

The status of the conjecture is that nobody seems to be able to prove it even for the symmetric group. On the other hand, the other groups are probably simpler, so once a proof for A_n appears, the other questions will most likely have been settled within a week or two.

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