Hardness of Approximating Problems on Cubic Graphs

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Abstract

Four fundamental graph problems, Minimum vertex cover, Maximum independent set, Minimum dominating set and Maximum cut, are shown to be APX-complete even for cubic graphs. This means that unless P=NP these problems do not admit any polynomial time approximation scheme on input graphs of degree bounded by three.

1 Introduction

Among combinatorial optimization problems that are computationally hard to solve, NP-hard optimization problems on graphs have a great relevance both from the theoretical and practical point of view.

Despite the apparent simplicity of cubic and at-most cubic graphs, several NP-hard graph problems remain NP-hard even if restricted to these classes of graphs, but become polynomial time solvable for graphs of degree 2 [10, 12].

Since one can be almost certain that NP-hard problems cannot be efficiently solved, one has to restrict oneself to compute approximate solutions. Therefore it would be desirable to identify if and how much boundedness of the graph degree is helpful in approximation.

It is well known that the variation of NP-hard graph problems in which the degree of the graph is bounded by a constant often allows to achieve different results with respect to the approximation properties. Namely, problems that for general graphs cannot be approximated within constant approximation ratio (e.g. Maximum independent set, Minimum dominating set and Minimum independent dominating set) have been shown to be in APX (i.e. approximable within *some* constant) for bounded degree graphs. For some NP-hard optimization problems that are approximable for general graphs (e.g. Minimum vertex cover) better approximation ratios have been achieved for graphs of low degree [4, 5, 6, 7, 13, 15, 16, 17, 18].

Nevertheless, many graph problems are APX-hard even if the degree of the graph is bounded by some constant, and therefore they can be approximated within some constant factor of the optimum, but cannot be approximated within *any* constant (PTAS) [14, 15, 16, 18].

Some problems are known to be APX-hard even for cubic or at-most-cubic graphs (e.g. Maximum 3-dimensional matching and Maximum independent dominating set [14, 15]). For several other graph problems it is just known that they are APX-hard for graphs of some bounded degree greater than 3 [18].

In this work we show APX-hardness results for several optimization problems on cubic or at-most-cubic graphs, namely for Minimum vertex cover (MIN VERTEX COVER), Maximum independent set (MAX IND SET), Minimum dominating set (MIN DOM SET), and Maximum cut (MAX CUT).

Surprisingly simple reductions are used for most of our results, but for showing the APX-completeness of MAX CUT on cubic graphs we need a quite complicated structure consisting of a chain of expander graphs. Expander graphs have been used in different ways in approximation preserving reductions [18,

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3, 2, 9], and seem to be very useful. For a description of expander graphs and an algorithm constructing expander graphs we refer to Ajtai [1].

The remainder of the paper is organized as follows. In Section 2, we state basic definitions and notations. In Section 3, we show the APX-completeness of MIN VERTEX COVER, MAX IND SET and MIN DOM SET on cubic graphs. In Section 4, we prove the APX-completeness of MAX CUT on cubic graphs.

2 Definitions

Although various reductions preserving approximability within constants have been proposed (see [8]), the L-reduction defined in [18] is perhaps the easiest one to use. Given two NP optimization problems F and G and a polynomial time transformation f from instances of F to instances of G, we say that f is an *L-reduction* if there are positive constants α and β such that for every instance x of F

1. $opt_G(f(x)) \leq \alpha \cdot opt_F(x)$,
2. for every feasible solution y of $f(x)$ with objective value $m_G(f(x),y) = c_2$ we can in polynomial time find a solution y' of x with $m_F(x,y') = c_1$ such that $|opt_F(x) - c_1| \leq \beta\, |opt_G(f(x)) - c_2|$.

Using L-reductions (or reductions similar to them) one can show that a problem F is APX-complete, i.e., F is approximable within c for some c and every approximable problem can be L-reduced to F[1]. In this paper we will consider the following APX-complete problems.

MAX CUT$-B$

INSTANCE: Graph $G = (V,E)$ of degree bounded by B.
SOLUTION: A partition of V into two parts: a red part P_R and a green part P_G.
MEASURE: Cardinality of the set of edges that are cut, i.e. edges with one end point in P_R and one end point in P_G.

MAX IND SET$-B$

INSTANCE: Graph $G = (V,E)$ of degree bounded by B.
SOLUTION: An independent set for G, i.e., a subset $V' \subseteq V$ such that no two vertices in V' are joined by an edge in E.
MEASURE: Cardinality of the independent set, i.e., $|V'|$.

MIN DOM SET$-B$

INSTANCE: Graph $G = (V,E)$ of degree bounded by B.
SOLUTION: A dominating set for G, i.e., a subset $V' \subseteq V$ such that for all $u \in V - V'$ there is a $v \in V'$ for which $(u,v) \in E$.
MEASURE: Cardinality of the dominating set, i.e., $|V'|$.

MIN VERTEX COVER$-B$

INSTANCE: Graph $G = (V,E)$ of degree bounded by B.
SOLUTION: A vertex cover for G, i.e., a subset $V' \subseteq V$ such that for all $(u,v) \in E$ at least one of u and v is included in V'.
MEASURE: Cardinality of the vertex cover, i.e., $|V'|$.

MAX E3-SAT$-B$

INSTANCE: Set of variables X, set of disjunctive clauses C over the variables X, where each clause consists of exactly three variables, and each variable occurs in at most B clauses.
SOLUTION: Truth assignment of X.
MEASURE: Cardinality of the set of clauses from C that are satisfied by the truth assignment.

[1] Previously the notion MAX SNP-completeness was used, but it is now more accurate to talk about APX-completeness. Every MAX SNP-complete problem is also APX-complete, even if a slightly different reduction has to be used, see for example [8].

3 APX-completeness of some problems on cubic graphs

MIN VERTEX COVER$-B$, MAX IND SET$-B$ and MIN DOM SET$-B$ were known to be APX-complete for some bounded degree B, and explicit proofs have been obtained for MIN VERTEX COVER-4, MAX IND SET-4 and MIN DOM SET-8 [18, 15].

In the following we will shown that these problems remain APX-complete even if the degree of the graphs is bounded by 3.

Theorem 3.1 MIN VERTEX COVER-3 *is* APX-*complete.*

Proof. It is well known that MIN VERTEX COVER can be approximated within 2, and thus is included in APX.

Now we show that MIN VERTEX COVER-3 is APX-hard. Let f be the following L-reduction from MIN VERTEX COVER-4 to MIN VERTEX COVER-3.

Given a graph $G = (V,E)$ of bounded degree 4 construct an at-most-cubic graph $G' = (V',E')$ in the following way. Let v be a vertex of degree 4. Split $v \in V$ into two vertices v_1 and v_2 of degree 2, then add an extra vertex u and edges (v_1,u), and (v_2,u), see figure 1.

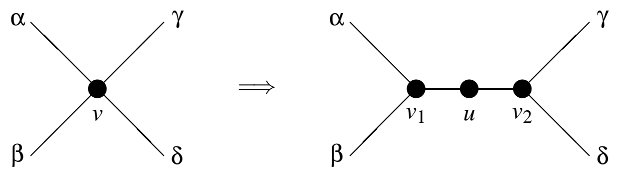


Figure 1: The transformation of a degree 4 vertex used both in the reduction from MIN VERTEX COVER-4 to MIN VERTEX COVER-3 and the reduction from MAX IND SET-4 to MAX IND SET-3.

Given a vertex cover $C' \subseteq V'$ of $G' = f(G)$ we transform it back to a vertex cover $C \subseteq V$ of G in the following manner. Notice that for each triple of vertices $v_1, v_2, u \in V'$ coming from a vertex $v \in V$ of degree 4: either $u \in C'$, or at least two vertices (possibly one of them is u) belong to C'. If at least two vertices among v_1, v_2 and u are in C', we can substitute such vertices in C' by the pair v_1, v_2 and we still have a vertex cover of at most the same size. Then from every vertex cover C' we can construct a vertex cover C of size $|C'| - s$, where s is the number of vertices of degree 4 in G. Include in C any vertex of degree less than 4 that belongs to C' and any vertex $v \in V$ of degree 4 such that the vertices $v_1, v_2, \in V'$ belong to C'.

It is easy to see that from every vertex cover $C \subseteq V$ of G we can construct a vertex cover $C' \subseteq V'$ of $G' = f(G)$ of size exactly $|C| + s$. In C' we include every vertex in C that has degree smaller than 4, and for each vertex $v \in V$ of degree 4 we do as follows. If $v \in C$ then $v_1, v_2 \in C'$, if $v \notin C$ then $u \in C'$. Since G has bounded degree 4 we have $|C| \geq |V|/5 \geq s/5$. We see that $|C'| = |C| + s \leq 6 \cdot |C|$

Thus $opt(f(G)) \leq 6 \cdot opt(G)$ and we have shown that f is an L-reduction with $\alpha = 6$ and $\beta = 1$. □

Theorem 3.2 MAX IND SET-3 *is* APX-*complete.*

This result was recently proved by Berman and Fujito [5] using a complex reduction from MAX E3-SAT$-B$. We can give a much simpler proof of this result using the same reduction as in the proof of Theorem 3.1. Analogously the reduction by Berman and Fujito could be used to show that MIN VERTEX COVER-3 is APX-complete.

Proof. (Outline) Since MAX IND SET-4 is APX-complete and since the complement of any vertex cover is an independent set the same transformation as above can be used to prove the theorem. We get an L-reduction from MAX IND SET-4 to MAX IND SET-3 with $\alpha = 6$ and $\beta = 1$. □

Theorem 3.3 MIN DOM SET-3 *is* APX-*complete.*

Proof. It is well known that the variation of MIN DOM SET in which the degree of the graph is bounded by a constant belongs to APX. In the following we will prove that it remains NP-hard even if the degree of the graph is bounded by 3. Since L-reductions compose [18], we first give an L-reduction f_1 from MIN VERTEX COVER-3 to MIN DOM SET-6 and then an L-reduction f_2 from MIN DOM SET-6 to MIN DOM SET-3.

Given an at-most-cubic graph $G=(V,E)$ construct a graph $G'=(V',E')$ of bounded degree 6 in the following way. For each edge (u,v) in the former graph insert an extra vertex w and edges (u,w), (v,w), see figure 2.

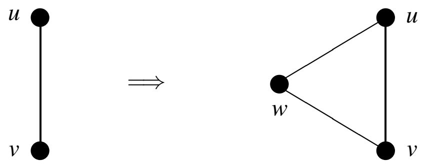


Figure 2: The transformation of an edge in the reduction from MIN VERTEX COVER-3 to MIN DOM SET-6.

It is easy to see that every dominating set $D \subseteq V'$ of $G'=f_1(G)$ can be transformed into an equally good or better vertex cover $C \subseteq V$ of G by including in C the following vertices. For each vertex $v \in D$ such that $v \in V$, include v in C. For each vertex $w \in D$ such that $w \notin V$, choose a vertex v such that $(v,w) \in E'$ and include v in C.

Consider now a vertex cover $C \subseteq V$ of G. We can construct a dominating set $D \subseteq V'$ of G' of the same size by including in D exactly the same vertices. Thus f_1 is an L-reduction with $\alpha=\beta=1$

Now we describe the L-reduction f_2 from MIN DOM SET-6 to MIN DOM SET-3. Given a graph $G=(V,E)$ of bounded degree 6 construct an at-most-cubic graph $G'=(V',E')$ in the following manner. Let $v \in V$ be a vertex of degree 4. We split v into two vertices v_1 and v_2 of degree 2. Then we add five extra vertices $u_{1,1}$, $u_{1,2}$, $u_{2,1}$, $u_{2,2}$, w, and edges $(v_1,u_{1,1})$, $(u_{1,1},u_{1,2})$, $(u_{1,2},w)$, $(v_2,u_{2,1})$, $(u_{2,1},u_{2,2})$, $(u_{2,2},w)$, see figure 3.

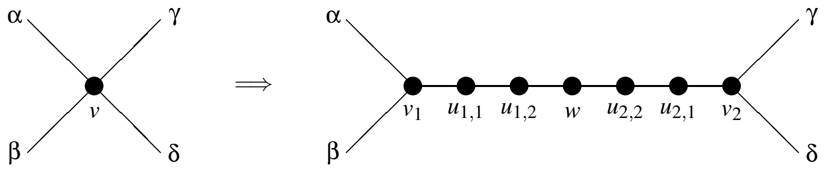


Figure 3: The transformation of a degree 4 vertex in the reduction from MIN DOM SET-6 to MIN DOM SET-3.

For each vertex $v \in V$ of degree 5 or 6 we do in a similar way except for the fact that we split v into three vertices, instead of two and extend the previous construction with a third leg from the center vertex w. More precisely we split v into v_1 and v_2 of degree 2 and v_3 of degree p, where p is 1 if the degree of v is 5 and 2 if the degree of v is 6. Then we add seven extra vertices $u_{1,1}$, $u_{1,2}$, $u_{2,1}$, $u_{2,2}$, $u_{3,1}$, $u_{3,2}$, w, and edges $(v_1,u_{1,1})$, $(u_{1,1},u_{1,2})$, $(u_{1,2},w)$, $(v_2,u_{2,1})$, $(u_{2,1},u_{2,2})$, $(u_{2,2},w)$, $(v_3,u_{3,1})$, $(u_{3,1},u_{3,2})$, $(u_{3,2},w)$.

It is easy to see that any dominating set $D' \subseteq V'$ of $G'=f_2(G)$ can be transformed back to a dominating set $D \subseteq V$ of G as follows. For each vertex $v \in V$ of degree 4: $v \in D$ if $|\{v_1,\, v_2,\, u_{1,1},\, u_{1,2},\, u_{2,1},\, u_{2,2}, w\} \cap D'| \geq 3$ and $v \notin D$ if $|\{v_1,\, v_2,\, u_{1,1},\, u_{1,2},\, u_{2,1},\, u_{2,2}, w\} \cap D'| = 2$. For each vertex $v \in V$ of degree greater than 4: $v \in D$ if $|\{v_1,\, v_2,\, v_3,\, u_{1,1},\, u_{1,2},\, u_{2,1},\, u_{2,2},\, u_{3,1},\, u_{3,2}, w\} \cap D'| \geq 4$ and $v \notin D$ if $|\{v_1,\, v_2,\, v_3,\, u_{1,1},\, u_{1,2},\, u_{2,1},\, u_{2,2},\, u_{3,1},\, u_{3,2}, w\} \cap D'| = 3$. It is clear that D is a dominating set of size $|D| \leq |D'| - 2 \cdot s_1 - 3 \cdot s_2$, where s_1 and s_2 are the number of vertices of degree 4 and greater than 4 in V, respectively.

Finally, given a dominating set $D \subseteq V$ of G we can construct a dominating set $D' \subseteq V'$ of $G' = f_2(G)$ such that $|D'| = |D| + 2 \cdot s_1 + 3 \cdot s_2$. Since G has bounded degree 6, we have $|D| \geq |V|/7$. Therefore $|D'| \leq |D| + 3 \cdot (s_1 + s_2) \leq 22 \cdot |D|$.

Thus, $opt(f_2(G)) \leq 22 \cdot opt(G)$ and we have shown that f_2 is an L-reduction with $\alpha = 22$ and $\beta = 1$. □

The above results are still valid for cubic graphs, that is for graphs where every vertex has degree exactly three.

We simply show this for MIN VERTEX COVER−3.

Split each vertex v of degree two into two vertices, v_1 and v_2, of degree 1. Then add two extra vertices, u_1 and u_2, and edges (v_1, u_1), (v_1, u_2), (u_1, u_2), (v_2, u_1), and (v_2, u_2). For each vertex v of degree 1 add 6 extra vertices u_1, u_2, u_3, u_4, u_5, and u_6 and edges (v, u_1), (v, u_2), (u_1, u_2), (u_1, u_3), (u_2, u_5), (u_3, u_4), (u_3, u_6), (u_4, u_5), (u_5, u_6), and (u_4, u_6), see figure 4.

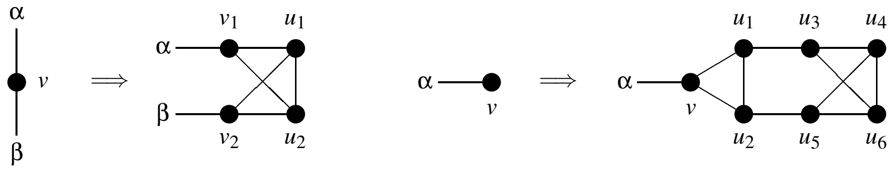


Figure 4: The transformation of vertices of degree 2 and 1 in the reduction from at-most-cubic graphs to cubic graphs.

From every solution of size c in the at-most-cubic graph we can construct a solution in the cubic graph of size exactly $4s_1 + 2s_2 + c$, where s_1 and s_2 are the number of vertices of degree 1 and 2 in the at-most-cubic graph, respectively.

4 APX-completeness of MAX CUT on cubic graphs

MAX CUT can be approximated within 1.139, and is therefore included in APX [11]. We will first show that MAX CUT is APX-hard for multigraphs of degree 6, and then for simple graphs of degree 3.

Theorem 4.1 MAX CUT−6 *for multigraphs is* APX-*complete.*

Proof. We will construct an L-reduction from MAX E3-SAT−3 to MAX CUT−6 for multigraphs.

Suppose we are given an instance of MAX E3-SAT−3 with n variables and m clauses. Without loss of generality we can assume that every variable occurs positively in at least one clause and negatively in at least one clause. This problem is known to be APX-complete [19].

Construct a multigraph with a vertex set consisting of two vertices named x_i and $\overline{x}_i$ (the *variable vertices*) for each variable, and four vertices named y_j, $\overline{y}_j$, b_{2j-1} and b_{2j} for each clause. For each clause $c_j = l_1 \vee l_2 \vee l_3$ we construct eight edges: (b_{2j-1}, l_1), (b_{2j}, y_j), (l_1, y_j), (l_2, l_3), $(l_2, \overline{y}_j)$, $(l_3, \overline{y}_j)$, and two parallel edges $(y_j, \overline{y}_j)$, see figure 5. We also, for every variable x_i, include two parallel edges between the vertices x_i and $\overline{x}_i$. The degree of the graph is 6.

Consider a solution where all the b_i vertices in the graph are placed in the same part, say the red part. If we look at the subgraph in figure 5 we can see that if all the three l_k vertices are in the red part, then at most 4 edges can be cut, but if at least one of the l_k vertices is in the green part, then it is always possible to choose parts for y_j and $\overline{y}_j$ so that six edges are cut.

If x_i and $\overline{x}_i$ are placed in the same part we can move one of them to the other part without decreasing the number of cut edges. This is because at least one of x_i and $\overline{x}_i$ occurs just in one clause, and has therefore just two edges except the two edges connecting x_i and $\overline{x}_i$.

Now we have a correspondence between the values of the variables and the partition of the variable vertices—if the variable x_i is true then the vertex x_i is green and $\overline{x}_i$ is red, and if the variable x_i is false then

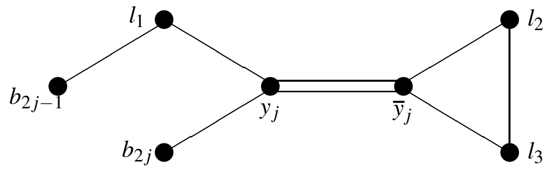


Figure 5: The constructed edges from a clause $l_1 \vee l_2 \vee l_3$.

the vertex x_i is red and $\overline{x}_i$ is green. Thus the number of cut edges will be $2n+4m+2s$ where s is the number of satisfied clauses in the corresponding MAX E3-SAT problem instance.

Recall that the above reasoning is valid only if all the b_j vertices are in the same part. In order to obtain this we construct a bipartite cubic expander between the b_j vertices and an equivalently large set of new vertices, called $c_{1,j}$. We then construct a chain of bipartite cubic expanders between $\{c_{i,j}\}$ and $\{c_{i+1,j}\}$ for $1 \le i < k$, where k is a constant to be decided later. We thus have $k+1$ layers of vertices that are connected by expanders. The degree of each vertex is at most 6. Let N be the number of b_j vertices (which means that N is the number of vertices in any of the $k+1$ layers).

Ajtai has shown that cubic bipartite expander graphs of size N can be constructed in polynomial time in N [1]. Such an expander $G = (A \cup B, E)$ (where A and B are the two parts in the bipartition) has the property that for any subset $A' \subseteq A$ with $|A'| \le |A|/2$, A' is connected to at least $(1+\alpha)|A'|$ vertices in B, and vice versa. α is some fixed positive constant.

Consider a solution and define the red part P_R as the part that contains most of the b_j vertices. We will show that we can move all the b_j vertices that are in the green part P_G to P_R without decreasing the size of the cut.

By moving the b_j vertices in P_G to P_R the number of cut edges not in the expander chain is decreased by at most $|\{b_j\} \cap P_G|$. On the other hand by putting all the vertices in the layers $\{b_j\}$ and $\{c_{i,j}\}$ for any even i in P_R, and all the vertices in the layers $\{c_{i,j}\}$ for any odd i in P_G, we will show that the cardinality of the cut in the expander chain is increased by at least the same number.

To show this, we calculate the gain that is guaranteed between any layers of vertices achieved by putting all the vertices as specified above. We will make use of the property of bipartite expanders described above.

We first consider the number g_0 of uncut edges between the b_j vertices and the $c_{1,j}$ vertices. By changing the partition as described above, the uncut edges will be cut, so the gain will be g_0. Let $m_0 = |\{b_j\} \cap P_G|$ and $m_1 = |\{c_{1,j}\} \cap P_R|$. By construction $m_0 \le N/2$. We need to consider a few cases.

Case 1. $m_1 \le N/2$

The m_0 vertices in $\{b_j\} \cap P_G$ are connected to at least $(1+\alpha) \cdot m_0$ vertices in $\{c_{1,j}\}$. Of these vertices at least $(1+\alpha) \cdot m_0 - m_1$ must be in $\{c_{1,j}\} \cap P_G$, which means that at least $(1+\alpha) \cdot m_0 - m_1$ of the edges from $\{b_j\} \cap P_G$ are uncut. Similarly $(1+\alpha) \cdot m_1 - m_0$ of the edges from $\{c_{1,j}\} \cap P_R$ to $\{b_j\}$ are uncut. Therefore $g_0 \ge (1+\alpha) \cdot m_0 - m_1 + (1+\alpha) \cdot m_1 - m_0 \ge \alpha(m_0 + m_1)$.

There are two subcases depending on if m_1, the number of red vertices in the $\{c_{1,j}\}$ layer, is greater or smaller than $m_0/(1+\alpha)$.

Case 1a. $m_1 \ge m_0/(1+\alpha)$

$$g_0 \ge \alpha \cdot (m_0 + (1-\alpha)m_0) \ge (2 \cdot \alpha - \alpha^2) \cdot m_0 \ge \alpha \cdot m_0.$$

Case 1b. $m_1 < m_0/(1+\alpha)$

If we just look at the first type of noncut edges we get

$$g_0 \ge (1+\alpha) \cdot m_0 - m_1 \ge (1 + \alpha - (1 - \alpha + \alpha^2)) \cdot m_0 = (2 \cdot \alpha - \alpha^2) \cdot m_0 \ge \alpha \cdot m_0.$$

Case 2. $m_1 > N/2$

Since the expander property is valid only for subsets of size at most $N/2$ we just look at the noncut edges from a subset of size $N/2$ of $\{c_{1,j}\} \cap P_R$. Then we get

$$g_0 \geq (1+\alpha) \cdot (N/2) - m_0 \geq (1+\alpha) \cdot m_0 - m_0 = \alpha \cdot m_0.$$

Thus in all cases the number of uncut edges between the two layers of vertices is at least $\alpha \cdot m_0$.

By similarly counting noncut edges in the rest of the expander chain we will obtain that if at layer i the number m_i of vertices placed in the "wrong" part of the cut is no greater than $N/2$, the gain g_i between layer i and layer $i+1$ is at least $\alpha \cdot m_i$.

If in some layer i we have $m_i > N/2$, the gain g_i between layer i and layer $i+1$ can be calculated exactly as above, but with respect to the vertices placed in the right part instead of the wrong part. The gain then becomes $g_i \geq \alpha \cdot (N - m_i)$.

Therefore, if $(2/3) \cdot m_0 \leq m_i \leq N - (2/3) \cdot m_0$ for all i we will gain at least $(2/3)\alpha \cdot m_0$ in each layer. If we choose $k \geq 3/(2 \cdot \alpha)$ the total gain will become at least m_0.

In order to consider the cases where m_i is small or large for some i we compute the gain in another way. We observe that if $m_0 > m_1$ at least $3(m_0 - m_1)$ of the edges from $\{b_j\} \cap P_G$ to $\{c_{1,j}\}$ must be uncut. In the same way we get the gain $g_j \geq \max\{3(m_j - m_{j+1}), 0\}$ for the layer j. Suppose $m_i < (2/3) \cdot m_0$ for some i. Summing over all layers we get a total gain of at least $3(m_0 - m_i) > m_0$.

Finally, if we for some i have $m_i > N - (2/3) \cdot m_0$ we can do as for small m_i, but work in the other direction. We then get a total gain of at least $3(N - (2/3) \cdot m_0 - (N - m_0)) > m_0$, which completes the proof. □

Theorem 4.2 MAX CUT-3 *is* APX*-complete.*

Proof. We simply give a reduction from MAX CUT-6 for multigraphs to MAX CUT-3 for simple graphs.

For a vertex v of degree $d, 2 \leq d \leq 4$ we do like follows: split the vertex into d *split vertices* of degree 1. Then add d extra vertices and construct a ring where every first vertex is one of the split vertices and every second vertex is a new vertex, see figure 6.

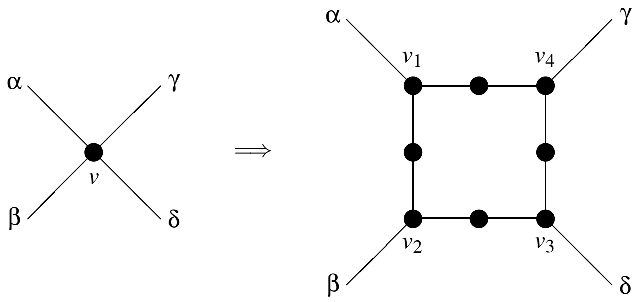


Figure 6: The transformation of a degree 4 vertex in the reduction from MAX CUT-6 to MAX CUT-3.

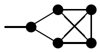
It is easy to see that if the split vertices are put in the same part of the partition and the new vertices are put in the other part, then every edge in the ring will be cut. Otherwise at least two of the edges in the ring will be uncut, and we can move one or two split vertices to the other part of the partition without decreasing the size of the cut.

For a vertex of degree 5 or 6 we do in a similar way, except that we need to construct two rings instead of just one. For every split vertex v_i we add two new vertices (l_i and r_i) and edges (v_i, l_i) and (v_i, r_i). We then put the l_i vertices in a ring, interleaved with new vertices, and similarly we put the r_i vertices in another ring, interleaved with new vertices. It is possible to show that given a solution we can without decreasing the size of the cut modify it so that all the 5 or 6 split vertices, originating from the same vertex, are in the same part of the partition.

When putting all split vertices in the same part as the majority of them at most 3 of the edges from the split vertices to the rest of the graph may be uncut in the worst case. We will show how to gain at least four edges in the internal ring structure by putting all split vertices in the same part.

First we examine how the vertices in the two rings are partitioned, and observe that the size of the cut will not decrease if we partition the vertices in the two rings in the same way (by using the partition of the "best cut" ring for both rings), so we do that. Now, if the vertices in the rings (including the split vertices) are not partitioned alternately red and green we will gain at least two edges per ring in the cut by partition them so, and thereby the split vertices will become in the same part of the partition.

This is an L-reduction with $\alpha = 13$ and $\beta = 1$, and the constructed graph is clearly simple and of degree 3. $\square$

MAX CUT is APX-hard even for cubic graphs. To show this we just have to extend the graph constructed in the above proof. For each node of degree 1 or 2 we simply add edges to gadgets of the following type: 

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