Comment on "Second derivative ridges are straight lines and the implications for computing Lagrangian Coherent Structures, Physica D 2012.05.006"

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Abstract

The finite-time Lyapunov exponent (FTLE) has become a standard tool for analyzing unsteady flow phenomena, partly since its *ridges* can be interpreted as Lagrangian coherent structures (LCS). While there are several definitions for ridges, a particular one called *second derivative ridges* has been introduced in the context of LCS, but subsequently received criticism from several researchers for being over-constrained. Among the critics are Norgard and Bremer 2012 [1], who suggest furthermore that the widely used definition of *height ridges* was a part of the definition of *second derivative ridges*, and that *topological separatrices* were ill-suited for describing ridges. We show that (a) the definitions of height ridges and second derivative ridges are *not* directly related, and (b) there is an interdisciplinary consensus throughout the literature that topological separatrices describe ridges. Furthermore, we provide pointers to practically feasible and numerically stable ridge extraction schemes for FTLE fields.

For a scalar function $\sigma : D \subset \mathbb{R}^2 \to \mathbb{R}$, the following definition is given by Shadden et al. [2]. The Hessian of σ is denoted by $\Sigma_{ij} = \partial^2\sigma \big/ \partial x_i \partial x_j$.

Definition. A *second derivative ridge* of σ is an injective curve $\mathbf{c} : (a, b) \to D$ satisfying the following conditions for each $s \in (a, b)$:

SR1. The vectors $\mathbf{c}'(s)$ and $\nabla\sigma(\mathbf{c}(s))$ are parallel.

SR2. $\Sigma(\mathbf{n}, \mathbf{n}) = \min_{\|\mathbf{u}\|=1} \Sigma(\mathbf{u}, \mathbf{u}) < 0$, where $\mathbf{n}$ is a unit normal vector to the curve $\mathbf{c}(s)$ and Σ is thought of as a bilinear form evaluated at the point $\mathbf{c}(s)$.

It has been noted by several researchers that curves described by the conditions SR1 and SR2 are over-constrained [3, 1]. This can be seen by noting that both conditions prescribe a different tangent direction for the curve $\mathbf{c}$: SR1 describes $\mathbf{c}$ as a curve tangential to the gradient. The equation in SR2 describes $\mathbf{c}$ as a curve tangential to the eigenvector of the Hessian corresponding to the larger (signed) eigenvalue. Due to this over-constrained definition, second derivative ridges are theoretically infeasible.

This observation was recently also published by Norgard and Bremer [1], but unfortunately, it led them to several incorrect conclusions. Firstly, they claimed that omitting condition SR1 leads to the definition of a *height ridge*. However, SR2 does not define a height ridge but a *tensor field line*, where the tensor field is given by the Hessian Σ. The definition of a height ridge fundamentally differs from SR2, as it does not involve the curve normal (or tangent) direction, but only derivatives of σ. It is obtained by replacing in SR2 the curve normal by the cogradient of σ, i.e., the gradient of σ rotated by $\frac{\pi}{2}$.

Secondly, the authors of [1] made the following statement: "More generally the result proves that in general (height) ridge lines and integral lines (curves tangent to the gradient) are disjunct." Because of the incorrect definition of height ridges in [1], this statement would have to be rephrased for tensor field lines anyway. However, there is a more fundamental question underlying this statement: what is the relationship between locally defined ridges (e.g., height ridges) and globally defined ridges (e.g., watersheds, topological separatrices)? In fact, it is often even overlooked that the watershed definition is a valid ridge definition and even among the first ones to be proposed, see Maxwell [4]. The relationship between these mathematically different approaches (local vs. global definition) was debated vigorously in the computer vision community in the early 1990s (see, e.g., Koenderink and van Doorn [5]). Nowadays, there is a consensus that both approaches have their merits, and López et al. [6] provide an exhaustive evaluation of the equated local and global definitions. In this light, the criticism of [1] regarding modern discrete approaches to watershed extraction [7, 8] is unsubstantiated.

The definition of second derivative ridges is over-constrained, but a relaxed version of the condition SR1 is often rather desirable for locally defined ridges: SR1 demands that the curve is tangential to the gradient $\nabla\sigma$, and a relaxed version of this is to require that the angle between the curve tangent and the gradient is below some threshold. Such a threshold agrees with the intuition that ridges should roughly follow the slope lines. For example, this has been used by Peikert and Sadlo [9] as a filter criterion for height ridges, such that parts of the curves are pruned which do not follow the above intuition of a "ridge." Peikert and Sadlo [9] show that such a removal of false positives is essential in practice, since height ridges tend to have this kind of spurious parts at both ends, often after a sharp turn. As a final remark, for the special case of FTLE fields, the *C-ridge* [3] provides a numerically favorable alternative to height ridges. In C-ridges, the Cauchy-Green tensor is used instead of Σ. The defining equation for C-ridges is one of the properties of the *weak LCS* as proposed by Haller [10, thm. 7].

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